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FINITE ELEMENT VIBRATION ANALYSIS OF COMPOSITE BEAMS BASED ON HIGHER-ORDER BEAM THEORY

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This paper presents a new finite element formulation for the free vibration analysis of composite beams based on the third-order beam theory. This work also studies the influence of the mass components resulting from higher-order displacements on the frequencies of flexural vibration. By using Hamilton's principle, the variational consistent equation of motion in matrix form corresponding to the third-order shear deformation theory is derived. The resulting mass matrices are decomposed into three parts, i.e., the usual part, including the rotary inertia, corresponding to first-order theory, the part resulting from higher-order displacement, and the part resulting from the coupling between the different components of the axial displacement. The numerical examples show that the higher-order and coupling mass matrices have a significant influence on the frequencies of high mode flexural vibration. The present element formulation for composite beams can be easily extended to composite plates and shells.

1. INTRODUCTION

Dynamic analysis of composite structures is of practical importance in many engineering applications. Because of the high ratio of tensile modulus to transverse shear modulus, the transverse shear deformations play an important rule not only in thick beam and plates analysis, but also in the high frequency vibration analysis of thin beams and plates under certain boundary and loading conditions. Thanks to the advantage that no shear correction factors are needed and the warping of the cross-section can be accounted for to a certain extent, higher-order shear deformation theories are widely used in the analysis of composite beams (references [1–4] among others). In finite element modelling, the accuracy of the dynamic behavior analysis is influenced by the formulation of the mass matrix. In many higher-order elements for dynamic analysis, the evaluation of the stiffness matrix is based on the higher-order theory, but the formulation of the mass matrices is either done merely according to the displacement field of the first-order

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theory or neglecting the coupling effect of the different order displacements [5]. Most papers on the dynamic analysis of composite beams and plates so far have only focused on the fundamental frequency of vibrations [2, 5, 6]. Because the high frequency vibrations could be important in many engineering applications, for instance, in the control of high frequency vibration of composite smart beams [7], it is desirable to analyze the frequencies of high mode vibrations of composite beams.

The objectives of this paper are twofold. One is to present a new variational consistent element formulation for the third-order shear deformation theory. The other is to study the influence of the mass components contributed from the higher-order displacement and the coupling of the different order components of the axial displacement on the accuracy of frequency analysis. By using Hamilton's Principle, the variational consistent mass matrices for the third-order theory are derived in this work. The resulting mass matrices can be decomposed into three parts, i.e., the usual part given by the first-order theory, the part resulting from higher-order displacement, and the part resulting from the coupling between the different order displacements. The stiffness matrix of a third-order beam element presented in the authors' previous paper [8] is used for the vibration analysis of composite beams. The two-noded higher-order composite beam element possesses a linear bending strain as opposed to the constant bending strain in the existing higher-order composite beam elements with the same number of nodal degrees of freedom [2–5]. The numerical examples show that the present element formulation is efficient and accurate compared with other higher-order beam elements using the same higher-order theory. The numerical results also illustrate that the mass matrices resulting from the higher-order displacement and the coupling of the different order displacements have a significant influence on the frequencies of high mode flexural vibration.

2. STRAINS AND VELOCITIES IN THE THIRD-ORDER BEAM THEORY

In a third-order beam or plate theory, the in-plane displacement is cubic in terms of the variable in the thickness direction to model the transverse shear deformations, but the deflection, i.e, transverse displacement, could be either higher-order (quadratic or cubic) or constant across the thickness. The shear deformation theories in which the in-plane displacement is of higher-order but the variation of the deflection across the thickness is neglected are called the simplified third-order theories [9]. Many researchers have shown that higher-order deflection does not have a significant influence on global solutions, such as strain energy, displacements, frequencies, etc., of a system (see the review paper of reference [9] among others). Therefore, the simplified third-order theory of shear flexible beams is adopted in this study.

