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A NOVEL APPROACH TO DETERMINE THE FREQUENCY EQUATIONS OF COMBINED DYNAMICAL SYSTEMS

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A novel approach is presented which can be used to reduce the generalized eigenvalue problem associated with the free vibration of a linear elastic structure carrying lumped elements at s distinct locations. Using N component modes in the assumed-modes method, the free vibration of such a combined dynamical system is governed by the solution of a generalized eigenvalue problem of order $N \times N$, whose stiffness and mass matrices consist of diagonal matrices modified by a total of s rank one matrices, where s corresponds to the number of attachment points. This generalized eigenvalue problem can be manipulated such that the natural frequencies governing free vibration can be calculated instead by solving a much smaller characteristic determinant of order $s \times s$. Interestingly enough, this smaller and simpler characteristic determinant can also be obtained by using the Lagrange multipliers formalism in conjunction with Lagrange's equations.

1. INTRODUCTION

The free vibration of combined dynamical systems which consist of linear elastic structures carrying lumped attachments has been studied by many authors over the years, and hence only a few selected references are given here [1–12]. The most common analytical approach used is the assumed-modes method [13], which is a procedure for discretizing a continuous system prior to obtaining the governing equations of motion. This method consists of assuming a solution of the free vibration problem in the form of a series composed of a linear combination of N spatial functions multiplied by the time varying generalized co-ordinates. The spatial functions must satisfy the boundary conditions of the unconstrained system, defined here as the system without the attachments. This series is then substituted into the expressions for the kinetic and potential energies, thus reducing them to discrete form, and the equations of motion are derived by means of Lagrange's equations. If the lumped elements are attached to the linear elastic structure at s distinct locations, then the mass and stiffness matrices of the

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combined system can be expressed as the sum of diagonal matrices and s rank one matrices. The modes of vibration of the combined system correspond to the eigensolution of an $N \times N$ generalized eigenvalue problem.

Other approaches have also been developed to analyze the dynamics of such a combined system. Dowell and others [11, 14–17] have exploited the power of Lagrange's equations and the use of Lagrange multipliers to study the dynamics of combined dynamical systems in terms of their component modes. This method is based on using the spatial functions of the unconstrained structure in a Rayleigh–Ritz analysis with the constraint conditions enforced by means of Lagrange multipliers. Using this particular approach, s Lagrange multipliers and s constraint equations are introduced into the analysis. Manipulating the equations of motion derived by using Lagrange's equations, the eigenvalues must satisfy the zeros of the s constraint equations in matrix form. While the final results obtained by the Lagrange multiplier approach are usually concise, the scheme is rather laborious to apply, because s Lagrange multipliers and s constraint equations need to be introduced. Due to its complexity, the method of Lagrange multipliers seems to have been used less for free vibration than other methods.

Nicholson and Bergman [18, 19] and Kukla [5] have also analyzed the dynamics of similar combined systems. They used the dynamic Green's function for the vibrating component systems to solve the generalized differential equations and to derive the frequency equations governing the free response. While the final results are exact, the approach they used is quite complicated, because the Green's function for the linear elastic structure needs first to be determined, which can be both tedious and time consuming. Moreover, the approach can only be used when the Green's function for the system can be derived.

Mathematicians have been developing efficient schemes for computing the eigenvalues of some modified matrix eigenvalue problems for years. Golub showed in reference [20] that the eigenvalues of a $N \times N$ diagonal matrix which is modified by a matrix of rank one can be calculated instead by solving the zeros of a simple secular equation which consists of the sum of N terms. In the following, the derivation given in reference [20] will first be extended, and it will be shown that when the matrices of a generalized eigenvalue problem consist of diagonal matrices modified by a total of s rank one matrices, the eigenvalues of the generalized eigenvalue problem, of size $N \times N$, can be determined instead by finding the zeros of a characteristic determinant of size $s \times s$. Then the utility of the proposed approach will be demonstrated by considering various example problems, and the results compared to known solutions.

2. THEORY

Consider a system whose free vibration is governed by the following generalized eigenvalue problem of size $N \times N$:

$$[\mathscr{K}]\boldsymbol{\eta} = \omega^2[\mathscr{M}]\boldsymbol{\eta}, \tag{1}$$

where matrices $[\mathscr{K}]$ and $[\mathscr{M}]$ consist of diagonal matrices, $[K^d]$ and $[M^d]$, modified by the sum of p and q rank one matrices, respectively, as follows:

$$[\mathscr{K}] = [K^d] + \sum_{i=1}^{p} k_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T,$$

$$[\mathscr{M}] = [M^d] + \sum_{i=1}^{q} m_i \boldsymbol{\phi}_{i+p} \boldsymbol{\phi}_{i+p}^T \tag{2, 3}$$

where

$$\boldsymbol{\phi}_i = [\phi_1(x_i), \ldots, \phi_j(x_i), \ldots, \phi_N(x_i)]^T \tag{4}$$

and $x_i \neq x_j$ for $i \neq j$. The eigenvalues of equation (1) must satisfy the following $N \times N$ characteristic determinant:

$$\det([\mathscr{K}] - \omega^2[\mathscr{M}]) = \det\left([K^d] + \sum_{i=1}^{p} k_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T - \omega^2[M^d] - \omega^2 \sum_{i=1}^{q} m_i \boldsymbol{\phi}_{i+p} \boldsymbol{\phi}_{i+p}^T\right)$$

$$= \det\left([K^d] - \omega^2[M^d] + \sum_{i=1}^{s} \sigma_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T\right) = 0, \tag{5}$$

where $s = p + q$ and

$$\sigma_i = \begin{cases} k_i, & 1 \leqslant i \leqslant p, \\ -\omega^2 m_{(i-p)}, & p+1 \leqslant i \leqslant s. \end{cases} \tag{6}$$

