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IMPOSING NODES TO THE NORMAL MODES OF A LINEAR ELASTIC STRUCTURE

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Spring–mass systems have long been used to suppress excess vibration in structural systems. In this paper, a chain of oscillators is used as a passive means of introducing nodes for the normal modes of a one-dimensional, arbitrarily supported, linear elastic structure, where a desired node can either coincide with the oscillator chain location or can be located elsewhere. It is shown that when the oscillator chain and the node are collocated, it is always possible to induce a node at any location along the structure for any given normal mode. When the oscillator chain and the node are not collected, however, it is only possible to induce a node for certain normal modes. Finally, a procedure to guide the proper selection of the oscillator chain parameters in order to induce nodes is outlined in detail.

1. INTRODUCTION

The free vibration of a linear elastic structure carrying oscillators has been studied by many authors over the years, and hence only a few selected historical references are given here [1–6]. In recent years, Dowell [7] used the Lagrange multiplier approach to obtain the frequency equation of a beam with an elastically mounted mass. He also examined the effects of the added oscillator on the natural frequencies of the combined system. Nicholson and Bergman [8] used the dynamic Green's function approach to derive the characteristic equation for the natural frequencies of a cantilevered beam connected at discrete points to oscillators with no rigid body degree of freedom, and of a simply supported beam connected at discrete points to oscillators with a rigid body degree of freedom. Ercoli and Laura [9] used the assumed-modes method to study the effects of concentrated masses elastically mounted to a beam on the frequencies of the system. Rossi *et al*. [10] examined the free vibrations of Timoshenko beams carrying elastically mounted concentrated masses. They solved the problem exactly and validated their solution using the finite element method. Kukla and Posiadała [11] investigated the free vibrations of beams with elastically mounted masses. They obtained closed form expressions for the frequency equations by means of the Green's functions. Gürgöze [12] examined the free vibration of a cantilever beam with an end mass to which an oscillator is attached. He used the Lagrange multiplier formalism in analyzing the free vibration of the system.

In all the above papers, the authors used various approaches to analyze the effects of oscillators on the free vibration and natural frequencies of the combined system, but none discussed the idea of using an oscillator chain as a possible means of imposing nodes to the normal modes of the combined system. Nicholson and Bergman [8] did mention that when the ith natural frequency of a combined system consisting of a beam connected to an oscillator with a rigid body degree of freedom is equal to the oscillator natural frequency, then the oscillator acts as a vibration absorber for the ith mode of vibration. However, they did not pursue the issue of how an oscillator can be utilized to induce nodes to the normal modes of the combined system.

In this paper, it will be shown that a chain of oscillators can be used to passively impose nodes to the normal modes of any one-dimensional elastic structure. By properly selecting values for the oscillator parameters and by properly choosing an attachment location for the oscillator chain, one can dictate the location of the nodes anywhere along the structure, and for any normal mode. This is important because it would allow us to place instruments sensitive to vibration at any desired location along the structure, i.e., at or near a node.

2. THEORY

2.1. FREQUENCY EQUATION FOR THE COMBINED SYSTEM

Consider an arbitrarily supported, one-dimensional, linear elastic structure to which is attached a chain of undamped oscillators (see Figure 1). Using the assumed-modes method [13], the physical deflection of the structure at a point x is given by

$$w(x, t) = \sum_{i=1}^{N} \phi_i(x)\eta_i(t), \tag{1}$$

where the $\phi_i(x)$ are the eigenfunctions of the unconstrained structure (i.e., the structure without any attachments), that serve as the basis functions for this approximate solution, the $\eta_i(t)$ are the corresponding generalized co-ordinates, and N is the number of modes used in the assumed-modes expansion. The total kinetic energy of the combined system can be expressed as

$$T = \frac{1}{2}\sum_{i=1}^{N} M_i\dot{\eta}_i^2(t) + \frac{1}{2}\sum_{i=1}^{M} m_i\dot{z}_i^2(t), \tag{2}$$

where the M_i are the generalized masses, m_i is the mass of the ith oscillator, $z_i(t)$ is its displacement, M is the total number of oscillators in the chain, and an overdot denotes a derivative with respect to time. The total potential energy is given by

$$V = \frac{1}{2}\sum_{i=1}^{N} K_i \eta_i^2(t) + \frac{1}{2}\sum_{i=1}^{M} k_i[z_i(t) - z_{i-1}(t)]^2, \qquad z_0(t) = w(x_1, t), \tag{3}$$

where the K_i are the generalized spring constants, k_i is the spring stiffness of the ith oscillator, and x_1 denotes the constraint location of the chain of oscillators.

Applying Lagrange's equations and assuming simple harmonic motion,

$$\eta_i(t) = \bar{\eta}_i\, \mathrm{e}^{\mathrm{j}\omega t}, \qquad z_i(t) = \bar{z}_i\, \mathrm{e}^{\mathrm{j}\omega t}, \tag{4}$$

where $\mathrm{j} = \sqrt{-1}$ and ω is the natural frequency of the combined system, the eigenvalue equation for the system of Figure 1 is given by

$$\begin{bmatrix} [\mathscr{K}] & [R] \\ [R]^T & [k] \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathbf{z}} \end{bmatrix} = \omega^2 \begin{bmatrix} [\mathscr{M}] & [0] \\ [0]^T & [m] \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathbf{z}} \end{bmatrix}, \tag{5}$$

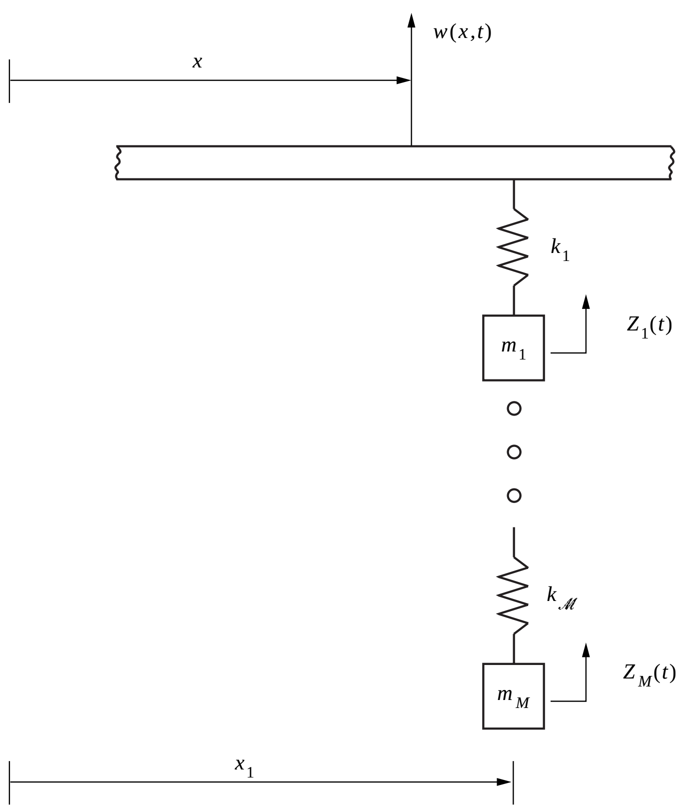

Figure 1. An arbitrarily supported, one-dimensional, linear elastic structure to which is attached a chain of undamped oscillators.