When the transverse shear deformation is chosen as an independent variable, the displacement in the third-order shear deformation beam theory [1, 2] is of the form

$$u(x, z, t) = u_0(x, t) - \left[\frac{\partial w_0}{\partial x} - \gamma(x, t)\right]z - \frac{4}{3h^2}\gamma(x, t)z^3, \tag{1}$$

$$w(x, z, t) = w_0(x, t), \tag{2}$$

where u_0 and w_0 are, respectively, the axial displacement and the deflection of a point on the beam reference plane; h is the beam thickness; and γ is the transverse shear deformation at the reference plane defined by

$$\gamma = \frac{\partial w_0}{\partial x} - \phi, \tag{3}$$

in which ϕ is the rotation of a normal to the reference plane about the y-axis. The influence of the normal strain in the y-direction on the behavior of composite beams is not considered here, since the emphasis here is to discuss the variational consistent finite element modelling of composite beams.

Equations (1) and (2) lead to the axial normal strain and the transverse shear strain as

$$e_x = \frac{\partial u_0}{\partial x} - \left(\frac{\partial^2 w_0}{\partial x^2} - \frac{\partial \gamma}{\partial x}\right)z - \alpha \frac{\partial \gamma}{\partial x} z^3, \qquad e_{xz} = \tfrac{1}{2}(1 - 3\alpha z^2)\gamma, \tag{4, 5}$$

where $\alpha = 4/3h^2$. The transverse shear strain defined in equation (5) satisfies the traction free condition on the top and bottom surfaces of a beam. However, when it is used directly to calculate the transverse shear stress, the resulting shear stress is discontinuous at the interfaces of the layers through the thickness. Nevertheless, this discontinuous transverse shear stress does not influence frequencies of vibration, a kind of global solution, very much as shown in many studies [9]. As a matter of fact, the total strain energy and frequencies given by the simplified higher-order theory used in this work have almost the same accuracy as those obtained from the complicated discrete-layer theories which satisfy the transverse shear stress continuity but need more computational work.

If we define, respectively, the membrane strain e_m, bending strain e_b, and higher-order transverse shear strain e_{hs} on the reference plane as

$$e_m = \frac{\partial u_0}{\partial x}, \qquad e_b = \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial \gamma}{\partial x}, \qquad e_{hs} = \frac{\partial \gamma}{\partial x}, \tag{6}$$

then the axial strain of a beam can be rewritten as

$$e_x = e_m - e_b z - \alpha e_{hs} z^3. \tag{7}$$

It follows from equations (1) and (2) that the velocities take the form

$$v_x = \frac{\partial u}{\partial t} = \frac{\partial u_0}{\partial t} - \frac{\partial}{\partial t}\left(\frac{\partial w_0}{\partial x} - \gamma\right)z - \alpha \frac{\partial \gamma}{\partial t} z^3, \qquad v_z = \frac{\partial w}{\partial t} = \frac{\partial w_0}{\partial t}. \tag{8, 9}$$

The bending strains in equation (6) are a function of the deflection and transverse shear deformation as opposed to the widely used rotation [2–5]. The advantage of the present bending strain is that a linear bending strain field can be achieved from a cubic deflection interpolation. Therefore, for the given number of nodal variables, the bending strain defined in this work gives a more accurate finite element solution than the constant bending strain in terms of the rotation [2–5].

3. HIGHER-ORDER COMPOSITE BEAM ELEMENT

Instead of deriving the differential equation of motion, the equation of motion is derived in terms of the element stiffness matrix and the mass matrix from Hamilton's Principle.

Let U_e and K_{ke} be the element strain energy and kinetic energy respectively, then Hamilton's Principle states that

$$\delta \sum_{elem} \int_{t_0}^{t} (U_e - K_{ke})\, \mathrm{d}t = 0, \tag{10}$$

in which the work done by external forces is neglected and the damping is not considered. The summation in equation (10) is over all the elements of the system. The Hamilton's Principle leads to the equilibrium equation of a system as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \tag{11}$$

where $\mathbf{M}$, $\mathbf{K}$, $\mathbf{q}$ and $\ddot{\mathbf{q}}$ are, respectively, the global mass matrix, stiffness matrix, nodal variable vector and acceleration vector of the system. Consequently, the frequency ω can be evaluated by

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{q} = \mathbf{0}. \tag{12}$$

The derivation of the variational consistent element stiffness matrix and mass matrix based on the third-order theory will be presented in the next section.

