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DYNAMIC ANALYSIS OF MULTISTEP PILES ON PASTERNAK SOIL SUBJECTED TO AXIAL TIP FORCES

M. A. De Rosa

Department of Structural Engineering, University of Basilicata, Via della Tecnica 3, 85100, Potenza, Italy

AND

M. J. Maurizi

Departamento de Ingenieria, Universidad Nacional del Sur, Bahía Blanca, Argentina

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The dynamic analysis of a foundation pile on a two-parameter elastic soil is performed, in the presence of non-classical boundary conditions. The soil discontinuities are simulated through the introduction of n step variations of the cross-section, whereas the partial restraints at the top and at the bottom are taken into account by imposing non-classical boundary conditions. Finally, the pile is supposed to be subjected to a conservative axial load at the tip. The analysis can be considered to be exact, in the framework of the Euler–Bernoulli hypothesis, the differential equation of motion is deduced and solved, and the frequency equation is derived for an arbitrary number of steps. Some numerical examples complete the paper.

1. INTRODUCTION

The dynamic analysis of foundation piles cannot be performed with the simple hypothesis of constant cross-section, because the presence of soil discontinuities and/or building imperfections is unavoidable. Consequently, it is mandatory to take into account the possibility of steps along the pile length.

Moreover, it is certainly possible to adopt the simple Winkler soil model [1], but its modulus of subgrade reaction should be assumed to vary from zero at the top to a maximum value at the bottom. A more realistic hypothesis assumes the existence of two soil parameters [2], which can be detected starting from simple *in situ* experiments, leading to the so-called Pasternak soil [3]. More generally, a well established bibliography exists on the *two-parameter elastic soil* [4–11].

The boundary conditions at the pile ends cannot be precisely stated, partly because of the unpredictable soil behaviour at the bottom, and partly because of the influence of superstructures at the top. In fact, the usual cantilever beam

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hypothesis is by no means satisfactory, and it is more convenient to introduce flexible ends, which can elastically react to transversal displacements and rotations. In this way, the partial restraints at the bottom due to the soil influence—and at the top due to some foundation block—can be easily accommodated. Finally, the axial force at the top can be considered to be conservative in nature, as in references [12–16].

It is evident, from the foregoing discussion, that every dynamic analysis must take into account a large number of parameters, because at least the four end flexibilities and the two soil parameters cannot be fixed *a priori*. Consequently, a time consuming parametric analysis becomes necessary, and, from this point of view, an exact approach seems to be quite useful.

In this paper the differential equation of motion is written and solved for a general foundation pile with piecewise constant cross-section, in the presence of elastically flexible ends and axial tip force. The number of cross-section steps is arbitrary, thus allowing the exact analysis of some realistic cases. Numerical examples complete the paper, in which the influence of the various parameters is taken into account.

2. EXACT ANALYSIS

A foundation pile with total span L, Young modulus E and mass density ρ, and assume that the cross-section is divided into N segments with length L_i, area A_i and moment of inertia I_i is considered. Moreover, the elastic soil along each segment is assumed to be defined by the (constant) parameters k_{Wi} and k_{Pi}.

It is convenient to define N reference frames, with origins at the bottom and at the $N-1$ intermediate steps, as sketched in Figure 1, whereas at the top a compressive axial force P is acting.

If the Euler–Bernoulli slender beam theory is adopted, then the following N equations of motion can be easily deduced by means of the Hamilton principle:

$$(EI_i)v_i''''(X_i, t) + [P - k_{Pi}]v_i''(X_i, t) + k_{Wi}v_i(X_i, t) + \rho A_i\ddot{v}_i(X_i, t) = 0, \qquad (1)$$

where the primes denote differentiation with respect to X_i, and the dot denotes differentiation with respect to time.

The solution can be sought in the following form:

$$v_i(x_i, t) = V_i(x_i)\,\mathrm{e}^{\mathrm{j}\omega t}, \qquad (2)$$

where $x_i = X_i/L$, ω is the circular frequency and $\mathrm{j} = \sqrt{-1}$.