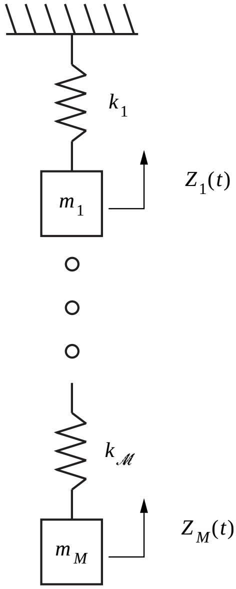

Figure 2. A grounded chain of undamped oscillators.

where $\bar{\boldsymbol{\eta}} = [\bar{\eta}_1 \;\; \bar{\eta}_2 \; . \; . \; . \; . \; \bar{\eta}_N]^T$ is the vector of generalized co-ordinates, $\bar{\mathbf{z}} = [\bar{z}_1 \;\; \bar{z}_2 \; . \; . \; . \; \bar{z}_M]^T$, the $M \times M$ matrices $[m]$ and $[k]$ are

$$[m] = \text{diag}\,[m_i], \qquad [k] = \text{tridiag}\,[-k_i \quad k_i + k_{i+1} \quad -k_{i+1}]; \tag{6}$$

the $N \times N$ matrices $[\mathscr{M}]$ and $[\mathscr{K}]$ are

$$[\mathscr{M}] = [M^d], \qquad [\mathscr{K}] = [K^d] + k_1 \boldsymbol{\phi}(x_1)\boldsymbol{\phi}^T(x_1), \tag{7}$$

where $[M^d]$ and $[K^d]$ are diagonal matrices whose ith diagonal elements are M_i and K_i, respectively, and

$$\boldsymbol{\phi}(x_1) = [\phi_1(x_1) \quad \phi_2(x_1) \cdots \phi_N(x_1)]^T; \tag{8}$$

the $N \times M$ matrices $[R]$ and $[0]$ are given by

$$[R] = [-k_1 \boldsymbol{\phi}(x_1) \quad \mathbf{0} \quad \mathbf{0} \cdots \mathbf{0}] \qquad \text{and} \qquad [0] = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \cdots \mathbf{0}], \tag{9}$$

where each of the $\mathscr{M}$ column vectors in the matrix is of length $\mathscr{N}$. Finally, note that $[\mathscr{M}]$ is a diagonal matrix, and $[\mathscr{K}]$ is a diagonal matrix modified by a simple rank one matrix.

By comparing the natural frequencies of the combined system of Figure 1 to those of the grounded oscillator chain of Figure 2, one can determine whether the constraint location, x_1, coincides with the node of a normal mode of the combined system. Consider for now the special case where the linear elastic structure is carrying a single oscillator ($M = 1$), with stiffness k and mass m. Then equation (5) simplifies to

$$\begin{bmatrix} [\mathscr{K}] & -k\boldsymbol{\phi}(x_1) \\ -k\boldsymbol{\phi}^T(x_1) & k \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{z} \end{bmatrix} = \omega^2 \begin{bmatrix} [\mathscr{M}] & \mathbf{0} \\ \mathbf{0}^T & m \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{z} \end{bmatrix}. \tag{10}$$

From the second of equations (10), one has

$$-k\boldsymbol{\phi}^T(x_1)\bar{\boldsymbol{\eta}} + k\bar{z} = \omega^2 m\bar{z} \tag{11}$$

or

$$(k - \omega^2 m)\bar{z} = k\boldsymbol{\phi}^T(x_1)\bar{\boldsymbol{\eta}} = k\sum_{i=1}^{N}\phi_i(x_1)\bar{\eta}_i. \tag{12}$$

Note that if $\omega^2 = k/m$, then

$$\sum_{i=1}^{N}\phi_i(x_1)\eta_i(t) = w(x_1, t) = 0. \tag{13}$$

Thus, if the simple oscillator natural frequency, $\sqrt{k/m}$, is equal to one of the combined natural frequencies of the system, say ω_j^{sys}, then the oscillator is attached at a node of the jth normal mode. Using the generalized differential equation for the combined system, Nicholson and Bergman [8] showed the above for the special case where the linear structure is an Euler–Bernoulli beam. Here, the assumed-modes method was used to reach the same conclusion for any linear elastic structure.

Let us now return to the general case of a structure carrying a chain of M oscillators. From equation (5), one has

$$[R]^T\bar{\boldsymbol{\eta}} + [k]\bar{\mathbf{z}} = \omega^2[m]\bar{\mathbf{z}}. \tag{14}$$

Rearranging the above yields

$$([k] - \omega^2[m])\bar{\mathbf{z}} = -[R]^T\bar{\boldsymbol{\eta}}. \tag{15}$$

For a given constraint location, x_1, assume one of the natural frequencies of the combined system satisfies the free vibration frequency equation for the grounded oscillator chain of Figure 2:

$$\det([k] - \omega^2[m]) = 0. \tag{16}$$

For non-trivial $\bar{\mathbf{z}}$, equation (16) holds if and only if

$$([k] - \omega^2[m])\bar{\mathbf{z}} = -[R]^T\bar{\boldsymbol{\eta}} = \mathbf{0} \tag{17}$$

or

$$w(x_1, t) = \sum_{i=1}^{N}\phi_i(x_1)\eta_i(t) = 0. \tag{18}$$

Thus, when a natural frequency of the combined system in Figure 1, say ω_j^{sys}, coincides with a natural frequency of the grounded oscillator chain of Figure 2, then the attachment location, x_1, is a node for the jth normal mode of the structure, i.e., the oscillator chain acts as a vibration absorber for the jth mode of vibration.

While it may appear that the free response of the combined system of Figure 1 requires one to solve a generalized eigenvalue problem of size $(N + M) \times (N + M)$ (see equation (5)), by simple manipulation one can reduce the generalized eigenvalue problem to a simple scalar equation consisting of the sum of N terms, resulting in substantial computational savings. From the last equation of equation (5), one has

$$k_M \bar{z}_{m-1} = \alpha_M \bar{z}_M \qquad \text{with} \qquad \alpha_M = k_M - \omega^2 m_M. \tag{19}$$

From the next to last equation of equation (5), one obtains

$$k_{M-1} \bar{z}_{M-2} = \alpha_{M-1} \bar{z}_{M-1} \quad \text{with} \quad \alpha_{M-1} = (k_{M-1} + k_M) - \frac{k_M^2}{\alpha_M} - \omega^2 m_{M-1}. \tag{20}$$

The above is repeated until one obtains

$$k_1 \boldsymbol{\phi}^T(x_1) \bar{\boldsymbol{\eta}} = \alpha_1 \bar{z}_1 \qquad \text{with} \qquad \alpha_1 = (k_1 + k_2) - \frac{k_2^2}{\alpha_2} - \omega^2 m_1. \tag{21}$$

The recursive formula for the coefficients α_i is thus given by

$$\alpha_i = (k_i + k_{i+1}) - \frac{k_{i+1}^2}{\alpha_{i+1}} - \omega^2 m_i, \qquad i = 1, 2, \ldots, M-1. \tag{22}$$

Equation (5) also gives

$$[\mathscr{K}]\bar{\boldsymbol{\eta}} - k_1 \boldsymbol{\phi}(x_1) \bar{z}_1 = \omega^2 [\mathscr{M}] \bar{\boldsymbol{\eta}}. \tag{23}$$