3.1. ELEMENT STIFFNESS MATRIX

Consider a straight beam element of length l with a rectangular cross-section $h \times b$ in which b is the beam width. The strain energy of an element, U_e, is of the form

$$U_e = \frac{b}{2} \int_l \int_{-h/2}^{h/2} (e_x Q_{xx} e_x + 4 e_{xz} Q_{xz} e_{xz})\, \mathrm{d}z\, \mathrm{d}x, \tag{13}$$

where Q_{xx} and Q_{xz} are, respectively, the longitudinal Young's modulus and transverse shear modulus, and they are functions of z. Substituting equations (5) and (7) into equation (13) leads to

$$U_e = \frac{1}{2} \int_l [e_m A_{xx} e_m + e_b D_{xx} e_b + \gamma S_{xx} \gamma + e_{hs} \alpha^2 H_{xx} e_{hs}$$

$$- 2 e_m \alpha B_{xx} e_b - 2 e_m \alpha E_{xx} e_{hs} + 2 e_b \alpha F_{xx} e_{hs}]\, \mathrm{d}x, \tag{14}$$

in which

$$(A_{xx}, B_{xx}, D_{xx}, E_{xx}, F_{xx}, H_{xx}) = b\int_{-h/2}^{h/2}(1, z, z^2, z^3, z^4, z^6)Q_{xx}\,\mathrm{d}z, \tag{15}$$

$$S_{xx} = b\int_{-h/2}^{h/2}(1 - 3\alpha z^2)^2 Q_{xz}\,\mathrm{d}z. \tag{16}$$

In the finite element modelling of displacement-based formulation, element strains in equation (14) can be expressed in terms of the element strain matrices and the nodal displacement vector of the element $\mathbf{q}_e$ as

$$e_m = \mathbf{B}_m\mathbf{q}_e, \qquad e_b = \mathbf{B}_b\mathbf{q}_e, \qquad 2e_s = \gamma = \mathbf{B}_s\mathbf{q}_e, \qquad e_{hs} = \mathbf{B}_{hs}\mathbf{q}_e, \tag{17}$$

Consequently, equation (14) becomes

$$\begin{aligned} U_e = \tfrac{1}{2}\mathbf{q}_e^T\int_l [&\mathbf{B}_b^T D_{xx}\mathbf{B}_b + \mathbf{B}_m^T A_{xx}\mathbf{B}_m + \mathbf{B}_s^T S_{xx}\mathbf{B}_s + \mathbf{B}_{hs}^T\alpha^2 H_{xx}\mathbf{B}_{hs} \\ &-(\mathbf{B}_m^T B_{xx}\mathbf{B}_b + \mathbf{B}_b^T B_{xx}\mathbf{B}_m) - (\mathbf{B}_m^T\alpha E_{xx}\mathbf{B}_{hs} + \mathbf{B}_{hs}^T\alpha E_{xx}\mathbf{B}_m) \\ &+(\mathbf{B}_b^T\alpha F_{xx}\mathbf{B}_{hs} + \mathbf{B}_{hs}^T\alpha F_{xx}\mathbf{B}_b)]\,\mathrm{d}x\,\mathbf{q}_e. \end{aligned} \tag{18}$$

Then Hamilton's Principle leads to the element stiffness matrix $\mathbf{K}_e$ as

$$\mathbf{K}_e = \mathbf{K}_b + \mathbf{K}_m + \mathbf{K}_s + \mathbf{K}_{hs} + \mathbf{K}_c, \tag{19}$$

with

$$\mathbf{K}_b = \int_l \mathbf{B}_b^T D_{xx}\mathbf{B}_b\,\mathrm{d}x, \qquad \mathbf{K}_m = \int_l \mathbf{B}_m^T A_{xx}\mathbf{B}_m\,\mathrm{d}x,$$

$$\mathbf{K}_s = \int_l \mathbf{B}_s^T S_{xx}\mathbf{B}_s\,\mathrm{d}x, \qquad \mathbf{K}_{hs} = \int_l \mathbf{B}_{hs}^T\alpha^2 H_{xx}\mathbf{B}_{hs}\,\mathrm{d}x,$$