Instead of solving the generalized eigenvalue problem of size $N \times N$, one can manipulate equation (5) such that the eigenvalues are given by the zeros of the product of the following characteristic determinants of order $N \times N$:

$$\det([K^d] - \omega^2[M^d]) \det\left([I] + \sum_{i=1}^{s} \sigma_i([K^d] - \omega^2[M^d])^{-1} \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T\right) = 0. \tag{7}$$

After some lengthy algebra, equation (7) can be shown to be identical to

$$\det([K^d] - \omega^2[M^d]) \det[B] = \left\{\prod_{i=1}^{N} (K_i - \omega^2 M_i)\right\} \det[B] = 0, \tag{8}$$

where K_i and M_i are the ith diagonal elements of $[K^d]$ and $[M^d]$, respectively, and the (i, j)th element of $[B]$, of size $s \times s$, is given by

$$b_{ij} = \sum_{r=1}^{N} \frac{\phi_r(x_i)\phi_r(x_j)}{K_r - \omega^2 M_r} + \frac{1}{\sigma_i} \delta_i^j, \qquad i, j = 1, \ldots, s, \tag{9}$$

where $\phi_r(x_i)$ denotes the rth function at x_i and δ_i^j is the Kronecker delta. Note that each element of $[B]$ consists of a sum of N terms. To compute the eigenvalues of equation (1), one can either solve a generalized eigenvalue problem of equation (1), of dimension $N \times N$, or equation (8), of size $s \times s$. Finally, for the special case of $s = 1$ (see Appendix A for detailed derivation), equation (8) simplifies to

$$\left\{\prod_{i=1}^{N} (K_i - \omega^2 M_i)\right\}\left(1 + \sigma_1 \sum_{i=1}^{N} \frac{\phi_i^2(x_1)}{K_i - \omega^2 M_i}\right) = 0. \tag{10}$$

When $\omega^2 \neq K_i/M_i$, the results obtained by Golub in reference [20] are obtained.

The free vibration of one- and two-dimensional linear elastic structures constrained by linear springs or carrying oscillators or concentrated masses has received considerable interest over the years. Interestingly, the stiffness and mass matrices of such systems are generally given by equations (2) and (3), i.e., the matrices consist of diagonal matrices modified by rank one matrices. Thus, instead of calculating the natural frequencies of these systems by solving an $N \times N$ generalized eigenvalue problem, one can determine them by solving the zeros of a reduced characteristic determinant of size $s \times s$.

Finally, it should be noted that the numbering scheme used in equation (6) is formulated under the implicit assumption that the lumped masses and lumped springs are attached at distinct locations. When the attachment points for these elements coincide at c locations, $s = p + q - c$, and equation (6) needs to be modified slightly. Regardless, once the generalized eigenvalue problem of equation (1) is manipulated into the form of equation (5) and the σ_i's are properly defined, the results of section 2 can be readily applied.

While the approach proposed here is rather straightforward, it can be shown to be applicable to a wide class of problems. In the following section, the natural frequencies governing the free vibration of various combined dynamical systems are derived by using the conventional assumed-modes method. Then the eigenvalue problems are manipulated so that the results of section 3 can be utilized. Whenever possible, our solutions, both analytical and numerical, will be compared to those given in literature, obtained by using other means.

3. RESULTS

First a simple combined system, shown in Figure 1 is considered, which consists of a linear elastic structure onto which an oscillator with no rigid body degree of freedom is attached. The same system has also been studied by Dowell [16] and Nicholson and Bergman [18]. Using the assumed-modes method, the lateral displacement of the combined system at point x can be expressed in the form of a finite series as

$$w(x, t) = \sum_{i=1}^{N} \phi_i(x)\eta_i(t), \tag{11}$$

where N represents the number of modes used in the expansion, $\phi_i(x)$ are the eigenfunctions of the unconstrained structure (i.e., the structure without any attachment), that serve as the basis functions for this approximate solution, and $\eta_i(t)$ are the generalized co-ordinates. Thus, the total kinetic and potential energies of the combined system of Figure 1 can be written as

$$T = \frac{1}{2}\sum_{i=1}^{N} M_i\dot{\eta}_i^2(t) + \frac{1}{2}m\dot{w}^2(x_1, t) = \frac{1}{2}\sum_{i=1}^{N} M_i\dot{\eta}_i^2(t) + \frac{1}{2}m\left(\sum_{i=1}^{N} \phi_i(x_1)\dot{\eta}_i(t)\right)^2, \tag{12}$$

where the M_i are the generalized masses, x_1 denotes the constraint location of the undamped oscillator, m is the mass of the oscillator, $w(x_1, t)$ is its displacement, and an overdot denotes a derivative with respect to time. The total potential energy is given by

$$V = \frac{1}{2}\sum_{i=1}^{N} K_i\eta_i^2(t) + \frac{1}{2}kw^2(x_1, t) = \frac{1}{2}\sum_{i=1}^{N} K_i\eta_i^2(t) + \frac{1}{2}k\left(\sum_{i=1}^{N} \phi_i(x_1)\eta_i(t)\right)^2, \tag{13}$$

