Journal of Sound and Vibration (1999) **219**(5), 925–926
Article No. jsvi.1998.1894, available online at http://www.idealibrary.com on IDEAL

NOTE ON A GENERALIZATION OF GAUSS LEAST SQUARES METHOD APPLIED TO ACTIVE NOISE REDUCTION SYSTEMS

L. M. B. C. Campos

Secção de Mecânica Aeroespacial, ISR, Instituto Superior Técnico, 1096 Lisboa Codex, Portugal

(*Received* 6 *September* 1996, *and in final form* 1 *August* 1998)

It is shown that the Gauss method of least squares can be generalized to a method of integer powers of quadratic functions, when choosing a cost function to optimize active noise reduction systems using transfer functions.

Consider a set of N loudspeakers or anti-noise sources, with output spectra $p_n(\omega)$ with $n = 1, \ldots, N$, attempting to cancel a background noise $r_\alpha(\omega)$ with the total sound field being monitored at M microphones, which have output spectra $q_\alpha(\omega)$, with $\alpha = 1, \ldots, M$, where the $N \times M$ matrix of transfer functions $T_{\alpha n}$ between loudspeakers and microphones is assumed to be known:

$$\alpha = 1, \ldots, M: \quad q_\alpha(\omega) = \sum_{n=1}^{N} T_{\alpha n}(\omega) p_n(\omega) + r_\alpha(\omega). \tag{1}$$

The usual choice of the cost function is the total acoustic power at the microphones:

$$E(\omega) = \sum_{\alpha=1}^{M} |q_\alpha(\omega)|^2 = \sum_{\alpha=1}^{M} q_\alpha(\omega) q_\alpha^*(\omega), \tag{2}$$

where an asterisk denotes the complex conjugate. The objective is to choose the ouputs of the loudspeakers $p_1(\omega), \ldots, p_N(\omega)$ so as to minimize the total acoustic power; this corresponds to the Gauss method of least squares. The Gauss method can be generalized to a method of least even powers by choosing as a cost function any integral power k of the acoustic power,

$$F(\omega) \equiv \{E(\omega)\}^k = \left\{ \sum_{\alpha=1}^{M} q_\alpha(\omega) q_\alpha^*(\omega) \right\}^k. \tag{3}$$

From equation (3) it follows that:

$$\mathrm{d}F = kE^{k-1}\,\mathrm{d}E, \qquad \mathrm{d}^2F = kE^{k-1}\,\mathrm{d}^2E + k(k-1)E^{k-2}(\mathrm{d}E)^2, \tag{4a, b}$$

so that the condition of minimum energy:

$$\mathrm{d}E = 0, \qquad \mathrm{d}^2E > 0, \tag{5a, b}$$

0022–460X/99/050925 + 02 $30.00/0

is equivalent to the condition of minimum cost function

$$k \geqslant 2: \quad \mathrm{d}F = 0, \qquad \mathrm{d}^2F > 0, \tag{6a, b}$$

for any $k \geqslant 2$. Thus, the Gauss method of least squares (2) or the use as cost function of an integer power of a quadratic (3), gives exactly the same result.

The condition of stationary cost function, in either form (5a) or (6a), leads to:

$$0 = \partial E/\partial p_n = \sum_{\alpha=1}^{M} q_\alpha^* \partial q_\alpha/\partial p_n = \sum_{\alpha=1}^{M} T_{\alpha n}\left(\sum_{m=1}^{N} T_{\alpha m}^* p_m^* + r_\alpha^*\right), \tag{7}$$

which can be solved to specify (8a) the input to the loudspeakers:

$$p_n = -\sum_{n=1}^{N}\sum_{\alpha=1}^{M} \mathbf{A}_{nm}^{-1} T_{\alpha m} r_\alpha, \qquad \mathbf{A}_{nm} \equiv \sum_{\alpha=1}^{M} T_{\alpha n} T_{\alpha m}^*, \tag{8a, b}$$

where $\mathbf{A}_{nm}^{-1}$ is the inverse of the matrix (8b). In order to prove that (8a) is indeed the optimal input, which minimizes the cost function (2) or (3), it is sufficient to prove (5b) or (6b), starting with:

$$\mathrm{d}^2E = \sum_{n,m=1}^{N} (\partial^2 E/\partial p_n dp_m^*)\, \mathrm{d}p_n\, \mathrm{d}p_m^*, \tag{9}$$

and using equation (1):

$$\partial^2 E/\partial p_n \partial p_m^* = \sum_{\alpha=1}^{M} (\partial q_\alpha/\partial p_n)(\partial q_\alpha^*/\partial p_m^*) = \sum_{\alpha=1}^{M} T_{\alpha n} T_{\alpha m}^*. \tag{10}$$

It follows that:

$$\mathrm{d}^2E = \sum_{\alpha=1}^{M}\sum_{n\cdot m=1}^{N} T_{\alpha n} dp_n T_{\alpha m}^* \mathrm{d}p_m^* = \sum_{\alpha=1}^{M}\left|\sum_{n=1}^{N} T_{\alpha n} dp_n\right|^2 > 0, \tag{11}$$

so that the extremum (5a, 6a) is actually a minimum (5b, 6b). In conclusion, when considering an active noise reduction system, using M monitoring microphones of outputs $q_\alpha(\omega)$, and N loudspeaker inputs $p_n(\omega)$, to minimize the residual noise $r_\alpha(\omega)$, with known transfer functions $T_{\alpha n}(\omega)$ between loudspeakers and microphones, the optimum input is (8a), in the sense that it minimizes (6a, b) the cost function (3) for all powers k, of which the Gauss least squares method corresponds to the case $k = 1$. The question may be raised of what is the advantage of using $k \neq 1$, i.e., a cost function other than the Gaussian, since the same minimum is obtained for any k. As an example consider a practical situation, in which a minimum is sought numerically, and it is shallow. The use of a larger exponent k may render the minimum more noticeable, by making it less shallow and thus easier to locate accurately by a numerical procedure.