Equation (1) becomes:

$$V_i''''(x_i) + b_i V_i''(x_i) + c_i^4 V_i(x_i) = 0, \qquad (3)$$

where now the primes denote differentation with respect to x_i,

$$b_i = (P - k_{P_i})L^2/EI_i \qquad (4)$$

and

$$c_i^4 = (k_{Wi} - \rho A_i\omega^2)L^4/EI_i. \qquad (5)$$

The characteristic polynomial is given by:

$$r^4 + b_i r^2 + c_i^4 = 0 \tag{6}$$

and its general solution is:

$$V_i(x_i) = A_{i1}\,\mathrm{e}^{r_1 x_i} + A_{i2}\,\mathrm{e}^{r_2 x_i} + A_{i3}\,\mathrm{e}^{r_3 x_i} + A_{i4}\,\mathrm{e}^{r_4 x_i}, \tag{7}$$

where r_1, r_2, r_3 and r_4 are the roots of the polynomial equation (6).

In order to find the roots, it is important to take into account that: (a) b_i does not have a definite sign; (b) c_i^4 does not have a definite sign.

Defining

$$p = r^2, \tag{8}$$

equation (6) becomes a second order polynomial equation:

$$p^2 + b_i p + c_i^4 = 0. \tag{9}$$

The generic solution for the ith segment is given by:

$$r_{i(1,2,3,4)} = \pm(1/\sqrt{2})\sqrt{-b_i \pm \sqrt{b_i^2 - 4c_i^4}} \tag{10}$$

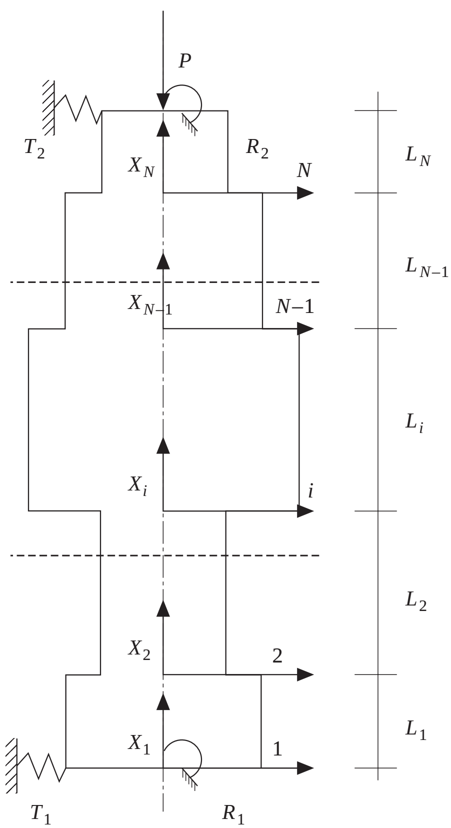

Figure 1. The structural system.

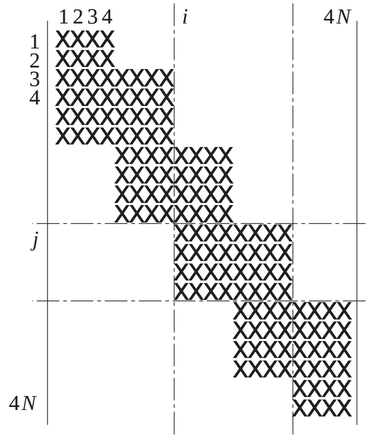

Figure 2. The skeleton of the matrix for the frequency equation.

and can be classified as follows:

2.1. CASE I: $b_i = 0$

Case Ia: $c_i^4 = 0$

$$V_i(x_i) = A_{i1} + A_{i2}x_i + A_{i3}x_i^2 + A_{i4}x_i^3. \tag{11}$$

Case Ib: $c_i^4 < 0$

$$V_i(x_i) = A_{i1}\cosh \alpha_i x_i + A_{i2}\sinh \alpha_i x_i + A_{i3}\cos \alpha_i x_i + A_{i4}\sin \alpha_i x_i, \tag{12}$$

where $\alpha_i = \sqrt{\sqrt{-c_i^4}}$.