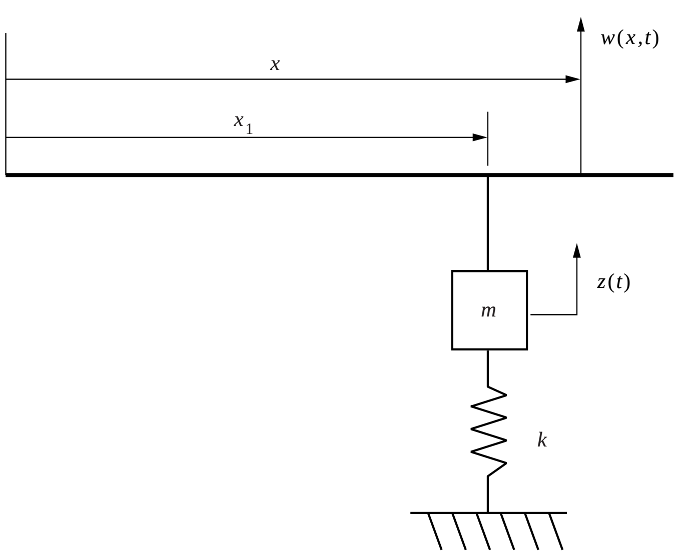

Figure 1. Combined dynamical system consisting of a linear elastic structure carrying an oscillator with no rigid body degree of freedom.

where the K_i are the generalized spring constants and k is the spring stiffness of the oscillator. Applying Lagrange's equations and assuming simple harmonic motion,

$$\eta_i(t) = \bar{\eta}_i \, \mathrm{e}^{\mathrm{j}\omega t}, \tag{14}$$

where $\mathrm{j} = \sqrt{-1}$ and ω is the natural frequency, the eigenvalue equation for the system of Figure 1 is obtained as

$$[\mathscr{K}]\boldsymbol{\eta} = \omega^2[\mathscr{M}]\bar{\boldsymbol{\eta}}, \tag{15}$$

where $\bar{\boldsymbol{\eta}} = [\bar{\eta}_1 \;\; \bar{\eta}_2 \;\; \cdots \;\; \bar{\eta}_N]^T$ is the vector of generalized co-ordinates, and

$$[\mathscr{M}] = [M^d] + m\boldsymbol{\phi}_1\boldsymbol{\phi}_1^T, \qquad [\mathscr{K}] = [K^d] + k\boldsymbol{\phi}_1\boldsymbol{\phi}_1^T. \tag{16}$$

Matrices $[M^d]$ and $[K^d]$ are both diagonal whose ith elements are given by M_i and K_i, respectively, and $\boldsymbol{\phi}_1$ is a vector of the eigenfunctions at the constraint location, x_1:

$$\boldsymbol{\phi}_1 = [\phi_1(x_1), \ldots, \phi_i(x_1), \ldots, \phi_N(x_1)]^T. \tag{17}$$

Note that both $[\mathscr{M}]$ and $[\mathscr{K}]$ consist of a diagonal matrix modified by a rank one matrix, thus the results previously derived in section 3 can be readily applied. For the system of Figure 1, while $p = 1$ and $q = 1$, $s = 1$ since the lumped mass and the lumped spring are attached at the same point, namely at x_1. Evoking equation (10) and setting $\sigma_1 = k - m\omega^2$, one immediately obtains the frequency equation for the system of Figure 1 as follows:

$$\prod_{i=1}^{N} (K_i - \omega^2 M_i)\left(1 + (k - m\omega^2) \sum_{r=1}^{N} \frac{\phi_i^2(x_1)}{K_i - \omega^2 M_i}\right) = 0. \tag{18}$$

When the constraint location is not located at the node of any of the component modes, the eigenvalues of the constrained and unconstrained systems must be distinct; thus $\omega^2 \neq K_i/M_i$, and equation (18) reduces to

$$1 + (k - m\omega^2) \sum_{r=1}^{N} \frac{\phi_i^2(x_1)}{K_i - \omega^2 M_i} = 0. \tag{19}$$

Dowell [16] analyzed the system of Figure 1 by using the Lagrange multiplier approach. In doing so, he formulated a constraint equation and introduced a Lagrange multiplier. Comparing equation (18) and (7) of reference [16] (which coincidentally, is identical to equation (19)), the absence of the product terms is noticed. When the constraint location, x_1, is coincident with any node of the unconstrained component modes, equation (18) must be used since equation (19) fails to generate all the natural frequencies of the combined system. In reference [15], Dowell circumvented the difficulty by artificially disassembling the structure and recovering all the "lost" modes. Alternatively, the "lost" modes can also be recovered by multiplying equation (19) through by

$$\prod_{i=1}^{N} (K_i - \omega^2 M_i),$$

the common denominator of the summation terms, in which case equation (18) is covered. Expanding equation (18), one gets

$$\prod_{i=1}^{N} (K_i - \omega^2 M_i) + (k - m\omega^2)\left(\sum_{r=1}^{N} \left[\phi_r^2(x_1) \prod_{i=1, i \neq r}^{N} (K_i - \omega^2 M_i) \right] \right) = 0, \quad (20)$$

which leads to a polynomial of order N in ω^2, giving us N eigenvalues for the constrained system. When x_1 is at a node of the ith normal mode, i.e., when $\phi_i(x_1) = 0$, some of the eigenvalues will correspond to those of the unconstrained structure. Physically, when the oscillator is attached to a node of the unconstrained normal modes, there will be certain modes of vibration of the unconstrained system that will be unaffected by the presence of the oscillator. Thus, the corresponding natural frequencies are expected to be unaltered when the spring–mass system is added. If $\phi_j(x_1)$ in the summation is zero, then one of the natural frequencies of the combined system will be identical to $\sqrt{K_j/M_j}$, the natural frequency of the jth unconstrained normal mode. The remaining natural frequencies can still be extracted by solving equation (19). Note that the solution obtained by Dowell has been recovered by using the more conventional and simpler assumed-modes method. In addition, the resultant frequency equation obtained can be used even when x_1 is at a node of the unconstrained normal modes.