Solving for $\bar{z}_1$ by using equation (21) and substituting its expression into equation (23) yields

$$([K^d] + \alpha_0 \boldsymbol{\phi}(x_1) \boldsymbol{\phi}^T(x_1)) \bar{\boldsymbol{\eta}} = \omega^2 [M^d] \bar{\boldsymbol{\eta}}, \tag{24}$$

where the coefficient α_0 is given by

$$\alpha_0 = k_1 - \frac{k_1^2}{\alpha_1} \tag{25}$$

and it depends on ω and all the oscillator parameters, the k_i's and m_i's, for $i = 1, \ldots, M$. Note that by simple algebraic manipulation, the $(N + M) \times (N + M)$ generalized eigenvalue problem of equation (5) has been reduced to one of size $N \times N$. For non-trivial $\bar{\boldsymbol{\eta}}$, the eigenvalues, ω^2, must make the following $N \times N$ characteristic determinant equal to zero:

$$\det \{[K^d] + \alpha_0 \boldsymbol{\phi}(x_1) \boldsymbol{\phi}^T(x_1) - \omega^2 [M^d]\} = 0. \tag{26}$$

Upon rearranging, equation (26) becomes

$$\det \{[K^d] - \omega^2 [M^d]\} \det \{[I] + \alpha_0 ([K^d] - \omega^2 [M^d])^{-1} \boldsymbol{\phi}(x_1) \boldsymbol{\phi}^T(x_1)\} = 0. \tag{27}$$

After some algebra, equation (27) can be shown to be identical to:

$$\prod_{i=1}^{N} (K_i - \omega^2 M_i) \left(1 + \alpha_0 \sum_{i=1}^{N} \frac{\phi_i^2(x_1)}{K_i - \omega^2 M_i} \right) = 0. \tag{28}$$

The above scalar equation gives the natural frequencies of the combined system of Figure 1. When x_1 does not coincide with any node of the unconstrained component modes, the eigenvalues of the constrained and unconstrained systems must be distinct; thus $K_i \neq \omega^2 M_i$ and the frequency equation of Figure 1 simplifies to

$$1 + \alpha_0 \sum_{i=1}^{N} \frac{\phi_i^2(x_1)}{K_i - \omega^2 M_i} = 0, \tag{29}$$

the solution of which can be very easily obtained. However, when x_1 is coincident with a node of the ith unconstrained component mode, then one of the natural frequencies of the combined system will be identical to $\sqrt{K_i/M_i}$, which is the natural frequency of the ith unconstrained component mode, and the remaining natural frequencies can still be extracted by solving equation (29).

2.2. OSCILLATOR CHAIN AND NODAL POINT ARE COLLOCATED

The above results show that the simple oscillator chain behaves as a vibration absorber. As long as a natural frequency of the combined system coincides with a natural frequency of the grounded chain of oscillators, the oscillator chain attachment location, x_1, is a node. For this case, we say that the oscillator chain and the node are *collocated*.

For a given application, suppose it is desired that the combined system have a node at x_1 only for certain normal modes. This requirement precludes us from using a rigid support at this location, since a node would then be introduced for all the normal modes. The results obtained in the previous section show that having a node at x_1 for certain normal modes can be achieved by simply attaching an oscillator chain, with appropriate oscillator parameters, at that location.

Assuming that x_1 does not coincide with any node of the mode shapes of the unconstrained structure, the frequency equation for the system of Figure 1 is given by equation (29), where the coefficient α_0 can be expressed as

$$\alpha_0 = \frac{N(\omega)}{D(\omega)}. \tag{30}$$

It can be shown that by setting the denominator polynomial, $D(\omega)$, equal to zero, one obtains the free vibration frequency equation for the grounded chain of oscillators of Figure 2. Thus, when a natural frequency of the combined system of Figure 1, ω_j^{sys}, equals a natural frequency of the grounded oscillator chain of Figure 2, ω_r^{osc}, equation (29) simplifies to

$$\sum_{i=1}^{N} \frac{\phi_i^2(x_1)}{K_i - (\omega_r^{osc})^2 M_i} = 0. \tag{31}$$

For a given x_1, equation (31) allows us to determine a set of ω_r^{osc}. These ω_r^{osc} correspond to the possible natural frequencies of the grounded oscillator chain for which x_1 is a node. Depending on which normal modes one wishes to have a node for at x_1, one selects the k_i and m_i accordingly. For instance, suppose one requires a node at x_1 only for the third normal mode. In this case, one can attach a single oscillator, of parameters k and m, such that the third natural frequency of the combined system is identical to the grounded oscillator natural frequency, i.e.,

$\omega_3^{sys} = \sqrt{k/m}$. To impose a node at x_1 for multiple normal modes, one can attach a chain of oscillators whose parameters, the k_i's and m_i's, are chosen such that the natural frequencies of the desired normal modes are identical to the natural frequencies of the grounded chain of oscillators. The selection of the k_i's and m_i's requires one to solve an inverse eigenvalue problem governing the free vibration of the grounded oscillator chain of Figure 2, i.e., knowing the desired natural frequencies of the oscillator chain of Figure 2, one determines the required k_i's and m_i's.

Finally, because a node can be imposed at x_1 for the jth normal mode as long as $\omega_j^{sys} = \sqrt{k/m}$, the selection of the oscillator parameters is not unique. The actual choice is generally dictated by limitations placed on the vibration amplitude of the oscillator mass.

2.3. OSCILLATOR CHAIN AND NODAL POINT ARE NOT COLLOCATED

The eigenfunction or mode shape, $\mathscr{W}(x)$, of the combined system of Figure 1 can be obtained from equation (1) as follows:

$$w(x, t) = \sum_{i=1}^{N} \phi_i(x)\eta_i(t) = \mathscr{W}(x)\zeta(t). \tag{32}$$

Using the Lagrange multiplier approach [7], it can be shown that the ith element for the jth eigenvector of the combined system of Figure 1 is given by

$$\bar{\eta}_i^j = \frac{\phi_i(x_1)}{K_i - (\omega_j^{sys})^2 M_i}, \tag{33}$$

where ω_j^{sys} is the jth natural frequency of the overall system. Thus, the jth mode shape of the combined system is given by

$$\mathscr{W}_j(x) = \sum_{i=1}^{N} \frac{\phi_i(x)\phi_i(x_1)}{K_i - (\omega_j^{sys})^2 M_i}. \tag{34}$$

When a node is imposed at $x = x_1$ for the jth normal mode, equation (34) simplifies to

$$\mathscr{W}_j(x_1) = \sum_{i=1}^{N} \frac{\phi_i^2(x_1)}{K_i - (\omega_j^{sys})^2 M_i} = 0. \tag{35}$$

Comparing equations (31) and (35), it is noted that when $\omega_j^{sys} = \omega_r^{osc}$, the two equations become identical. Thus, it has again been demonstrated that if the jth natural frequency of Figure 1 is equal to an oscillator natural frequency of Figure 2, the oscillator chain is attached to a node of the jth normal mode.