Case Ic: $c_i^4 > 0$

$$V_i(x_i) = A_{i1}\cosh \beta_i x_i \cos \beta_i x_i + A_{i2}\sinh \beta_i x_i \cos \beta_i x_i + A_{i3}\cosh \beta_i x_i \sin \beta_i x_i + A_{i4}\sinh \beta_i x_i \sin \beta_i x_i, \tag{13}$$

TABLE 1

Non-dimensional critical loads for a pile with constant cross-section, clamped at bottom and free at the tip, for varying values of the non-dimensional soil parameters

	K_{pi}			
K_{wi}	0	0·5	1	2·5
0	2·4674	7·4022	12·3370	27·1414
1	2·6499	7·5847	12·5195	27·3239
100	11·9964	16·9312	21·8660	36·6704
10 000	100·0123	104·9471	109·8820	124·6864
1 000 000	999·9999	1004·9348	1009·8695	1024·6740

TABLE 2

Non-dimensional critical loads for a two-stepped pile with $x_{1,3} = 0{\cdot}4$, $x_2 = 0{\cdot}2$, $\zeta_{1,3} = 1$, $\zeta_2 = 1{\cdot}44$, *clamped at bottom and free at the tip, for varying values of the non-dimensional soil parameters*

	K_{pi}			
K_{wi}	0	0·5	1	2·5
0	2·7463	8·7144	14·5560	31·4525
1	2·9602	8·9136	14·7410	31·5999
100	13·4850	18·6238	23·7355	38·9535
10 000	101·2131	106·2065	111·1983	126·1648
1 000 000	1000·0024	1004·9372	1009·8720	1024·6764

TABLE 3

Non-dimensional critical loads for a two-stepped pile with $x_{1,3} = 0{\cdot}1$, $x_2 = 0{\cdot}8$, $\zeta_{1,3} = 1$, $\zeta_2 = 1{\cdot}44$, *clamped at bottom and free at the tip, for varying values of the non-dimensional soil parameters*

	K_{pi}			
K_{wi}	0	0·5	1	2·5
0	4·1896	13·3811	22·4780	49·1320
1	4·5420	13·7165	22·7957	49·3932
100	20·9572	29·2086	37·3770	61·3802
10 000	139·1708	144·9155	150·6443	167·7406
1 000 000	1034·7125	1039·7626	1044·8124	1059·9604

where:

$$\beta_i = c_i/\sqrt{2}.$$

2.2. CASE II: $b_i > 0$

Case IIa: $c_i^4 = 0$

$$V_i(x_i) = A_{i1} + A_{i2}x_i + A_{i3}\cos d_{1i}x_i + A_{i4}\sin d_{1i}x_i, \tag{14}$$

where $d_{1i} = \sqrt{b_i}$.

Case IIb: $c_i^4 < 0$

$$V_i(x_i) = A_{i1}\cosh \alpha_i' x_i + A_{i2}\sinh \alpha_i' x_i + A_{i3}\cos \alpha_i'' x_i + A_{i4}\sin \alpha_i'' x_i, \tag{15}$$

where:

$$\alpha_i' = (1/\sqrt{2})\sqrt{-b_i + \sqrt{b_i^2 - 4c_i^4}}, \qquad \alpha_i'' = (1/\sqrt{2})\sqrt{b_i + \sqrt{b_i^2 - 4c_i^4}}. \tag{16, 17}$$

Case IIc: $c_i^4 > 0$

IIc1: $b_i^2 = 4c_i^4$

$$V_i(x_i) = (A_{i1} + A_{i2}x_i)\cos d_{2i}x_i + (A_{i3} + A_{i4}x_i)\sin d_{2i}x_i, \tag{18}$$

where

$$d_{2i} = \sqrt{b_i}/2.$$

IIc2: $b_i^2 < 4c_i^4$

$$V_i(x_i) = A_{i1}\cosh\gamma_i x_i \cos\gamma_i' x_i + A_{i2}\sinh\gamma_i x_i \cos\gamma_i' x_i + A_{i3}\cosh\gamma_i x_i \sin\gamma_i' x_i + A_{i4}\sinh\gamma_i x_i \sin\gamma_i' x_i, \tag{19}$$

where

$$\gamma_i = \sqrt{\sqrt{c_i^4/4} - b_i/4}, \qquad \gamma_i' = \sqrt{\sqrt{c_i^4/4} + b_i/4}. \tag{20, 21}$$