Nicholson and Bergman also analyzed the system of Figure 1 in reference [18], where the linear elastic structure is a uniform cantilevered beam and x_1 is not at a node of the component normal modes. They solved the frequency equation exactly by using the Green's function approach, and verified their solution to those obtained by using a 10 and 14 term Galerkin's method and the finite element method (see reference [18] for a detailed discussion of their finite element results). In order to compare with their solution, let $\phi_i(x)$ of equation (19) be the normalized (with respect to the mass per unit length, ρ, of the beam) eigenfunctions of a uniform cantilevered Euler–Bernoulli beam of length L:

$$\phi_i(x) = \frac{1}{\sqrt{\rho L}} \left(\cos \beta_i x - \cosh \beta_i x + \frac{\sin \beta_i L - \sinh \beta_i L}{\cos \beta_i L + \cosh \beta_i L} (\sin \beta_i x - \sinh \beta_i x) \right), \quad (21)$$

such that the generalized masses and stiffnesses are given by

$$M_i = 1 \qquad \text{and} \qquad K_i = (\beta_i L)^4 EI/(\rho L^4), \quad (22)$$

where E is the Young's modulus, I is the moment of inertia of the cross-section of the beam, and $\beta_i L$ satisfies the transcendental equation

$$\cos \beta_i L \cosh \beta_i L = -1. \quad (23)$$

TABLE 1

The first eight natural frequencies, normalized with respect to the fundamental natural frequency of a uniform cantilevered beam, of a cantilevered beam carrying an undamped oscillator with no rigid body degree of freedom (see Figure 1) at $x_1 = 0{\cdot}78L$. *The mass and spring stiffness of the oscillator are* $m = \rho L$ *and* $k = 30EI/L^3$, *respectively. The natural frequencies are non-dimensionalized by dividing by* $\sqrt{EI/(\rho L^4)}$. *The exact, Galerkin and FEM results are obtained from reference* [17]

Natural frequency	Exact	Galerkin ($N = 14$)	FEM	Equation (19) ($N = 14$)
1	1·39370	1·39370	1·3937	1·39370
2	6·26599	6·26599	6·2660	6·26599
3	14·93879	14·94144	14·939	14·94106
4	28·61698	28·62937	28·618	28·62758
5	52·94585	52·96782	52·953	52·96463
6	84·73575	84·73908		84·73859
7	110·88502	111·15113		111·11206
8	139·59270	140·12006		140·03976

Table 1 compares the results obtained by solving equation (19), for $N = 14$, and those given in reference [18]. From Table 1, note the excellent agreement between the results of equation (19) and the exact solution given in reference [18]. The slight discrepancy in results between the Galerkin or assumed-modes method and the present approach is due to the solver used in reference [18]. When the assumed-modes method is applied using a double precision version of the subroutine rsg.f in EISPACK [21], the results between the Galerkin and the present scheme become identical.

Consider now the combined system of Figure 2, which consists of a linear elastic structure carrying an oscillator with a rigid body degree of freedom. This system

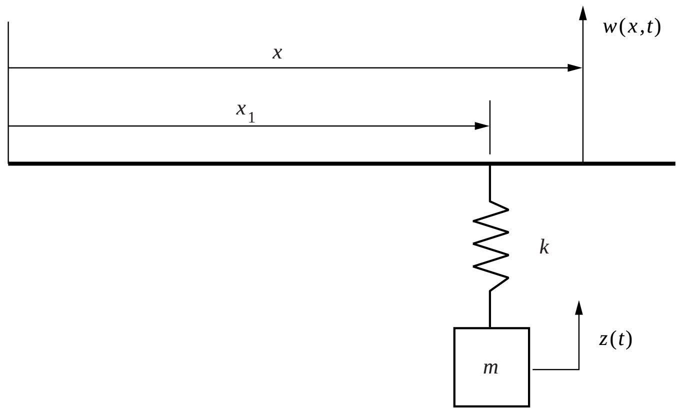

Figure 2. Combined dynamical system consisting of a linear elastic structure carrying an oscillator with a rigid body degree of freedom.

has also been analyzed by Dowell [16] and Nicholson and Bergman [18]. The total kinetic and potential energies of the system of Figure 2 are given by

$$T = \frac{1}{2} \sum_{i=1}^{N} M_i \dot{\eta}_i^2(t) + \frac{1}{2} m \dot{z}^2(t), \tag{24}$$

$$V = \frac{1}{2} \sum_{i=1}^{N} K_i \eta_i^2(t) + \frac{1}{2} k (w(x_1, t) - z(t))^2 = \frac{1}{2} \sum_{i=1}^{N} K_i \eta_i^2(t)$$
$$+ \frac{1}{2} k \left(\sum_{i=1}^{N} \phi_i(x_1) \eta_i(t) - z(t) \right)^2, \tag{25}$$

where $z(t)$ represents the displacement of the oscillator. Applying Lagrange's equations and assuming simple harmonic motion for $\eta_i(t)$ and $z(t)$, one obtains the generalized eigenvalue problem

$$\begin{bmatrix} [\mathscr{K}] & -k\boldsymbol{\phi}_1 \\ -k\boldsymbol{\phi}_1^T & k \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{z} \end{bmatrix} = \omega^2 \begin{bmatrix} [\mathscr{M}] & \mathbf{0} \\ \mathbf{0}^T & m \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{z} \end{bmatrix}, \tag{26}$$

where $\bar{\boldsymbol{\eta}}$ is a vector of generalized co-ordinates, and the $N \times N$ matrices $[\mathscr{M}]$ and $[\mathscr{K}]$ are

$$[\mathscr{M}] = [M^d], \qquad [\mathscr{K}] = [K^d] + k\boldsymbol{\phi}_1\boldsymbol{\phi}_1^T. \tag{27}$$