$$\begin{aligned} \mathbf{K}_c = \int_l [&-(\mathbf{B}_m^T B_{xx}\mathbf{B}_b + \mathbf{B}_b^T B_{xx}\mathbf{B}_m) - (\mathbf{B}_m^T\alpha E_{xx}\mathbf{B}_{hs} + \mathbf{B}_{hs}^T\alpha E_{xx}\mathbf{B}_m) \\ &+ (\mathbf{B}_b^T\alpha F_{xx}\mathbf{B}_{hs} + \mathbf{B}_{hs}^T\alpha F_{xx}\mathbf{B}_b)]\,\mathrm{d}x, \end{aligned} \tag{20}$$

where $\mathbf{K}_b$, $\mathbf{K}_m$, $\mathbf{K}_s$, $\mathbf{K}_{hs}$ and $\mathbf{K}_c$ are the element bending, membrane, transverse shear, higher-order shear and coupling stiffness matrices, respectively.

3.2. CONSISTENT MASS MATRIX FOR HIGHER-ORDER BEAM THEORY

The kinetic energy of an element K_{ke} corresponding to the higher-order theory takes the form

$$K_{ke} = \frac{b}{2}\int_l \int_{-h/2}^{h/2} (v_z^2 + v_x^2)\rho(z)\,\mathrm{d}z\,\mathrm{d}x = \frac{b}{2}\int_l \int_{-h/2}^{h/2} \left[\left(\frac{\partial w}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]\rho\,\mathrm{d}z\,\mathrm{d}x$$

$$= \frac{b}{2}\int_l \int_{-h/2}^{h/2} \left[\left(\frac{\partial w_0}{\partial t}\right)^2 + \left(\frac{\partial u_0}{\partial t}\right)^2 + z^2\left(\frac{\partial^2 w_0}{\partial t\,\partial x}\right)^2 + \alpha^2 z^6\left(\frac{\partial \gamma}{\partial t}\right)^2\right.$$

$$\left. - 2z\,\frac{\partial u_0}{\partial t}\frac{\partial^2 w_0}{\partial t\,\partial x} - 2z^3\,\frac{\partial u_0}{\partial t}\frac{\partial \gamma}{\partial t} + 2\alpha z^4\,\frac{\partial^2 w_0}{\partial t\,\partial x}\frac{\partial \gamma}{\partial t}\right]\rho\,\mathrm{d}z\,\mathrm{d}x, \tag{21}$$

where $\rho(z)$ is the density across the beam thickness. By defining

$$(J_A, J_B, J_D, J_E, J_F, J_H) = b\int_{-h/2}^{h/2} (1, z, z^2, z^3, z^4, z^6)\rho\,\mathrm{d}z, \tag{22}$$

the element kinetic energy K_{ke} can be written as

$$K_{ke} = \frac{1}{2}\int_l \left[J_A\left(\frac{\partial w_0}{\partial t}\right)^2 + J_A\left(\frac{\partial u_0}{\partial t}\right)^2 + J_D\left(\frac{\partial^2 w_0}{\partial t\,\partial x}\right)^2 + \alpha^2 J_H\left(\frac{\partial \gamma}{\partial t}\right)^2\right.$$

$$\left. -2J_B\,\frac{\partial u_0}{\partial t}\frac{\partial^2 w_0}{\partial t\,\partial x} - 2\alpha J_E\,\frac{\partial u_0}{\partial t}\frac{\partial \gamma}{\partial t} + 2\alpha J_F\,\frac{\partial^2 w_0}{\partial t\,\partial x}\frac{\partial \gamma}{\partial t}\right]\mathrm{d}x. \tag{23}$$

The equation above shows that similar to the stretching and bending coupling in the stiffness matrix there is also an axial and rotary velocity coupling in the mass matrix when the density is not symmetric about the reference plane of the composite beams. The coupling of the transverse shear velocity and the deflection slope velocity is always non-zero as long as the transverse shear deformation is not zero.