TABLE 4

Non-dimensional critical loads for a two-stepped pile with $x_{1,3} = 0{\cdot}4$, $x_2 = 0{\cdot}2$, $\zeta_{1,3} = 1{\cdot}44$, $\zeta_2 = 1$, *clamped at bottom and free at the tip, for varying values of the non-dimensional soil parameters*

	K_{pi}			
K_{wi}	0	0·5	1	2·5
0	2·0204	6·4530	10·8401	23·6940
1	2·1775	6·6174	11·0118	23·8877
100	9·9474	14·7573	19·5584	33·8949
10 000	98·7751	103·6423	108·5071	123·0871
1 000 000	999·9894	1004·9241	1009·8587	1024·6630

TABLE 5

Non-dimensional critical loads for a two-stepped pile with $x_{1,3} = 0{\cdot}1$, $x_2 = 0{\cdot}8$, $\zeta_{1,3} = 1{\cdot}44$, $\zeta_2 = 1$, *clamped at bottom and free at the tip, for varying values of the non-dimensional soil parameters*

	K_{pi}			
K_{wi}	0	0·5	1	2·5
0	1·3244	4·2025	7·0197	15·1681
1	1·4383	4·3237	7·1475	15·3119
100	7·3098	10·6303	13·9003	23·4052
10 000	65·5437	69·6744	73·7864	86·0010
1 000 000	945·5941	949·8481	954·0770	966·5934

TABLE 6

First non-dimensional free frequency for the pile in Table 3, for various values of the ratio $\gamma = \lambda/\lambda_c$

		K_{pi}			
K_{wi}	γ	0	0·5	1	2·5
0	0·0	1·8780	2·4565	2·7534	3·2559
	0·2	1·7824	2·3425	2·6322	3·1207
	0·4	1·6648	2·2007	2·4819	2·9571
	0·6	1·5101	2·0104	2·2795	2·7421
	0·8	1·2751	1·7122	1·9571	2·4004
1	0·0	1·9169	2·4745	2·7663	3·2638
	0·2	1·8199	2·3603	2·6452	3·1289
	0·4	1·7004	2·2182	2·4950	2·9658
	0·6	1·5430	2·0272	2·2927	2·7515
	0·8	1·3034	1·7275	1·9697	2·4107
100	0·0	3·2999	3·4624	3·5872	3·8595
	0·2	3·2054	3·3705	3·4948	3·7586
	0·4	3·0799	3·2518	3·3791	3·6397
	0·6	2·8920	3·0767	3·2132	3·4839
	0·8	2·5455	2·7442	2·8971	3·2086
10 000	0·0	10·0376	10·0559	10·0727	10·1161
	0·2	9·9358	9·9626	9·9858	10·0417
	0·4	9·6713	9·7300	9·7800	9·8896
	0·6	9·0298	9·1160	9·1966	9·4074
	0·8	7·8005	7·8887	7·9732	8·2070
1 000 000	0·0	31·6256	31·6265	31·6274	31·6299
	0·2	31·3045	31·3173	31·3297	31·6113
	0·4	30·2750	30·2962	30·3173	31·5041
	0·6	28·2847	28·3108	28·3367	31·2211
	0·8	24·4951	24·5219	24·5485	30·7818

IIc3: $b_i^2 > 4c_i^4$

$$V_i(x_i) = A_{i1} \cos \alpha_i''' x_i + A_{i2} \sin \alpha_i''' x_i + A_{i3} \cos \alpha_i'' x_i + A_{i4} \sin \alpha_i'' x_i, \tag{22}$$

where

$$\alpha_i''' = (1/\sqrt{2})\sqrt{b_i - \sqrt{b_i^2 - 4c_i^4}}. \tag{23}$$

2.3. CASE III $b_i < 0$

Case IIIa: $c_i^4 = 0$

$$V_i(x_i) = A_{i1} + A_{i2}x_i + A_{i3} \cosh d_{3i}x_i + A_{i4} \sinh d_{3i}x_i, \tag{24}$$

where $d_{3i} = \sqrt{-b_i}$.