Suppose one wishes to impose a node at x_2 for a selected set of normal modes. However, due to various physical constraints, one cannot attach a chain of oscillators at that location, but instead at some other location x_1. For this case, the oscillator chain and the node are said to be *not collocated*, since $x_1 \neq x_2$. Of great interest then is the possible oscillator chain attachment location, x_1, and the corresponding oscillator parameters, the m_i's and k_i's, that one ought to select in order to have a node at x_2. From equation (34), a node at x_2 for the jth normal mode requires that

$$\mathscr{W}_j(x_2) = \sum_{i=1}^{N} \frac{\phi_i(x_2)\phi_i(x_1)}{K_i - (\omega_j^{sys})^2 M_i} = 0. \tag{36}$$

To determine the possible oscillator chain locations and the oscillator parameters which lead to a node at x_2, the following steps are required. (1) For a desired node location, x_2, x_1 (the oscillator chain attachment location) is varied and the set of ω_j^{sys} (the natural frequencies of the combined system) which satisfies equation (36) is computed. (2) The ω_j^{sys} dictate the normal modes for which one can have a node at x_2. If a given ω_j^{sys} falls outside the frequency band for which the desired normal mode lies, it is impossible to impose a node at x_2 for that particular normal mode (this is discussed in more detail in section 4). (3) For a given x_1 and ω_j^{sys} (ω_j^{sys} denotes the natural frequency of the jth normal mode for which one wishes to have a node at x_2), one then solves for the values of m_i and k_i of the oscillator chain which satisfy equation (29). (4) For a given x_1 and ω_j^{sys}, the selection of m_i and k_i is not unique; the actual choice is generally limited by the vibration amplitudes of the m_i.

3. RESULTS

Suppose one wishes to impose a node at x_1 for certain normal modes of an arbitrarily supported one-dimensional structure. This can be easily realized by attaching a chain of oscillators, with appropriate values of k_i and m_i, such that the oscillator chain and the node are collocated. To illustrate this method, a simply supported uniform Euler–Bernoulli beam is considered, whose normalized eigenfunctions (with respect to the linear density of the beam) and eigenvalues are given by

$$\phi_i(x) = \sqrt{\frac{2}{\rho L}} \sin\frac{i\pi x}{L} \qquad \text{and} \qquad \lambda_i = (i\pi)^4 \frac{EI}{(\rho L^4)}, \tag{37}$$

where ρ is the linear density or the mass per unit length of the beam and L is its length. Then

$$M_i = \int_0^L \rho\phi_i^2(x)\,\mathrm{d}x = 1 \qquad \text{and} \qquad K_i = \int_0^L EI\left(\frac{\mathrm{d}^2\phi_i}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = \lambda_i, \tag{38}$$

where E is the Young's modulus and I is the moment of inertia of the cross-section of the uniform beam. Equation (31) thus reduces to

$$\sum_{i=1}^{N} \frac{\sin^2(i\pi x_1/L)}{(i\pi)^4 - (\bar{\omega}_r^{osc})^2} = 0, \tag{39}$$

where $\bar{\omega}_r^{osc} = \omega_r^{osc}/\sqrt{EI/(\rho L^4)}$ is the rth dimensionless natural frequency which satisfies equation (39). Equation (39) allows one to vary the node location, x_1, and to compute the required $\bar{\omega}_r^{osc}$. To impose a node at x_1 for the jth normal mode, a single oscillator is attached, whose grounded natural frequency satisfies equation (39) and is equal to $\bar{\omega}_j^{sys} = \omega_j^{sys}/\sqrt{EI/(\rho L^4)}$. To impose a node at x_1 for p normal modes simultaneously, a chain of p oscillators is attached, whose grounded natural frequencies satisfy the solution of equation (39), and are equal to the natural frequencies of the p normal modes.

Figure 3 shows a plot that could be used to design such combined systems. It depicts the dimensionless oscillator natural frequency, $\bar{\omega}_r^{osc}$, versus the node location, x_1/L, when the oscillator chain and the node are collocated, for $N = 20$ (in all the subsequent analyses, $N = 20$). Because the simply supported beam is symmetric about its midspan, a node at x_1/L also leads to a node at $(1 - x_1/L)$. Thus, the x-axis is only shown from $x_1/L = 0$ to 1/2. Dowell noted in reference [7] that if a spring–mass combination (which by itself has a rigid body degree of freedom) is attached to another system, the frequencies that were originally higher than the spring–mass natural frequency are increased, those that were originally lower are decreased, and a new natural frequency appears between the original pair of frequencies nearest the oscillator natural frequency. Thus one would expect the $\bar{\omega}_r^{osc}$ of equation (39) to lie between the natural frequencies of a simply supported Euler–Bernoulli beam. These natural frequencies are given by $\bar{\omega}_i^{beam} = (i\pi)^2$, and they are represented by the horizontal lines in Figure 3. As anticipated, the range of oscillator natural frequency within which one can impose a node for a given normal mode is indeed banded. Specifically, $\bar{\omega}_r^{beam} \leqslant \bar{\omega}_r^{osc} \leqslant \bar{\omega}_{r+1}^{beam}$.

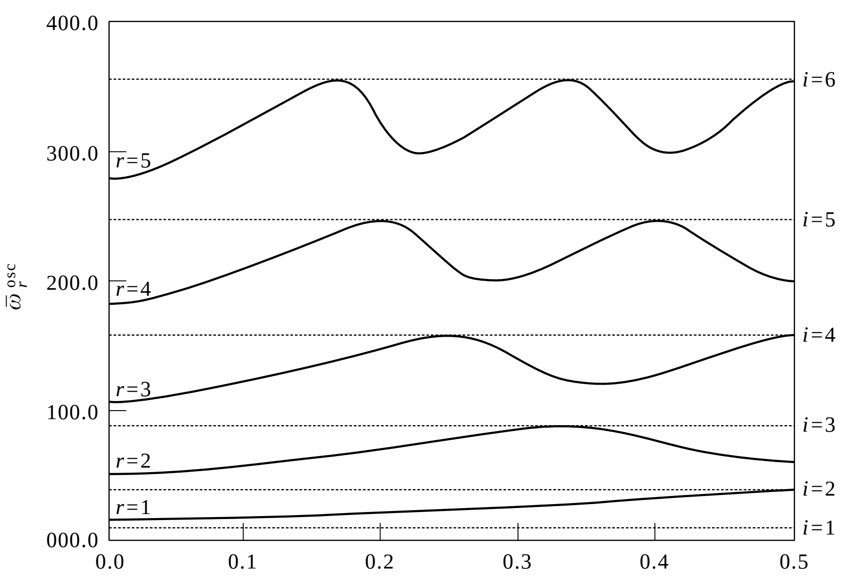

Figure 3. The rth oscillator natural frequency, $\bar{\omega}_r^{osc}$, of equation (39) versus the node location, x_1/L, when the oscillator and the node are collocated. The horizontal line, $\bar{\omega}_i^{beam} = (i\pi)^2$, represents the ith natural frequency of a simply supported Euler–Bernoulli beam.

TABLE 1

The required oscillator natural frequency, $\bar{\Omega}_l$, in order to induce a node at 0·37L for the lth normal mode only. The ω_j^{sys}'s are the natural frequencies of the combined system for the chosen $\bar{\Omega}_l$, and $\bar{\omega}_i^{beam}$ is the ith natural frequency of a simply supported uniform Euler–Bernoulli beam. The natural frequencies are all non-dimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$. The oscillator and node locations are collocated.