From the second equation of equation (26), one has

$$-k\boldsymbol{\phi}_1^T\bar{\boldsymbol{\eta}} + k\bar{z} = \omega^2 m\bar{z}. \tag{28}$$

Solving for $\bar{z}$ one obtains

$$\bar{z} = -\frac{k\boldsymbol{\phi}_1^T\bar{\boldsymbol{\eta}}}{m\omega^2 - k}. \tag{29}$$

Substituting equation (29) into the first equation of (26) yields

$$\left([\mathscr{K}] + \frac{k^2\boldsymbol{\phi}_1\boldsymbol{\phi}_1^T}{m\omega^2 - k} \right) \bar{\boldsymbol{\eta}} = \omega^2 [\mathscr{M}] \bar{\boldsymbol{\eta}}. \tag{30}$$

After some algebra, it is found that the natural frequencies of Figure 2 must satisfy the $N \times N$ characteristic determinant

$$\det \left([K^d] - \omega^2 [M^d] + \frac{km\omega^2}{m\omega^2 - k} \boldsymbol{\phi}_1\boldsymbol{\phi}_1^T \right) = 0, \tag{31}$$

TABLE 2

The first eight natural frequencies, normalized with respect to the fundamental natural frequency of a simply supported beam, of a uniform simply supported Euler–Bernoulli beam carrying an undamped oscillator with a rigid body degree of freedom (see Figure 2) at $x_1 = 0{\cdot}78L$. The mass and spring stiffness of the oscillator are $m = \rho L$ and $k = 780EI/L^3$, respectively. The natural frequencies are non-dimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$. The exact, Galerkin and FEM results are obtained from reference [17]

Natural frequency	Exact	Galerkin ($N = 14$)	FEM	Equation (32) ($N = 14$)
1	0·70947	0·70948	0·70947	0·70948
2	2·23736	2·23762	2·2374	2·23762
3	5·23911	5·23978	5·2393	5·23978
4	9·51339	9·51376	9·5138	9·51376
5	16·04630	16·04633	16·048	16·04633
6	25·01936	25·01937		25·01937
7	36·09847	36·09853		36·09853
8	49·10021	49·10028		49·10028

which appears in the form of equation (5). Thus, using equation (10), the frequency equation of Figure 2 is immediately written as

$$\prod_{i=1}^{N} (K_i - \omega^2 M_i)\left(1 + \frac{km\omega^2}{m\omega^2 - k}\sum_{i=1}^{N}\frac{\phi_i^2(x_1)}{K_i - \omega^2 M_i}\right) = 0. \tag{32}$$

When x_1 does not coincide with any node of the unconstrained component modes, equation (32) reduces to equation (7a) derived by Dowell in reference [16]. Again, it should be emphasized that equation (32) was obtained by reducing the generalized eigenvalue problem associated with free vibration, while Dowell used the rather complicated Lagrange multiplier formalism in determining the same frequency equation.

Nicholson and Bergman [18] also examined the system of Figure 2, where the linear elastic structure consists of a uniform simply supported Euler–Bernoulli beam. They used the Green's function approach in their derivation of the frequency equation, and they validated their solution by using a Galerkin's approach and the finite element method. In order to compare with the results given in reference [18], $\phi_i(x)$ of equation (32) is defined as

$$\phi_i(x) = \sqrt{\frac{2}{\rho L}}\sin\frac{i\pi x}{L}, \tag{33}$$

which are the normalized (with respect to the mass per unit length, ρ, of the beam) eigenfunctions of a simply supported Euler–Bernoulli beam, so that the generalized masses and stiffness become

$$M_i = 1 \qquad \text{and} \qquad K_i = (i\pi)^4 EI/(\rho L^4). \tag{34}$$

Table 2 compares the results of equation (32) to those given in Table 2 of reference [18]. Note again the excellent agreement between the solutions.

Consider next the free vibration of a two-dimensional structure onto which an oscillator with a rigid body degree of freedom is attached at (x_1, y_1) (see Figure 3). Using the assumed-modes method and the results of section 3, the frequency equation for the aforementioned constrained two-dimensional system is given by (the steps to derive the frequency equation are identical to those outlined for the sample problem of Figure 2, hence they are omitted here for brevity)

$$\prod_{i=1}^{N} (K_i - \omega^2 M_i)\left(1 + \frac{km\omega^2}{m\omega^2 - k}\sum_{i=1}^{N}\frac{\phi_i^2(x_1, y_1)}{K_i - \omega^2 M_i}\right) = 0, \tag{35}$$

where $\phi_i(x_1, y_1)$ are the eigenfunctions of the unconstrained two-dimensional structure.

Nicholson and Bergman [19] also analyzed the system of Figure 3, where the two-dimensional structure consists of a simply supported rectangular plate. They calculated the natural frequencies of the combined system by using the Green's function approach. In order to compare with their results, let the $\phi_i(x, y) = \phi_{pq}(x, y)$ be the normalized eigenfunctions of a simply supported plate:

$$\phi_{pq}(x, y) = \frac{2}{\sqrt{\rho ab}} \sin\frac{p\pi x}{a} \sin\frac{q\pi y}{b}, \tag{36}$$

where ρ is the mass per unit area of the plate; a and b are the lengths of the plate in the x and y directions, respectively. Then the generalized masses and stiffnesses become

$$M_i = M_{pq} = 1 \qquad \text{and} \qquad K_i = K_{pq} = \pi^4\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right)^2 \frac{D}{\rho}, \tag{37}$$