Case IIIb: $c_i^4 < 0$

$$V_i(x_i) = A_{i1} \cosh \alpha_i' x_i + A_{i2} \sinh \alpha_i' x_i + A_{i3} \cos \alpha_i'' x_i + A_{i4} \sin \alpha_i'' x_i. \tag{25}$$

Case IIIc: $c_i^4 > 0$

IIIc1: $b_i^2 = 4c_i^4$

$$V_i(x_i) = (A_{i1} + A_{i2}x_i)\cosh d_{4i}x_i + (A_{i3} + A_{i4}x_i)\sinh d_{4i}x_i, \tag{26}$$

where $d_{4i} = \sqrt{-b_i/2}$.

IIIc2: $b_i^2 > 4c_i^4$

$$V_i(x_i) = A_{i1}\cosh\alpha_i' x_i + A_{i2}\sinh\alpha_i' x_i + A_{i3}\cosh\delta_i x_i + A_{i4}\sinh\delta_i x_i, \tag{27}$$

where

$$\delta_i = (1/\sqrt{2})\sqrt{-b_i - \sqrt{b_i^2 - 4c_i^4}}. \tag{28}$$

IIIc3: $b_i^2 < 4c_i^4$

$$V_i(x_i) = A_{i1}\cosh\gamma_i x_i\cos\gamma_i' x_i + A_{i2}\sinh\gamma_i x_i\cos\gamma_i' x_i + A_{i3}\cosh\gamma_i x_i\sin\gamma_i' x_i + A_{i4}\sinh\gamma_i x_i\sin\gamma_i' x_i. \tag{29}$$

TABLE 7

First non-dimensional free frequency for the pile in Table 4, for various values of the ratio $\gamma = \lambda/\gamma_c$.

		K_{pi}			
K_{wi}	γ	0	0·5	1	2·5
0	0·0	1·8408	2·3617	2·6215	3·0701
	0·2	1·7982	2·3109	2·5659	3·0030
	0·4	1·7517	2·2553	2·5051	2·9301
	0·6	1·7003	2·1936	2·4379	2·8499
	0·8	1·6429	1·1244	2·3625	2·7607
1	0·0	1·8790	2·3800	2·6349	3·0784
	0·2	1·8358	2·3291	2·5792	3·0113
	0·4	1·7886	2·2734	2·5184	2·9384
	0·6	1·7365	2·2116	2·4512	2·8583
	0·8	1·6782	2·1421	2·3758	2·7691
100	0·0	3·2356	3·3674	3·4661	3·6284
	0·2	3·1979	3·3384	3·4237	3·5694
	0·4	3·1555	3·2857	3·3779	3·5061
	0·6	3·1067	3·2379	3·4237	3·4373
	0·8	3·0491	3·1831	3·2714	3·3615
10 000	0·0	9·7372	9·7625	9·7843	9·8327
	0·2	9·6728	9·7035	9·7307	9·7927
	0·4	9·5891	9·6247	9·6569	9·7341
	0·6	9·4829	9·5230	9·5596	9·6501
	0·8	9·3465	9·3918	9·4332	9·5354
1 000 000	0·0	29·3143	29·3189	29·3234	29·3370
	0·2	29·1666	29·1709	29·1751	29·1878
	0·4	28·9966	29·0005	29·0043	29·0157
	0·6	28·7974	28·8006	28·8038	28·8134
	0·8	28·5574	28·5597	28·5620	28·5687

TABLE 8

Non-dimensional critical loads for a pile with central step $\zeta_1 = 1{\cdot}44$, $\zeta_2 = 1$, $k_{wi} = 10$, $k_{pi} = 1$, clamped at bottom and free at the tip, for varying values of the non-dimensional flexibility parameter T_2

T_2	λ_c	T_2	λ_c
0	19·0362	5	7·9233
0·05	12·9709	10	7·8920
0·1	10·7252	50	7·6870
0·5	8·4771	100	7·8639
1	8·1715	∞	7·8608