	$\bar{\Omega}_2 = 31{\cdot}0875$	$\bar{\Omega}_3 = 84{\cdot}2826$	$\bar{\Omega}_4 = 121{\cdot}7408$	$\bar{\Omega}_5 = 237{\cdot}2121$	$\bar{\omega}_i^{beam} = (i\pi)^2$
$\bar{\omega}_1^{sys}$	5·8965	5·9571	5·9621	5·9655	$\bar{\omega}_1^{beam} = 9{\cdot}8696$
$\bar{\omega}_2^{sys}$	<u>31·0875</u>	33·1385	33·2833	33·3794	$\bar{\omega}_2^{beam} = 39{\cdot}4784$
$\bar{\omega}_3^{sys}$	61·2517	<u>84·2862</u>	85·0820	85·4992	$\bar{\omega}_3^{beam} = 88{\cdot}8264$
$\bar{\omega}_4^{sys}$	90·6679	114·7106	<u>121·7408</u>	126·3606	$\bar{\omega}_4^{beam} = 157{\cdot}9137$
$\bar{\omega}_5^{sys}$	164·6456	211·3124	230·2278	<u>237·2121</u>	$\bar{\omega}_5^{beam} = 246{\cdot}7401$
$\bar{\omega}_6^{sys}$	247·6237	259·2890	285·6916	321·3917	$\bar{\omega}_6^{beam} = 355{\cdot}3058$

Outside the frequency band, one cannot impose any additional node for that given normal mode. The points of intersection between the $(r+1)$th horizontal line and the $\bar{\omega}_r^{osc}$ curve correspond to the node locations of the $(r+1)$th normal mode of a simply supported Euler–Bernoulli beam. At these locations, no oscillators are needed to induce nodes. Conversely, at these locations, an oscillator with any natural frequency will lead to a node. Not surprisingly, for a given node location, the oscillator natural frequency increases with the normal mode number, implying that in order to induce a node for a high mode, one needs to attach an oscillator with a large spring–mass natural frequency. Finally since $\bar{\omega}_1^{osc} > \bar{\omega}_1^{beam}$, it is not possible to induce a node for the first normal mode of the combined system considered here.

To impose a node at x_1 for a given normal mode, a single oscillator is attached, whose natural frequency can be extracted from the design plot of Figure 3. Let $\bar{\Omega}_l$ denote the required oscillator natural frequency for which the lth normal mode of the combined system has a node at x_1. Table 1 shows the required $\bar{\Omega}_l$ in order to induce a node at $x_1 = 0{\cdot}37L$, when the oscillator and the node are collocated. For the system under consideration, since it is impossible to induce a node for the first normal mode, $\bar{\Omega}_{r+1} = \bar{\omega}_r^{osc}$, $r = 1, 2, \ldots$, which implies that for a given x_1, the rth root of equation (39), $\bar{\omega}_r^{osc}$, gives rise to a node at x_1 for the $(r+1)$th the normal mode of the combined system. Moreover, observe that the natural frequencies of a simply supported Euler–Bernoulli beam (see the entries in the sixth column) that are lower than the oscillator natural frequency are decreased, while those that are higher are increased, and a new natural frequency is embedded between the original pair of frequencies nearest to the oscillator natural frequency. This is consistent with the results found by Dowell [7]. Figure 4 illustrates the first five normal modes of the combined system when an oscillator of frequency $\bar{\Omega}_l$, given in Table 1, is attached at $0{\cdot}3\mathscr{L}$. Note that when $\bar{\omega}_j^{sys} = \bar{\Omega}_j$, the oscillator gives rise to a node at its attachment location for the jth normal mode, as clearly depicted in Figure 4.

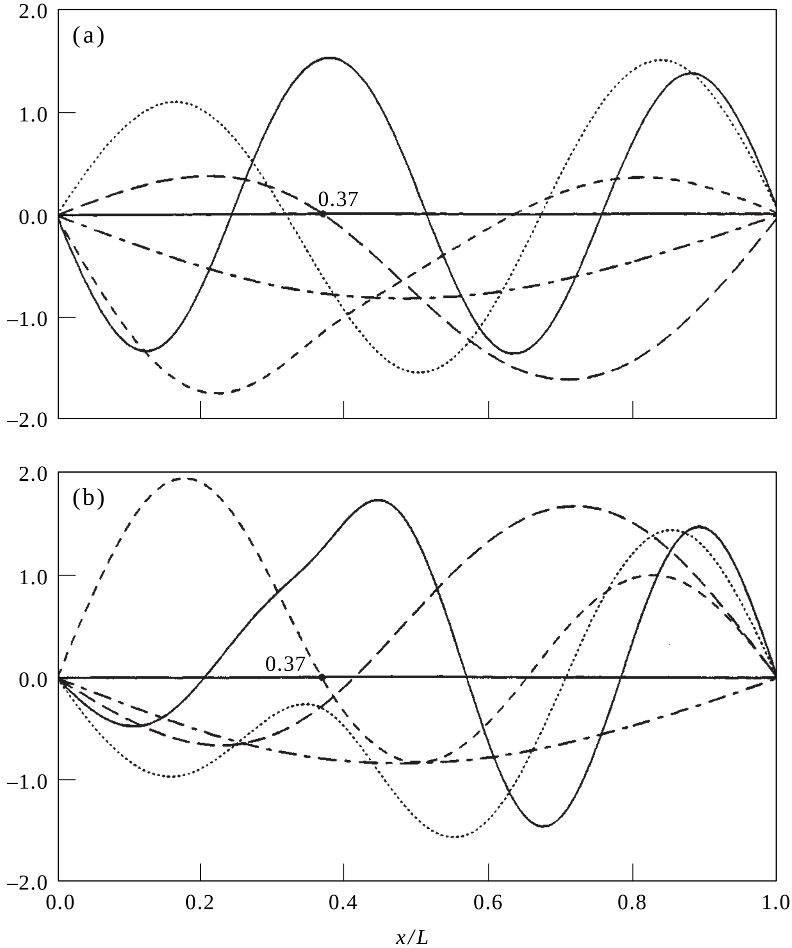

Figure 4. The first five normal modes of a combined system consisting of a simply supported Euler–Bernoulli beam to which is attached an undamped oscillator at 0·37L, of natural frequency $\bar{\Omega}_2 = 31{\cdot}0875$ (a) and $\bar{\Omega}_3 = 84{\cdot}2826$ (b). Modes 1 to 5 correspond to curves – · –, – – –, - - -, ..., ——, respectively. The oscillator and the node are collocated.

To impose a node at x_1 for p normal modes simultaneously, a chain of p oscillators is attached, with appropriate oscillator parameters, at x_1. For definiteness, suppose one wishes to impose a node at 0·23L for the second and fourth normal modes. To impose a node at x_1 for the second normal mode only, a single oscillator is attached at x_1, whose natural frequency is given by $\bar{\Omega}_2 = \bar{\omega}_1^{osc}$. To impose a node at x_1 for the fourth normal mode only, a single oscillator is attached at x_1 of frequency $\bar{\Omega}_4 = \bar{\omega}_3^{osc}$. To impose a node for the second and fourth normal modes simultaneously, a chain of two oscillators is attached at x_1, whose grounded oscillator natural frequencies are chosen such that $\bar{\Omega}_2 = \bar{\omega}_1^{osc} = 22{\cdot}6414$ and $\bar{\Omega}_4 = \bar{\omega}_2^{osc} = 74{\cdot}9470$ (see Figure 3 for $x_1 = 0{\cdot}23L$). The subscript of $\bar{\Omega}_l$ for the second oscillator natural frequency is no longer given by $l = n + 1$, where n is the normal mode number for which we wish to have a node at x_1 (as was the case for imposing a node for a single normal mode), but by $l = n + 2$, since a new

TABLE 2

The first six natural frequencies of the combined system for the two sets of stiffness values, $(\bar{k}_1^1, \bar{k}_2^1) = (1155{\cdot}4604, 2485{\cdot}6120)$ *and* $(\bar{k}_1^2, \bar{k}_2^2) = (4971{\cdot}2339, 579{\cdot}2302)$*, that induce a node at* $0{\cdot}23L$ *for the second and fourth normal modes. The oscillator masses are* $\bar{m}_1 = \bar{m}_2 = 1$*. The dimensionless parameters are defined as follows:* $\bar{m}_i = m_i/(\rho L)$, $\bar{k}_i^j = k_i^j/(EI/L^3)$*, and* $\bar{\omega}_p^{sys} = \omega_p^{sys}/\sqrt{EI/(\rho L^4)}$*. The oscillator and the node are collocated*