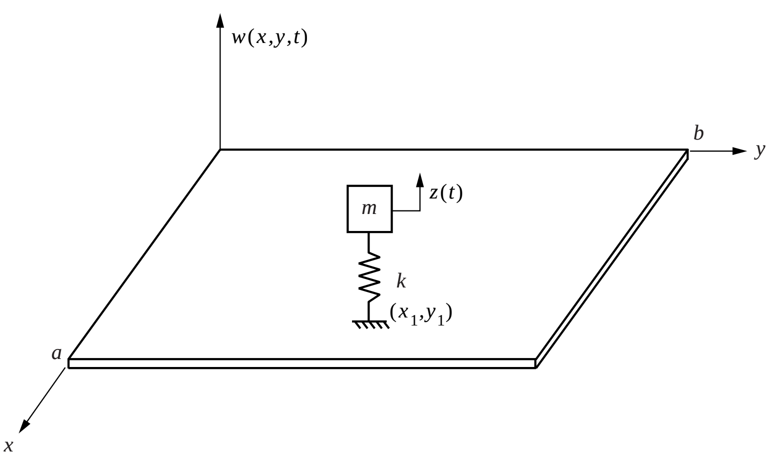

Figure 3. Combined dynamical system consisting of an oscillator with a rigid body degree of freedom attached to a plate.

TABLE 3

The first six natural frequencies of a uniform simply supported rectangular plate carrying an undamped oscillator with a rigid body degree of freedom (see Figure 3) at $(x_1, y_1) = (0{\cdot}225a, 0{\cdot}275a)$. The ratio of the plate lengths is given by $b/a = 0{\cdot}75$. The mass and spring stiffness of the oscillator are $m = \rho a^2$ and $k = 100a^2/D$, respectively. The natural frequencies are non-dimensionalized by dividing by $\sqrt{D/\rho}$. The exact results are obtained from reference [18]

Natural frequency	Exact	Equation (38) ($r = n = 10$)
1	8·09799	8·10038
2	30·13446	30·14057
3	60·45673	60·46554
4	80·83303	80·83596
5	107·36171	107·36422
6	111·56776	111·57855

where $D = Eh^3/[12(1 - \mu^2)]$ is the bending stiffness, h is the plate thickness and μ is the Poisson's ratio. Since the attachment location considered by Nicholson and Bergman [19] does not coincide with a node, the frequency equation of equation (35) reduces to

$$1 + \frac{km\omega^2}{m\omega^2 - k} \sum_{p=1}^{r} \sum_{q=1}^{n} \frac{4(\sin^2 p\pi x_1/a)(\sin^2 q\pi y_1/b)}{\rho ab(\pi^4(p^2/a^2 + q^2/b^2)^2 D/\rho - \omega^2)} = 0. \tag{38}$$

Table 3 shows the results obtained by solving equation (38), for $r = n = 10$, and those given in Table 1 of reference [19]. Note the excellent agreement between the solutions.

As a final example, let us consider the free vibration of a linear elastic structure carrying various lumped attachments, as shown in Figure 4, which consists of linear translational springs of stiffnesses k_1 and k_2 at x_1 and x_5, a linear rotational spring of stiffness c at x_3, a concentrated mass m at x_2, and a linear undamped oscillator of mass M and stiffness k_M at x_4. The total kinetic and potential energies of the system are

$$T = \frac{1}{2} \sum_{i=1}^{N} M_i \dot{\eta}_i^2(t) + \frac{1}{2} m\dot{w}^2(x_2, t) + \frac{1}{2} M\dot{z}^2(t), \tag{39}$$

$$V = \frac{1}{2} \sum_{i=1}^{N} K_i \eta_i^2(t) + \frac{1}{2} k_1 w^2(x_1, t) + \frac{1}{2} k_2 w^2(x_5, t) + \frac{1}{2} c\theta^2(x_3, t)$$

$$+ \frac{1}{2} k_M (z(t) - w(x_4, t))^2, \tag{40}$$

where $w(x, t)$ and $\theta(x, t)$ represent the lateral and rotational displacements, respectively, of the linear elastic structure at point x, and they can be expressed in the form of finite series as

$$w(x, t) = \sum_{i=1}^{N} \phi_i(x)\eta_i(t), \qquad \theta(x, t) = \sum_{i=1}^{N} \psi_i(x)\eta_i(t). \tag{41}$$

The functions $\phi_i(x)$ and $\psi_i(x)$ are the transverse and rotational eigenfunctions of the unconstrained system that serve as the basis functions for this approximate solution. Applying Lagrange's equations and assuming simple harmonic motion, the generalized eigenvalue problem governing free vibration for the system of Figure 4 can be written as

$$\begin{bmatrix} [\mathscr{K}] & -k_M\boldsymbol{\phi}_4 \\ -k_M\boldsymbol{\phi}_4^T & k_M \end{bmatrix}\begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{z} \end{bmatrix} = \omega^2\begin{bmatrix} [\mathscr{M}] & \mathbf{0} \\ \mathbf{0}^T & M \end{bmatrix}\begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{z} \end{bmatrix}. \tag{42}$$