TABLE 9

First and second non-dimensional free frequencies for a pile with $x_{1,3} = 0{\cdot}3$, $x_{2,4} = 0{\cdot}1$, $x_5 = 0{\cdot}2$, $\zeta_{1,3,5} = 1$, $\zeta_{2,4} = 1{\cdot}44$ for various values of the non-dimensional flexibility parameters $R_2 = T_2$

$R_2 = T_2$	Ω_1	Ω_2	$R_2 = T_2$	Ω_1	Ω_2
0	5·0035	8·1079	5	2·8630	5·4454
0·0005	4·9255	7·7719	50	2·8533	5·4273
0·005	4·3342	6·4024	500	2·8523	5·4254
0·05	3·2918	5·8691	5000	2·8522	5·4252
0·5	2·9379	5·5751	∞	2·8522	5·4252

Finally, the general solution can be expressed as:

$$V_i(x_i) = A_{i1} V_{i,1} + A_{i2} V_{i,2} + A_{i3} V_{i,3} + A_{i4} V_{i,4}, \tag{30}$$

where the terms $V_{i,k}$, $k = 1, 4$ assume different values according to the above classification.

3. BOUNDARY CONDITIONS

The boundary conditions are by no means intuitive, and it is necessary to use an energy based approach, in order to be sure of not missing some term. They are:

at $x_1 = 0$

$$R_1 V_1''(0) = -V_1'(0), \qquad T_1(V_1'''(0) + b_1 V_1'(0)) = V_1(0), \tag{31}$$

at $x_{i-1} = L_{i-1}/L$ and $x_i = 0$

$$V_{i-1}(x_{i-1}) = V_i(0), \qquad V_{i-1}'(x_{i-1}) = V_i'(0), \qquad EI_{i-1} V_{i-1}(x_{i-1}) = EI_i V_i(0),$$

$$EI_{i-1}(V_{i-1}'''(x_{i-1}) + b_{i-1} V_{i-1}'(x_{i-1})) = EI_i(V_i'''(0) + b_i V_i'(0)). \tag{32}$$

at $x_N = L_N/L$

$$R_2 V_N''(x_N) = V_N'(x_N), \qquad T_2(V_N'''(x_N) + b_N V_N'(x_N)) = -V_N(x_N). \tag{33}$$

where

$$R_1 = EI_1/k_{R1}L, \quad R_2 = EI_N/k_{R2}L, \quad T_1 = EI_1/k_{T1}L^3, \quad T_2 = EI_N/k_{T2}L^3 \tag{34}$$

are the non-dimensional rotational stiffnesses and axial stiffnesses, respectively, proportional to the rotational stiffnesses k_{R1}, k_{R2} and the axial stiffnesses k_{T1}, k_{T2}.

The above derived linear homogeneous system has non-trivial solutions if the determinant of the coefficients is equal to zero. The first two rows of the determinant refer to the presence of the flexible constraint at the bottom, and they are given in Appendix 1, together with the last two rows, which refer to the constraint at the top. The other terms of the determinant can be easily described by looking at the matrix sketched in Figure 2. More particularly, the terms of the ith step are given by the rows $(4i - 1, 4i + 2)$, and are reported in Appendix 2. All the other entries of the matrix are equal to zero.

4. NUMERICAL EXAMPLES

Now the following non-dimensional coefficients are defined:

$$K_{Wi} = k_{Wi}L^4/EI_i; \qquad K_{Pi} = k_{Pi}L^2/\pi^2 EI_i; \qquad A_i/A_1 = \zeta_i; \qquad I_i/I_1 = \zeta_i^2 \tag{35}$$

and the non-dimensional axial load:

$$\lambda = PL^2/EI_1 \tag{36}$$

which has its critical value at:

$$\lambda_c = P_c L^2/EI_1. \tag{37}$$

Finally, it is convenient to express the results in the terms of the non-dimensional frequency parameter:

$$\Omega_i = \sqrt{\sqrt{\rho A_1 \omega_i^2 L^4/EI_1}}. \tag{38}$$

The following issues have been addressed:

4.1. THE INFLUENCE OF THE SOIL PARAMETERS ON THE CRITICAL LOAD

In Table 1 the non-dimensional critical loads are given for a pile with constant cross-section, $\zeta_1 = 1$ and $\zeta_1^2 = 1$. The pile is supposed to be clamped at the bottom and free at the top, as can be noted from the critical load in the absence of soil.