	$(\bar{k}_1^1, \bar{k}_2^1)$	$(\bar{k}_1^2, \bar{k}_2^2)$
$\bar{\omega}_1^{sys}$	5·6297	5·6703
$\bar{\omega}_2^{sys}$	<u>22·6414</u>	<u>22·6514</u>
$\bar{\omega}_3^{sys}$	57·9118	35·3202
$\bar{\omega}_4^{sys}$	<u>74·9470</u>	<u>74·9470</u>
$\bar{\omega}_5^{sys}$	103·6586	147·7930
$\bar{\omega}_6^{sys}$	158·5028	168·0244

frequency appears between the original pair of frequencies nearest the first oscillator natural frequency [7], pushing the combined system natural frequency count up by 1 to $n + 2$. The same bookkeeping applies when more than two normal modes are required to have a node at x_1. For simplicity, it is assumed $\bar{m}_1 = \bar{m}_2 = 1$, where $\bar{m}_i = m_i/(\rho L)$. Then by choosing values for $\bar{k}_1$ and $\bar{k}_2$ such that the grounded chain of oscillators has natural frequencies $\bar{\Omega}_2$ and $\bar{\Omega}_4$ (in effect, by solving an inverse eigenvalue problem), one obtains the following two sets of stiffness values: $(\bar{k}_1^1, \bar{k}_2^1) = (1155{\cdot}4604, 2485{\cdot}6120)$ and $(\bar{k}_1^2, \bar{k}_2^2) = (4971{\cdot}2339, 579.2302)$, where $\bar{k}_i^j = k_i^j/(EI/L^3)$ is the dimensionless spring stiffness for the ith spring in the oscillator chain for the jth set of stiffness values. Table 2 shows the dimensionless natural frequencies of the combined system for the above sets of stiffness values. Note that for both cases considered, the second and fourth natural frequencies of the overall system, $\bar{\omega}_2^{sys}$ and $\bar{\omega}_4^{sys}$, coincide with the oscillator natural frequencies chosen, $\bar{\Omega}_2$ and $\bar{\Omega}_4$. Figure 5 shows the first five normal modes of the combined system, whose oscillator stiffnesses are given by $(\bar{k}_1^1, \bar{k}_2^1)$ (see Figure 5(a)) and $(\bar{k}_1^2, \bar{k}_2^2)$ (see Figure 5(b)). While the normal modes are quite different, the second and fourth normal modes have a node at $0{\cdot}23L$ for both sets of the stiffness values.

Consider now the case where the oscillator chain and the node are not collocated. To impose a node at x_2 for a given normal mode only, a single oscillator is attached, of parameters $\bar{k} = k/(EI/L^3)$ and $\bar{m} = m/(\rho L)$, at x_1. For a given x_1, one first solves for the set of $\bar{\omega}_j^{sys}$ which satisfies equation (36). These $\bar{\omega}_j^{sys}$ immediately reveal for which normal modes one can impose a node at x_2 when the oscillator chain is attached at x_1. Figure 6 shows the $\bar{\omega}_j^{sys}$ as a function of x_1/L

for $x_2 = 0{\cdot}23L$. The ith horizontal line denotes the ith natural frequency of a simply supported Euler–Bernoulli beam. From Figure 6 the following are observed: (1) For a given x_1, when no $\bar{\omega}_j^{sys}$ lies between $\bar{\omega}_i^{beam}$ and $\bar{\omega}_{i+1}^{beam}$, a node cannot be imposed at x_2 for the $(i+1)$th normal mode. (2) The points of intersection between the $\bar{\omega}_j^{sys}$ curves and the ith horizontal line correspond to the node locations of the ith normal mode of a simply supported Euler–Bernoulli beam. (3) When x_1 and x_2 lie close to one another, it is possible to impose a node at x_2 for every normal mode. However, when x_1 and x_2 are far apart, the number of normal modes for which one can impose a node decreases significantly, and only the higher modes can be made to have a node at x_2 when the oscillator is attached at x_1. (4) Figure 6 also allows one to determine the possible range of attachment

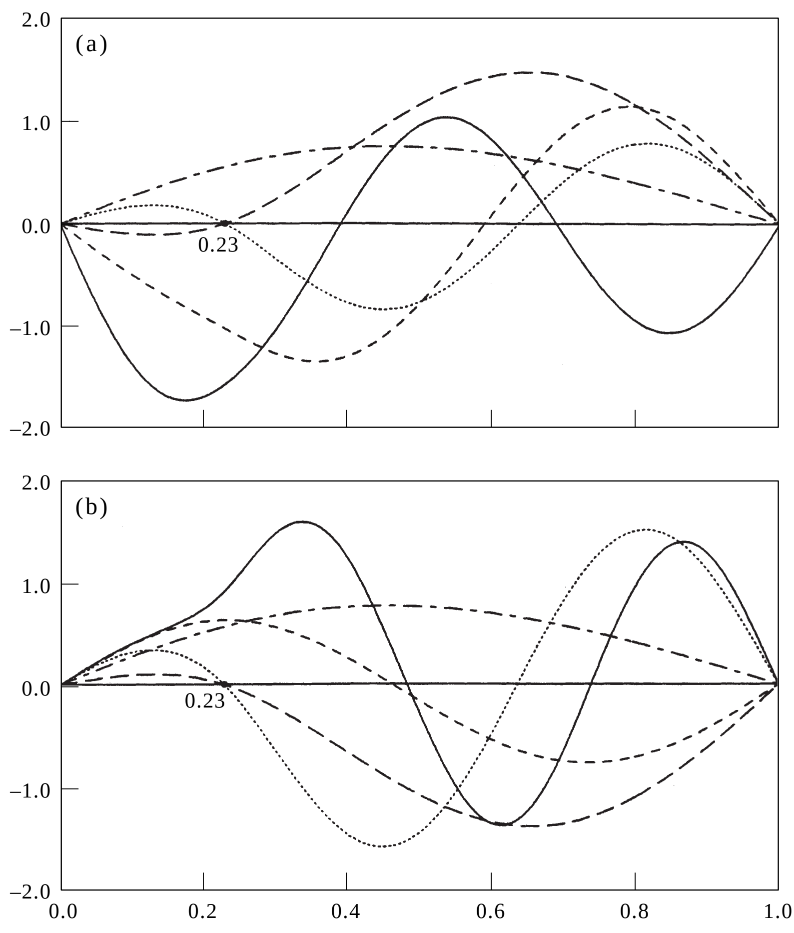

Figure 5. The first five normal modes of a combined system consisting of a simply supported Euler–Bernoulli beam to which is attached a chain of two undamped oscillators at $0{\cdot}23L$. The oscillator parameters, $\bar{m}_1 = \bar{m}_2 = 1$, $(\bar{k}_1^1, \bar{k}_2^1) = (1155{\cdot}4604, 2485{\cdot}6120)$ (a) and $(\bar{k}_1^2, \bar{k}_2^2) = (4971{\cdot}2339, 579{\cdot}2302)$ (b) are chosen such that the second and fourth normal modes have a node at $0{\cdot}23L$. The grounded oscillator natural frequencies are $\bar{\Omega}_2 = 22{\cdot}6514$ and $\bar{\Omega}_4 = 74{\cdot}9470$. Key as in Figure 4.