In the finite element analysis, the element displacements can be interpolated in terms of the element nodal displacement vector $\mathbf{q}_e$ as

$$u_0 = \mathbf{N}_u\mathbf{q}_e, \quad w_0 = \mathbf{N}_w\mathbf{q}_e, \quad \frac{\partial w_0}{\partial x} = \mathbf{N}_{wx}\mathbf{q}_e, \qquad \text{and} \qquad \gamma = \mathbf{N}_\gamma\mathbf{q}_e, \tag{24}$$

where $\mathbf{N}_j$ ($j = u$, w and γ) are the interpolation matrices. The explicit expressions of these matrices will depend on the chosen element type and nodal variables. $\mathbf{N}_{wx}$ can be obtained by differentiating $\mathbf{N}_w$ with respect to x. By substituting equations (24) and (23) into equation (10), one obtains the consistent element mass matrix $\mathbf{M}_e$ as

$$\mathbf{M}_e = \mathbf{M}_w + \mathbf{M}_{uo} + \mathbf{M}_{wx} + \mathbf{M}_\gamma + \mathbf{M}_{uow} + \mathbf{M}_{uo\gamma} + \mathbf{M}_{wx\gamma}, \tag{25}$$

with

$$\mathbf{M}_w = \int_l \mathbf{N}_w^T J_A \mathbf{N}_w \, \mathrm{d}x,$$

$$\mathbf{M}_{uo} = \int_l \mathbf{N}_{uo}^T J_A \mathbf{N}_{uo} \, \mathrm{d}x,$$

$$\mathbf{M}_{wx} = \int_l \mathbf{N}_{wx}^T J_D \mathbf{N}_{wx} \, \mathrm{d}x,$$

$$\mathbf{M}_\gamma = \alpha^2 \int_l \mathbf{N}_\gamma^T J_H \mathbf{N}_\gamma \, \mathrm{d}x,$$

$$\mathbf{M}_{uow} = -\int_l (\mathbf{N}_{uo}^T J_B \mathbf{N}_w + \mathbf{N}_w^T J_B \mathbf{N}_{uo}) \, \mathrm{d}x,$$

$$\mathbf{M}_{uo\gamma} = -\alpha \int_l (\mathbf{N}_{uo}^T J_E \mathbf{N}_\gamma + \mathbf{N}_\gamma^T J_E \mathbf{N}_{uo}) \, \mathrm{d}x,$$

$$\mathbf{M}_{wx\gamma} = \alpha \int_l (\mathbf{N}_{wx}^T J_F \mathbf{N}\gamma + \mathbf{N}_\gamma^T J_F \mathbf{N}_{wx}) \, \mathrm{d}x. \tag{26}$$

$\mathbf{M}_w$, $\mathbf{M}_{uo}$ and $\mathbf{M}_{wx}$ are, respectively, the usual transverse, axial and rotary inertia matrices according to the first-order theory; $\mathbf{M}_\gamma$ is the mass matrix resulting from the higher-order displacement; and $\mathbf{M}_{uow}$, $\mathbf{M}_{uo\gamma}$ and $\mathbf{M}_{wx\gamma}$ are the coupling terms of different components of the axial displacement. It can be seen from equation (22) that $\mathbf{M}_{uow}$ and $\mathbf{M}_{uo\gamma}$ vanish when the density across the beam thickness is symmetric about the reference plane.

The variational consistent mass matrix here has two meanings: one is opposed to the lumped mass method, and the other sepcifies that the contribution of the higher-order displacement to the mass matrix is also taken into account.

4. NUMERICAL EXAMPLES

The efficiency and accuracy of the present element formulation based on the higher-order theory are demonstrated by some numerical examples in this section. The influence of the higher-order mass matrix will also be demonstrated by numerical results.

A two node beam element based on equations (1) and (2), named HQCB-8A, was developed for static analysis by the authors [8]. In this element, the element nodal degrees of freedom at node i, $\mathbf{q}_i$, take the form

$$\mathbf{q}_i = [u_i, w_i, w_{,xi}, \gamma_i]^T, \qquad i = 1, 2. \tag{27}$$

This is the simplest nodal variable vector corresponding to the strains defined in equations (6) and (3). The element displacement vector $\mathbf{q}_e$ of a beam with nodes 1 and 2 is of the form

$$\mathbf{q}_e = \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix}. \tag{28}$$

Consequently, a cubic deflection w_0 and a linear displacement u_0 as well as transverse shear deformation γ can be interpolated. The Hermite shape function is used here for $\mathbf{N}_w$ defined in equation (24). It follows from equation (6) that the present third-order composite beam element possesses a linear bending strain as opposed to the constant bending strain in the existing higher-order composite beam elements with the same nodal variables [2–5]. Therefore, this higher-order composite beam element is more accurate than the higher-order beam elements having a constant bending strain field although the same higher-order theory and the same number of nodal variables are employed [8]. Another advantage of HQCB-8A is that its element stiffness is given explicitly. The element stiffness matrix of HQCB-8A is used here for the free vibration analysis.