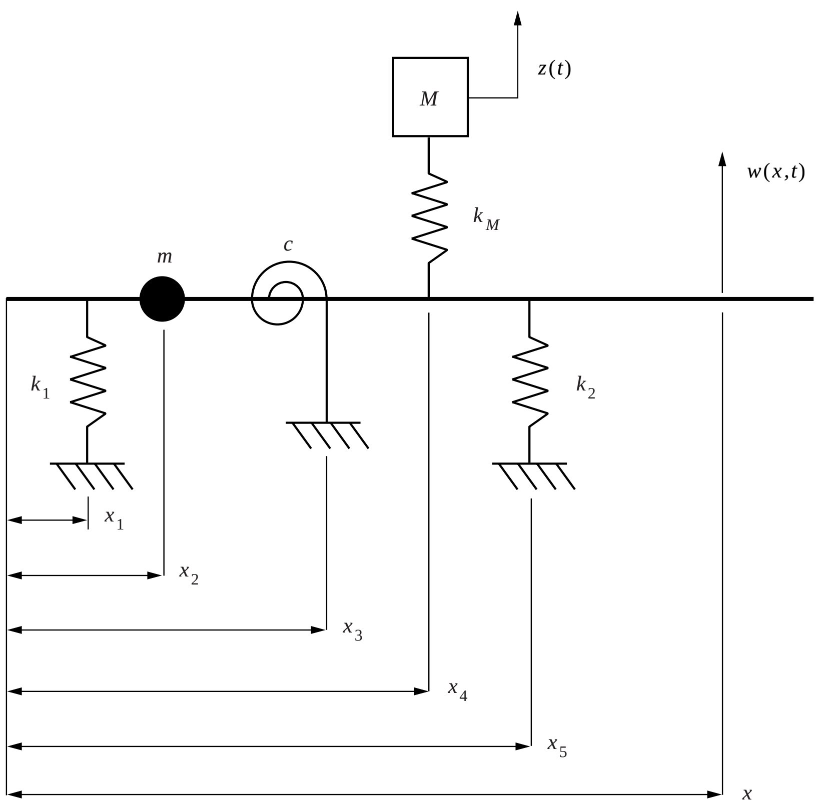

Figure 4. Combined dynamical system consisting of a linear elastic structure with various lumped attachments.

The $N \times N$ matrices $[\mathscr{M}]$ and $[\mathscr{K}]$ are

$$[\mathscr{M}] = [M^d] + m\boldsymbol{\phi}_2\boldsymbol{\phi}_2^T, \tag{43}$$

$$[\mathscr{K}] = [K^d] + k_1\boldsymbol{\phi}_1\boldsymbol{\phi}_1^T + k_2\boldsymbol{\phi}_5\boldsymbol{\phi}_5^T + c\boldsymbol{\psi}_3\boldsymbol{\psi}_3^T + k_M\boldsymbol{\phi}_4\boldsymbol{\phi}_4^T, \tag{44}$$

where

$$\boldsymbol{\phi}_i = [\phi_1(x_i), \ldots, \phi_j(x_i), \ldots, \phi_N(x_i)]^T,$$
$$\boldsymbol{\psi}_i = [\psi_1(x_i), \ldots, \psi_j(x_i), \ldots, \psi_N(x_i)]^T. \tag{45}$$

Solving for $\bar{z}$ using the second equation of equation (42) and substituting the resultant into the first equation of equation (42) yields

$$\left([\mathscr{K}] + \frac{k_M^2\boldsymbol{\phi}_4\boldsymbol{\phi}_4^T}{M\omega^2 - k_M}\right)\bar{\boldsymbol{\eta}} = \omega^2[\mathscr{M}]\bar{\boldsymbol{\eta}}. \tag{46}$$

Rearranging equation (46), one can write it alternatively as

$$\left([K^d] - \omega^2[M^d] + \sum_{i=1}^{5} \sigma_i\mathbf{u}_i\mathbf{u}_i^T\right)\bar{\boldsymbol{\eta}} = 0, \tag{47}$$

where

$$\sigma_1 = k_1, \qquad \sigma_2 = -m\omega^2, \qquad \sigma_3 = c, \qquad \sigma_4 = \frac{k_M M\omega^2}{M\omega^2 - k_M}, \qquad \sigma_5 = k_2, \tag{48}$$

and $\mathbf{u}_i$ is a vector of length N whose jth element is given by

$$u_j(x_i) = \begin{cases} \phi_j(x_i), & i = 1, 2, 4, 5, \\ \psi_j(x_i), & i = 3. \end{cases} \tag{49}$$

Since the characteristic determinant of equation (47) appears in the form of equation (5), the characteristic determinant governing the frequency equation of Figure 4 is immediately obtained by applying equations (8) and (9).

It should be emphasized that the above formulation is equally applicable regardless if the linear elastic structure consists of an Euler–Bernoulli beam or a Timoshenko beam. If the linear elastic structure is an Euler–Bernoulli beam, then the rotational and translational displacements are related by

$$\theta(x, t) = \frac{\partial w}{\partial x}(x, t). \tag{50}$$

Then $\psi_j(x_i)$ of equation (49) simplifies to $\psi_j(x_i) = \phi_j'(x_i)$, where the prime denotes a derivative with respect to x. When the attachment locations, the x_i, do not coincide with the nodes of the unconstrained component modes, the natural frequencies of the system of Figure 4 are given by the roots of the 5×5 characteristic determinant

$$\det[B] = 0, \tag{51}$$

where the (i, j)th element of $[B]$ is given by

$$b_{ij} = \sum_{r=1}^{N} \frac{u_r(x_i)u_r(x_j)}{K_r - \omega^2 M_r} + \frac{1}{\sigma_i}\delta_i^j, \qquad i, j = 1, \ldots, 5, \tag{52}$$

and the σ_i are defined in equation (48).

In reference [17] Posiadała analyzed the free vibration of a similar system which consists of an Euler–Bernoulli beam carrying additional elements. He derived the frequency equation for the combined system by means of the Lagrange multiplier approach, which required considerable effort. For the system of Figure 4, in the limit as $k_2 \to \infty$, the support against beam translation considered in reference [17] is obtained, and equation (51) reduces to equation (13) of reference [17] exactly. Since Posiadała had already verified his solution in reference [17], no such numerical validation will be made here.