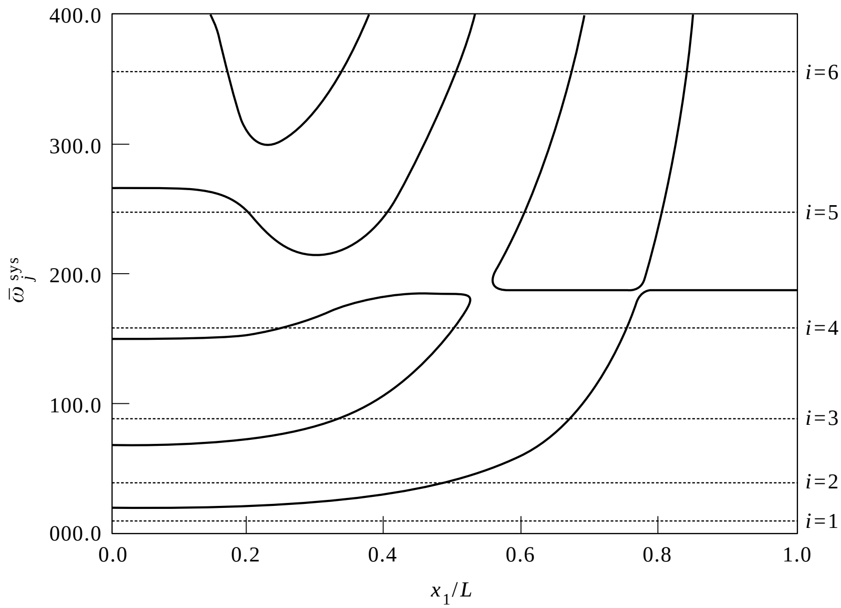

Figure 6. The natural frequencies of the combined system, the $\bar{\omega}_j^{sys}$, versus the attachment location, x_1/L, in order to have a node at $x_2 = 0{\cdot}23L$. The horizontal line, $\bar{\omega}_i^{beam} = (i\pi)^2$, represents the ith natural frequency of a simply supported Euler–Bernoulli beam.

location, x_1, for which one can successfully induce a node at x_2 for a given normal mode. For example, to impose a node at x_2 for the second normal mode whose natural frequency lies between $\bar{\omega}_1^{beam}$ and $\bar{\omega}_2^{beam}$, one can attach an oscillator anywhere between $0 < x_1 < L/2$. The permissible ranges to impose a node at x_2 for the third normal mode are given by $0 < x_1 < L/3$ and $L/2 < x_1 < 2L/3$; for the fourth normal mode, one has $0 < x_1 < L/4$, $L/3 < x_1 < L/2$ and $2L/3 < x_1 < 3L/4$. Similar ranges to impose a node at x_2 can be found for all the other normal modes.

The $\bar{\omega}_j^{sys}$ must also satisfy the frequency equation of the combined system of Figure 1, which for $M = 1$ can be expressed as

$$(\bar{\omega}_j^{sys})^2 - (\bar{\omega}^{osc})^2 + 2\bar{m}(\bar{\omega}_j^{sys})^2(\bar{\omega}^{osc})^2 \sum_{i=1}^{N} \frac{\sin^2 (i\pi x_1/L)}{(i\pi)^4 - (\bar{\omega}_j^{sys})^2} = f(\bar{m}, \bar{\omega}_j^{sys}, \bar{\omega}^{osc}) = 0, \tag{40}$$

where $\bar{\omega}^{osc} = \sqrt{k/m}/\sqrt{EI/(\rho L^4)}$. For any $\bar{\omega}_j^{sys}$ which satisfies equation (36), one can vary $\bar{\omega}^{osc}$ and plot $f(\bar{m}, \bar{\omega}_j^{sys}, \bar{\omega}^{osc})$ as a function of $\bar{m}$. By varying $\bar{m}$ one obtains a family of curves which reveal the range of $\bar{m}$ values for which a node can be induced at x_2 when the oscillator is attached at x_1. For a given $\bar{m}$, if $f(\bar{m}, \bar{\omega}_j^{sys}, \bar{\omega}^{osc})$ fails to have a zero crossing in the range of $\bar{\omega}_{osc}$ considered, then it is not possible to have a node at x_2 in that oscillator natural frequency range for the normal mode corresponding to $\bar{\omega}_j^{sys}$. Table 3 shows the solution of equation (36), i.e., the values of $\bar{\omega}_j^{sys}$, when $x_1 = 0{\cdot}15L$ and $x_2 = 0{\cdot}23L$. The corresponding normal mode number is also indicated by comparing $\bar{\omega}_j^{sys}$ to the natural frequencies of a simply supported

TABLE 3

The required oscillator natural frequency, $\bar{\Omega}_l = \Omega_l / \sqrt{EI/(\rho L^4)}$, for a given system natural frequency, $\bar{\omega}_l^{sys} = \omega_l^{sys}/\sqrt{EI/(\rho L^4)}$, in order to induce a node at $x_2 = 0{\cdot}23L$ for the lth normal mode when the oscillator is at $x_1 = 0{\cdot}15L$ (the oscillator and the node are not collocated). The mass of the oscillator is $\bar{m} = m/(\rho L) = 0{\cdot}05$

$\bar{\omega}_l^{sys}$	Mode number, l	$\bar{\Omega}_l$
21·3403	2	21·4378
71·0634	3	74·4099
151·3873	4	367·2292
262·2095	6	199·2957
390·5823	7	336·5236

Euler–Bernoulli beam. Note that for the chosen x_1 and x_2, it is impossible to impose a node at $0{\cdot}23L$ for the fifth normal mode, since no $\bar{\omega}_j^{sys}$ exists between $\bar{\omega}_4^{beam}$ and $\bar{\omega}_5^{beam}$. The corresponding oscillator natural frequencies required in order to impose a node for the lth normal mode, the $\bar{\Omega}_l = \bar{\omega}^{osc}$, obtained by solving equation (40) for $\bar{m} = 0{\cdot}05$ and a system natural frequency of $\bar{\omega}_l^{sys}$, are also shown in Table 3. While $\bar{\Omega}_l = \bar{\omega}_l^{sys}$ when the oscillator and the node are collocated, they are not equal when the oscillator and the node are not collocated. Table 4 shows the first seven natural frequencies of the combined system for the oscillator natural frequencies of Table 3. Figure 7 depicts the first five normal modes of the combined system when an oscillator of frequency $\bar{\Omega}_l$, given in Table 3, is attached at $x_1 = 0{\cdot}15L$. Note that when the oscillator natural frequency is properly chosen, an oscillator at x_1 gives rise to a node at x_2, as shown in Figure 7.

Figure 8 shows a sample plot of $f(\bar{m}, \bar{\omega}_j^{sys}, \bar{\omega}^{osc})$ versus $\bar{\omega}^{osc}$ for varying values of $\bar{m}$. Note that for $\bar{\omega}_j^{sys} = 151{\cdot}3873$ and for $0 < \bar{\omega}^{osc} < 1000$, in order to impose a node at $0{\cdot}23L$ when the oscillator is attached at $0{\cdot}15L$, one must select $\bar{m} \leqslant 0{\cdot}0588$.