By using a linear interpolation for u_0 and γ and a cubic interpolation for w_0, the mass matrices defined in equations (25) and (26) can be easily evaluated.

4.1. EXAMPLE 1. FREE VIBRATION OF SIMPLY SUPPORTED AS4/3501-6 GRAPHITE-EPOXY BEAMS

Two simply supported orthotropic beams with different aspect ratios are used in this example. The material properties are: $E_1 = 144{\cdot}9$ GPa, $E_2 = 9{\cdot}65$ GPa, $G_{12} = G_{13} = 4{\cdot}14$ GPa, $G_{23} = 3{\cdot}45$ GPa, $v_{12} = 0{\cdot}3$, $\rho = 1389{\cdot}23$ kg/m^3. The first five frequencies of the thin beam ($L/h = 120$) and thick beam ($L/h = 15$) are tabulated in Table 1, where L is the beam length. Twenty elements are used for the whole beam. The analytical solutions based on the first-order shear deformation theory

TABLE 1

Frequencies of simply supported AS4/3501-6 graphite-epoxy beams

		f (kHz)		
L/h	Mode	Present	Analytical [10]	FSDT FEM [11]
120	1	0·051	0·051	0·054
	2	0·202	0·203	0·213
	3	0·451	0·457	0·472
	4	0·794	0·812	0·801
	5	1·232	1·269	–
15	1	0·753	0·755	0·789
	2	2·537	2·548	2·656
	3	4·680	4·716	4·895
	4	6·868	6·960	7·168
	5	9·011	9·194	–

TABLE 2

Non-dimensional frequencies of symmetric [0/90/90/0] *beams with* $L/h = 15$

Formulation	BC's	Mode 1	2	3	4	5
Present	SS	2·4979	8·4364	15·5932	22·8974	30·0061
	CC	4·6194	10·4162	17·1724	24·2001	31·2144
	CS	3·5264	9·4736	16·4201	23·5591	30·6107
	CF	0·9199	4·9054	11·4886	18·6886	25·9931
Analytical [10]	SS	2·5023	8·4812	15·7558	23·3089	30·8386
	CC	4·5940	10·2906	16·9559	24·1410	31·2874
	CS	3·5254	9·4423	16·3839	23·6850	31·0569
	CF	0·9241	4·8925	11·4400	18·6972	26·2118
Timo [4]	CF	0·923	4·941	11·656	19·180	27·038
HOBT4a [4]	CF	0·927	5·073	12·159	20·762	28·820
HOBT4b and HOBT5 [4]	CF	0·924	4·895	11·832	19·573	27·720

(FSDT) [10] and the numerical results obtained from the FSDT element in reference [11] are also given in the table for comparison. It should be noted that the shear correction factor used in reference [10] is 5/6 while that in reference [11] is 0·633. The table shows that the present results are slightly lower than the analytical solutions based on the first-order theory [10]. But in general the present solutions agree well with the FSDT analytical solutions. The numerical solutions obtained from the FSDT beam element in reference [11] are larger than the analytical solutions, even though a smaller shear correction factor of 0·633 is used.

4.2. EXAMPLE 2. FREQUENCIES OF SYMMETRIC [0/90/90/0] CROSS-PLY BEAMS WITH $L/h = 15$

The material properties defined in the previous example are used here again. Different boundary conditions are considered to study the performance of the present element formulation. The boundary conditions are represented by C for clamped edge, S for simply supported edge, F for free edge and

$$u_0 = w_0 = w_{,x} = \gamma = 0 \qquad \text{at clamped edge (C)},$$

$$u_0 = w_0 = 0 \qquad \text{at simply supported edge (S)}.$$

The first five non-dimensional frequencies of the beams with different boundaries are given in Table 2, and the non-dimensional frequencies are defined by

$$\varpi_i = \omega_i L^2 \sqrt{J_A / E_1 h^3}, \qquad i = 1, 2, 3 \ldots$$

The numerical results given in reference [4] for clamped edges are also listed in the table. In reference [4] different higher-order beam theories are used where 4 degrees of freedom (d.o.f.) per node are used in HOBT4a as well as HOBT4b, and 5 d.o.f. per node are used in HOBT5. The higher-order theory used in HOBT4b is the same as that used in this work. However, the element stiffness and mass matrices of the present element are given explicitly, while the numerical integration is employed in the evaluation of the stiffness and mass matrices in reference [4]. The results obtained from the different theories in reference [4] are quite different. In general the present results for CF are smaller than the analytical solutions [10], and the results given by all the elements based either on the Timoshenko theory or the higher-order theory in reference [4] are larger than the analytical solutions in reference [10].