4.2. THE INFLUENCE OF THE STEPS ON THE CRITICAL LOAD

In Tables 2, 3 the critical loads are given for a pile with two steps, with $\zeta_{1,3} = 1$ and $\zeta_2 = 1{\cdot}44$. In Table 2 the non-dimensional span of the first and third segments is equal to $x_{1,3} = 0{\cdot}4$, whereas the intermediate segment has length equal to $x_2 = 0{\cdot}2$, so modelling the presence of an intermediate defect. The resulting values are quite near to the corresponding values for constant cross-section.

Another case is given in Table 3, where the step locations have been changed, in such a way that $x_{1,3} = 0{\cdot}1$ and $x_2 = 0{\cdot}8$.

In Tables 4, 5 the step abscissae do not change, but the central segment is assumed to be more slender than the first and third segments, because it is assumed $\zeta_{1,3} = 1{\cdot}44$ and $\zeta_2 = 1$.

Obviously, the critical load increases for increasing values of the soil parameters.

4.3. THE INFLUENCE OF THE AXIAL LOAD AND OF THE SOIL PARAMETERS ON THE FIRST FREE FREQUENCY

It is well-known that the frequencies decrease with increasing axial loads, and that the first frequency is equal to zero when the axial load attains its critical value. Consequently, in Tables 7, 8 the first non-dimensional frequency is reported for various values of ($\gamma = \lambda/\lambda_c$) and for the cases in Tables 3, 4.

It is possible to deduce that: the resulting curves are typical divergence curves; the non-dimensional frequency increases with increasing values of the soil parameters K_{wi} and K_{pi}.

4.4. THE INFLUENCE OF THE END FLEXIBILITIES ON THE CRITICAL LOAD

In Table 8 the non-dimensional critical load has been reported for a pile with a central step, with $\zeta_1 = 1{\cdot}44$ and $\zeta_2 = 1$. The soil parameters are defined by $K_{wi} = 10$ and $K_{pi} = 1$. The pile is supposed to be clamped at the bottom end, whereas the other end is supposed to be elastically restrained against the translation. The values of the non-dimensional coefficient T_2 are allowed to vary between the limiting values 0 (clamped–simply supported pile) and ∞ (clamped–free pile). Accordingly, the critical load decreases with increasing flexibility values.

4.5. THE INFLUENCE OF THE END FLEXIBILITIES ON THE FREE FREQUENCIES

In Table 9 the first two non-dimensional frequencies have been given for a multistep pile defined by the step abscissae $x_{1,3} = 0{\cdot}3$, $x_{2,4} = 0{\cdot}1$ and $x_5 = 0{\cdot}2$ and by the cross-section parameters $\zeta_{1,3,5} = 1$ and $\zeta_{2,4} = 1{\cdot}44$. The pile is clamped at the bottom, whereas the other end is elastically restrained against the translation and the rotation. The two limiting cases $R_2 = T_2 = 0$ and $R_2 = T_2 = \infty$ give the clamped–clamped pile and the clamped–free pile, respectively. Finally, the non-dimensional soil parameters are given by $K_{wi} = 10$ and $K_{pi} = 1$. As can be immediately seen, the free frequency values decrease with increasing flexibility values.

5. CONCLUSIONS

The exact analysis of a foundation pile on Pasternak soil has been performed in the presence of rotationally and axially flexible ends. The cross-section of the pile is supposed to be defined by N segments with constant cross-sections, divided by $N-1$ intermediate steps. The equation of motion has been derived and solved, leading to a frequency equation in a highly regular form. Some numerical examples end the paper, where a parametric analysis has been performed for various parameter values.