4. CONCLUSIONS

A novel approach has been introduced to reduce the size of the characteristic determinant needed to calculate the natural frequencies of combined dynamical systems consisting of linear elastic structures carrying assorted lumped attachments at s distinct locations. Using the classical assumed-modes method in conjunction with the Lagrange's equations, it was found that the natural frequencies are obtained by solving the roots of an $N \times N$ characteristic determinant. Algebraically manipulating this characteristic determinant, it is reduced to a smaller one of size $s \times s$, the same solution that is obtained by applying the more complicated and often more tedious Lagrange multipliers formalism. Finally, it should be emphasized that while the results are obtained in the engineering context by considering the free vibration of combined dynamical systems, the results formulated can be extended to determine the eigenvalues of any diagonal matrix modified by a series of rank one matrices.

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APPENDIX A

The derivation of equation (9) is rather lengthy. For brevity, let us consider instead the special case of $s = 1$. The general case of arbitrary s is merely an extension of the below derivation. For $s = 1$, equation (5) can be manipulated as follows:

$$\det([\mathscr{K}] - \omega^2[\mathscr{M}]) = \det([K^d] - \omega^2[M^d] + \sigma\boldsymbol{\phi}\boldsymbol{\phi}^T)$$

$$= \det([K^d] - \omega^2[M^d]) \det([I] + \sigma([K^d] - \omega^2[M^d])^{-1}\boldsymbol{\phi}\boldsymbol{\phi}^T)$$

$$= \left\{\prod_{i=1}^{N} (K_i - \omega^2 M_i)\right\} \det([I] + \sigma([K^d] - \omega^2[M^d])^{-1}\boldsymbol{\phi}\boldsymbol{\phi}^T) = 0.$$

Consider now the remaining determinant:

$$\det\left([I] + \sigma([K^d] - \omega^2[M^d])^{-1}\boldsymbol{\phi}\boldsymbol{\phi}^T\right)$$

$$= \det\left\{[I] + \begin{bmatrix} \sigma/(K_1 - \omega^2 M_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma/(K_N - \omega^2 M_N) \end{bmatrix} \begin{bmatrix} \phi_1^2 & \phi_1\phi_2 & \cdots & \phi_1\phi_N \\ \phi_2\phi_1 & \phi_2^2 & \cdots & \phi_2\phi_N \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N\phi_1 & \phi_N\phi_2 & \cdots & \phi_N^2 \end{bmatrix}\right\}$$

$$= \det\begin{pmatrix} 1 + \sigma\phi_1^2/(K_1 - \omega^2 M_1) & \sigma\phi_1\phi_2/(K_1 - \omega^2 M_1) & \cdots & \sigma\phi_1\phi_N/(K_1 - \omega^2 M_1) \\ \sigma\phi_2\phi_1/(K_2 - \omega^2 M_2) & 1 + \sigma\phi_2^2/(K_2 - \omega^2 M_2) & \cdots & \sigma\phi_2\phi_N/(K_2 - \omega^2 M_2) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma\phi_N\phi_1/(K_N - \omega^2 M_N) & \sigma\phi_N\phi_2/(K_N - \omega^2 M_N) & \cdots & 1 + \sigma\phi_N^2/(K_N - \omega^2 M_N) \end{pmatrix}.$$

For columns $i = 1, \ldots, N-1$, multiply column $i+1$ by $-\phi_i/\phi_{i+1}$ and add the resultant to column i (the determinant will not change by this operation), then the above reduces to

$$\det\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \sigma\phi_1\phi_N/(K_1 - \omega^2 M_1) \\ -\phi_1/\phi_2 & 1 & 0 & \cdots & 0 & \sigma\phi_2\phi_N/(K_2 - \omega^2 M_2) \\ 0 & -\phi_2/\phi_3 & 1 & \ddots & 0 & \sigma\phi_3\phi_N/(K_3 - \omega^2 M_3) \\ 0 & 0 & -\phi_3/\phi_4 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi_{N-1}/\phi_N & 1 + \sigma\phi_N^2/(K_N - \omega^2 M_N) \end{pmatrix} = \det[A].$$

Expanding the above determinant along the last column yields

$$
\begin{aligned}
\det[A] = {} & \frac{\sigma\phi_1\phi_N}{K_1-\omega^2M_1}\left(-\frac{\phi_1}{\phi_2}\right)\left(-\frac{\phi_2}{\phi_3}\right)\cdots\left(-\frac{\phi_{N-1}}{\phi_N}\right) \\
& -\frac{\sigma\phi_2\phi_N}{K_2-\omega^2M_2}(1)\left(-\frac{\phi_2}{\phi_3}\right)\left(-\frac{\phi_3}{\phi_4}\right)\cdots\left(-\frac{\phi_{N-1}}{\phi_N}\right) \\
& +\cdots \\
& -\frac{\sigma\phi_{N-1}\phi_N}{K_{N-1}-\omega^2M_{N-1}}(1)(1)(1)\cdots\left(-\frac{\phi_{N-1}}{\phi_N}\right) \\
& +\left(1+\frac{\sigma\phi_N^2}{K_N-\omega^2M_N}\right)(1)(1)\cdots(1) \\
= {} & \frac{\sigma\phi_1^2}{K_1-\omega^2M_1}+\frac{\sigma\phi_2^2}{K_2-\omega^2M_2}+\frac{\sigma\phi_3^2}{K_3-\omega^2M_3}+\cdots+\frac{\sigma\phi_N^2}{K_N-\omega^2M_N}+1 \\
= {} & 1+\sum_{i=1}^{N}\frac{\sigma\phi_i^2}{K_i-\omega^2M_i}.
\end{aligned}
$$