4.3. EXAMPLE 3. UNSYMMETRIC [0/90/0/90] BEAMS WITH $L/h = 15$

The material properties of thick beams in this example are the same as those in Example 1. The first five non-dimensional frequencies of the beams with different boundaries are listed in Table 3. The results of both simply supported and clamped beams obtained from the present element are given in the table, while in reference [3] only the results for clamped beams are available. The higher-order theory in reference [3] is the same as the present one, but the bending strain in reference [3] is defined in terms of the rotation. As a result, its element bending strain is constant when 4 d.o.f. per node are used. The present results for the clamped–clamped beam are slightly lower than those in reference [3] except for mode 5. It appears that there is a typing error for this value in reference [3].

4.4. EXAMPLE 4. UNSYMMETRIC [0/90] BEAMS WITH $L/h = 10$

The two-layer cross-ply beam considered here exhibits strong coupling between the stretching and bending. This example is used to compare the present results with those obtained from the higher-order theory in which the in-plane displacement u is cubic and the deflection w is quadratic [12]. The material properties of the thick beams in this example are the same as those in Example 1. Twenty elements are used in the analysis in order to compare the results with those in reference [12]. But it should be noted that the higher-order element in reference [12] has 7 d.o.f. at each node. The first five non-dimensional frequencies of the beams with simply supported and clamped boundaries are given in Table 4. The results given by the first-order theory [13] are also listed in the table for comparison. The letter f in the table refers to flexural vibration, and the letter a refers to axial vibration. In the case of simply supported, SS, boundary, the fifth frequency is for the axial vibration. The frequencies of the axial vibration given in references [12, 13] are different. The present result for the axial vibration is close

TABLE 3

Non-dimensional frequencies of unsymmetric [0/90/0/90] *beams with* $L/h = 15$

Formulation	BC's	Mode 1	2	3	4	5
Present	SS	1·9619	6·6566	13·1225	19·9408	26·9840
	CC	3·6994	8·8119	15·0012	21·6318	28·3575
Reference [3]	CC	3·7244	8·9275	15·3408	22·3408	24·3155

TABLE 4

Non-dimensional frequencies of unsymmetric [0/90] beam with $L/h = 10$

Mode	SS				CC			
	Present	Reference [12]	Reference [13]		Present	Reference [12]	Reference [13]	
1	8·0369	8·0471	8·1439	$f(1)$	12·118	12·113	12·141	$f(1)$
2	21·479	21·598	21·661	$f(2)$	28·487	28·460	28·473	$f(2)$
3	42·676	43·632	43·778	$f(3)$	48·116	48·223	48·141	$f(3)$
4	62·989	63·630	63·787	$f(4)$	69·018	69·709	69·449	$f(4)$
5	85·485	85·992	89·313	$a(1)$	91·254	92·313	91·743	$f(5)$

TABLE 5

Non-dimensional frequencies of unsymmetric [0/90/0/90/0/90] *simply supported beams with* $L/h = 5$

Formulation	Mode 1	2	3	4	5
Present	1·4243	3·5227	5·5882	7·5298	9·5732
Timo [4]	1·432	3·597	5·750	7·856	9·994
HOBT4b [4]	1·434	3·614	5·870	8·114	10·462
HOBT5 [4]	1·416	3·531	5·625	7·795	10·021

to the value given in reference [12]. Table 4 shows that the present results agree well with those given by the quadratic w in reference [12]. Therefore, similar to the conclusions in reference [12], the present results also indicate that the influence of higher-order deflection on the natural frequencies of vibration is not significant, although the higher-order deflection does have a relatively larger influence on the high vibrating modes than its influence on the fundamental mode.