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APPENDIX 1

$$a_{1,1} = R_1 V''_{1,1} - V'_{1,1}, \qquad a_{1,2} = R_1 V''_{1,2} - V'_{1,2},$$

$$a_{1,3} = R_1 V''_{1,3} - V'_{1,3}, \qquad a_{1,4} = R_1 V''_{1,4} - V'_{1,4},$$

$$a_{2,1} = T_1(V'''_{1,1} + b_1 V'_{1,1}) + V_{1,1}, \qquad a_{2,2} = T_1(V'''_{1,2} + b_1 V'_{1,2}) + V_{1,2},$$

$$a_{2,3} = T_1(V'''_{1,3} + b_1 V'_{1,3}) + V_{1,3}, \qquad a_{2,4} = T_1(V'''_{1,4} + b_1 V'_{1,4}) + V_{1,4},$$

$$a_{4N-1,4N-3} = R_1 V''_{N,1} + V'_{N,1}, \qquad a_{4N-1,4N-2} = R_1 V''_{N,2} + V'_{N,2},$$

$$a_{4N-1,4N-1} = R_1 V''_{N,3} + V'_{N,3}, \qquad a_{4N-1,4N} = R_1 V''_{N,4} + V'_{N,4},$$

$$a_{4N,4N-3} = T_1(V'''_{N,1} + b_N V'_{N,1}) - V_{N,1}, \qquad a_{4N,4N-2} = T_1(V'''_{N,2} + b_N V'_{N,2}) - V_{N,2},$$

$$a_{4N,4N-1} = T_1(V'''_{N,3} + b_N V'_{N,3}) - V_{N,3}, \qquad a_{4N,4N} = T_1(V'''_{N,4} + b_N V'_{N,4}) - V_{N,4}.$$

APPENDIX 2

$$a_{j,k} = V_{i-1,1}, \qquad a_{j,k+1} = V_{i-1,2}, \qquad a_{j,k+2} = V_{i-1,3}, \qquad a_{j,k+3} = V_{i-1,4},$$

$$a_{j,k+4} = -V_{i,1}, \qquad a_{j,k+5} = -V_{i,2}, \qquad a_{j,k+6} = -V_{i,3}, \qquad a_{j,k+7} = -V_{i,4},$$

$$a_{j+1,k} = V'_{i-1,1}, \quad a_{j+1,k+1} = V'_{i-1,2}, \quad a_{j+1,k+2} = V'_{i-1,3}, \quad a_{j+1,k+3} = V'_{i-1,4},$$

$$a_{j+1,k+4} = -V'_{i,1}, \quad a_{j+1,k+5} = -V'_{i,2}, \quad a_{j+1,k+6} = -V'_{i,3}, \quad a_{j+1,k+7} = -V'_{i,4},$$

$$a_{j+2,k} = EI_{i-1}V''_{i-1,1}, \qquad a_{j+2,k+1} = EI_{i-1}V''_{i-1,2}, \qquad a_{j+2,k+2} = EI_{i-1}V''_{i-1,3},$$

$$a_{j+2,k+3} = EI_{i-1}V''_{i-1,4}, \qquad a_{j+2,k+4} = -EI_iV''_{i,1}, \qquad a_{j+2,k+5} = -EI_iV''_{i,2},$$

$$a_{j+2,k+6} = -EI_iV''_{1,3}, \qquad a_{j+2,k+7} = -EI_iV''_{1,4},$$

$$a_{j+3,k} = EI_{i-1}(V'''_{i-1,1} + b_{i-1}V'_{i-1,1}), \qquad a_{j+3,k+1} = EI_{i-1}(V'''_{i-1,2} + b_{i-1}V'_{i-1,2}),$$

$$a_{j+3,k+2} = EI_{i-1}(V'''_{i-1,3} + b_{i-1}V'_{i-1,3}), \quad a_{j+3,k+3} = EI_{i-1}(V'''_{i-1,4} + b_{i-1}V'_{i-1,4}),$$

$$a_{j+3,k+4} = -EI_i(V_{i,1} + b_iV'_{i,1}), \qquad a_{j+3,k+5} = -EI_i(V_{i,2} + b_iV'_{i,2}),$$

$$a_{j+3,k+6} = -EI_i(V_{i,3} + b_iV'_{i,3}), \qquad a_{j+3,k+7} = -EI_i(V_{i,4} + b_iV'_{i,4}).$$