TABLE 4

The natural frequencies of the combined system, the $\bar{\omega}_j^{sys}$, for the given x_1, x_2, $\bar{m}$ and oscillator natural frequencies, the $\bar{\Omega}_l$, of Table 3. The natural frequencies are all non-dimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$

	$\bar{\Omega}_2 = 21{\cdot}4378$	$\bar{\Omega}_3 = 74{\cdot}4099$	$\bar{\Omega}_4 = 367{\cdot}2293$	$\bar{\Omega}_6 = 199{\cdot}2957$	$\bar{\Omega}_7 = 336{\cdot}5236$
$\bar{\omega}_1^{sys}$	9·7429	9·7671	9·7688	9·7686	9·7688
$\bar{\omega}_2^{sys}$	21·3403	37·8291	38·2244	38·1922	38·2219
$\bar{\omega}_3^{sys}$	40·0069	71·0634	84·8020	84·2757	84·7640
$\bar{\omega}_4^{sys}$	89·0947	96·3967	151·3873	147·9846	151·1935
$\bar{\omega}_5^{sys}$	158·0482	160·0288	240·1361	217·6523	239·5334
$\bar{\omega}_6^{sys}$	246·7871	247·3760	351·3371	262·2095	349·0227
$\bar{\omega}_7^{sys}$	355·3120	355·3848	418·1369	356·2188	390·5823

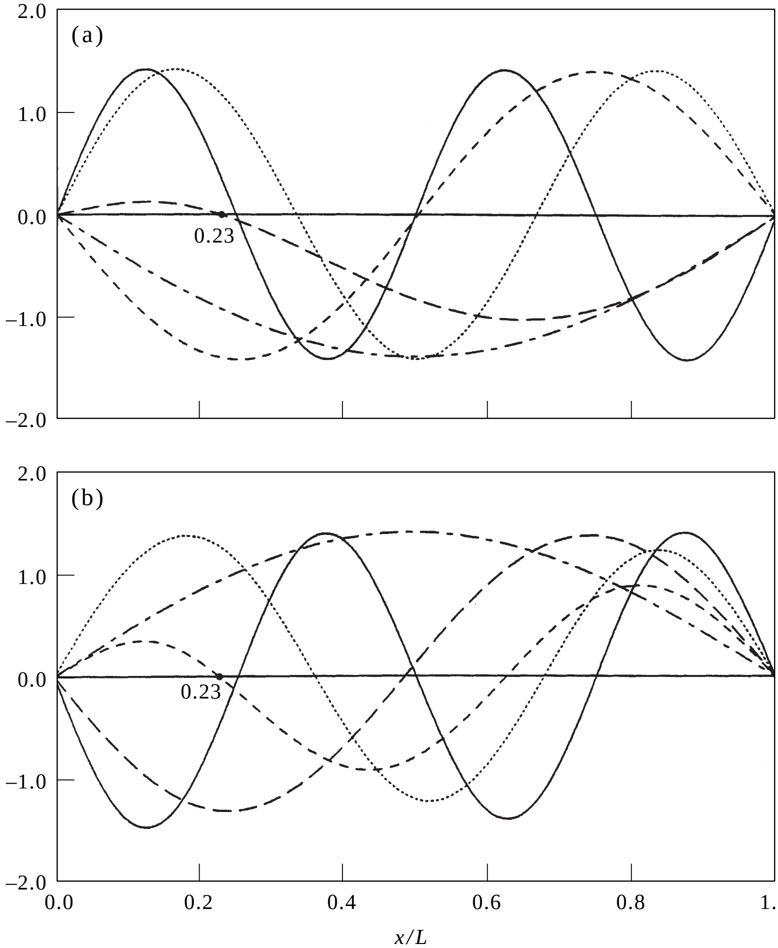

Figure 7. The first five normal modes of a combined system consisting of a simply supported Euler–Bernoulli beam to which is attached an undamped oscillator, of mass $\bar{m} = 0{\cdot}05$ and natural frequency $\bar{\Omega}_2 = 21{\cdot}4378$ (a) and $\bar{\Omega}_3 = 74{\cdot}4099$ (b). The attachment and node locations are $x_1 = 0{\cdot}15L$ and $x_2 = 0{\cdot}23L$, respectively. Key as in Figure 4.

Similar plots can be obtained for any $\bar{\omega}_j^{sys}$. These plots can be used to determine the possible range of $\bar{m}$ values for which a node can be induced at x_2 when the oscillator is attached at x_1.

4. CONCLUSIONS

A chain of oscillators can be used as a passive means of imposing nodes for the normal modes of a linear elastic structure at any location along the structure. To induce a node at x_1 for the jth normal mode only, a single oscillator can always be attached at x_1 whose natural frequency can be extracted from the design plot of Figure 3 and is identical to $\bar{\omega}_j^{sys}$. To induce a node at x_1 for l normal modes simultaneously, a chain of l oscillators is attached at x_1 whose grounded natural frequencies can also be extracted from Figure 3, and are equal to the natural

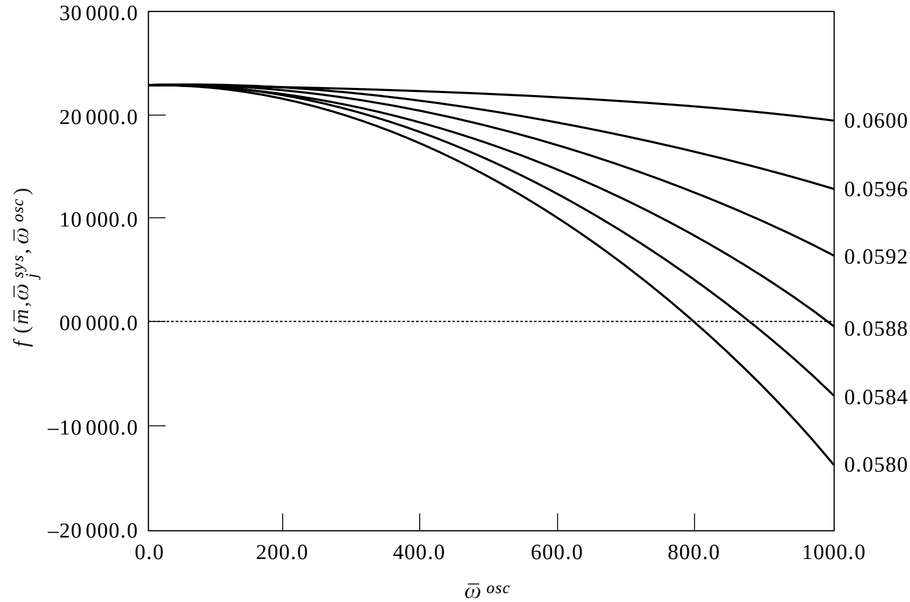

Figure 8. A sample plot of $f(\bar{m}, \bar{\omega}_j^{sys}, \bar{\omega}^{osc})$ versus $\bar{\omega}^{osc}$ for $\bar{\omega}_j^{sys} = 151{\cdot}3873$. The $\bar{m}$ parameter varies from 0·0580 to 0·0600 with an increment of 0·0004.

frequencies of the l normal modes for which we wish to have a node at x_1. A chain of oscillators can also be attached at x_1 and a node imposed at some other location x_2. A design plot such as Figure 6 allows us to determine the possible range of oscillator locations, x_1 in order to induce a node at x_2. When the oscillator and node are not collocated, not all the normal modes can be made to have a node at x_2.

Finally, while the focus of this paper has been on imposing nodes to the free vibration of a linear elastic structure, the approach could be extended to forced vibration. Future work will be concerned with minimizing forced vibration at a given location by attaching damped oscillators of appropriate parameters.

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