4.5. EXAMPLE 5. UNSYMMETRIC [0/90/0/90/0/90] SIMPLY SUPPORTED THICK BEAMS WITH $L/h = 5$

A thick composite beam with a length to thickness ratio of 5 is considered in this example. The material and geometric properties are: $E_1 = 0{\cdot}762\mathrm{E}8$ psi, $E_2 = 90{\cdot}3048\mathrm{E}7$ psi, $G_{13} = 0{\cdot}1524\mathrm{E}7$ psi, $v_{12} = 0{\cdot}3$, $\rho = 0{\cdot}7257\mathrm{E} - 4\ \mathrm{lbs}^2/\mathrm{in}^4$, $h = 6$ in, $b = 1$ in. The first eight frequencies of the beams were evaluated in reference [4]. By recalling that the error of the frequencies computed by the beam or plate theory would be very large when the wavelength of a vibration exceeds its thickness [1], only the first five non-dimensional frequencies of flexural vibration of the beams are listed in Table 5. The present results are closer to those given by elements of Timo and HOBT5 in reference [4] rather than those given by HOBT4b although the present element and HOBT4b in reference [4] are based on the same theory. HOBT4b gives larger frequencies than those obtaind by the present element and by the elements of Timo and HOBT5 in reference [4].

4.6. EXAMPLE 6. SYMMETRIC [0/0/90/90/0/0] SIMPLY SUPPORTED THICK BEAMS WITH $L/h = 5$

This example concerns the influence of the mass matrices contributed by the higher-order displacement and the coupling of the different order displacements on the frequencies of flexural vibrations. The material and geometric properties of this thick beam are the same as those in the previous example. The first five non-dimensional frequencies of flexural vibration of the beams are listed in Table 6. The present results obtained from the variational consistent mass matrices are smaller than the results given by HOBT4a and between those given by elements of Timo and HOBT5 in reference [4]. The numerical results clearly show that the mass matrices resulting from higher-order displacement and the coupling of different order displacements have a negligible influence on the fundamental

TABLE 6

Non-dimensional frequencies of symmetric [0/0/90/90/0/0] *simply supported beams with* $L/h = 5$

	Present			Reference [4]		
Mode	Consistent mass	No $\mathbf{M}_{wx\gamma}$	No $\mathbf{M}_{\gamma}$ and $\mathbf{M}_{wx\gamma}$	Timoshenko	HOBT4a	HOBT4b and HOBT5
1	1·6455	1·6297	1·6323	1·639	1·736	1·656
2	3·8601	3·6622	3·7229	3·810	4·125	3·923
3	6·0710	5·4262	5·6203	5·912	6·439	6·191
4	8·3407	6·9831	7·3752	7·988	8·722	8·470
5	10·7068	8·4129	9·0392	10·100	11·042	10·803

frequencies but these higher-order and coupling mass matrices have a significant influence on the frequencies of high mode flexural vibration. For example, the difference for the frequency of mode 5 is more than 20%.

5. SUMMARY AND CONCLUSIONS

By using Hamilton's Principle, this work presents the derivation of the variational consistent stiffness and mass matrices for the finite element modelling of a composite beam based on the third-order shear deformation theory. The influence of the mass components resulting from the higher-order displacements on the frequencies of flexural vibration is also studied. Similar to that, the stiffness matrix can be decomposed into three terms corresponding to, respectively, the first-order theory, the higher-order shear and the coupling of the different components of the axial strains; the variational consistent mass matrices can also be decomposed into three parts. These are: the usual part including the rotary inertia given by the first-order theory, the part resulting from the higher-order displacement, and the part resulting from the coupling between the different order components of the axial displacement. The bending strain is expressed in terms of the deflection and transverse shear deformation in this work. As a result, the present two-noded higher-order beam element possesses a linear bending strain as opposed to the constant bending strain in the existing higher-order composite beam elements when the same number of nodal degrees of freedom is used. The numerical examples show that the present element formulation is efficient and accurate. The numerical examples also show that the higher-order and coupling mass matrices have a negligible influence on the fundamental frequencies, but they have a significant influence on the frequencies of high mode flexural vibration. The methodology for the variational consistent element formulation of composite beams presented in this work can be easily extended to composite plates and shells.

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