Journal of Sound and Vibration (1999) **220**(1), 171–177
Article No. jsvi.1998.1897, available online at http://www.idealibrary.com on IDEAL

LETTERS TO THE EDITOR

THREE-DIMENSIONAL FREE VIBRATION OF THICK LAMINATED CYLINDRICAL SHELLS WITH CLAMPED EDGES

KEWEI DING AND LIMIN TANG

Department of Engineering Mechanics, Dalian University of Technology, Dalian 116023, *Liaoning, People's Republic of China*

(*Received* 7 *April* 1998, *and in final form* 13 *July* 1998)

1. INTRODUCTION

The dynamic analysis of cylindrical shells has attracted the attention of many researchers. For example, Prasad and Jain [1] investigated the free vibration problem of finite cylindrical shells, Chou and Achenbach [2] stuided the three-dimensional vibration problem of orthotropic cylinders, and Heyliger and Jilani [3] considered the free vibration of inhomogeneous elastic cylinders and spheres. With no initial assumptions regarding stress and deformation models, the three vibration problems of simply supported homogeneous isotropic, orthotropic, and laminated thick cylindrical shells were solved by Soldatos *et al*. [4–6]. In Soldatos' papers, the thick shells were divided into N fictitious subcylinders in order to simplify the variable coefficient differential equations into a set of simpler ones that was solved by using a method of successive approximations. Recently, based on the analysis developed in references [4–6], three-dimensional vibrations of laminated cylinders and cylindrical panels with a symmetric or an antisymmetric cross-ply lay-up and three-dimensional static, dynamic, thermoelastic and buckling analysis of homogeneous and laminated composite cylinders have been studied by Soldatos and Ye [7, 8]. A successive approximation approach was oulined by Soldatos [8] which is suitable for corresponding analyses of hollow cylinders having fixed edge boundaries. Ding *et al*. [9] gave the exact solution for axisymmetric vibration and buckling of laminated cylindrical shells having a simply supported edge boundary, by means of the state–space method.

To the authors' knowledge, the exact free vibration analysis for the quite thick cylindrical shell with clamped edges is so difficult that few references have been found. In this paper however, by introducing the Hellinger–Reissner variational principle, the mixed state Hamilton equations are presented. A three-dimensional solution is expressed for the free vibration problem of thick laminated closed cylindrical shells with two clamped edges by means of a transfer matrix and the successive approximation method [4]. Numerical results are obtained and compared with those of FEM calculated using SAP5.

0022–460X/99/060171 + 07 $30.00/0

2. MIXED STATE EQUATION

The transient Hellinger–Reissner variational principle can be shown to be of the form

$$\Pi^* = \iiint_{\Omega} \left\{ \frac{\partial u}{\partial r}\tau_{rx} + \frac{\partial v}{\partial r}\tau_{r\theta} + \frac{\partial w}{\partial r}\sigma_r - H \right\} r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}x - \iint_{S_u} [p_r(w - \bar{w})$$

$$+ p_\theta(v - \bar{v}) + p_x(u - \bar{u})] \, \mathrm{d}s - \iint_{S_\sigma} (\bar{p}_r w + \bar{p}_\theta v + \bar{p}_x u) \, \mathrm{d}s, \qquad (1)$$

in which the usual index notation is used. S_σ and S_u denote respectively the portion of the edge boundary where tractions $\bar{p}_i$ are prescribed and where displacements $\bar{u}_i$ are prescribed. The quadratic form of the Hamilton function H can be written as

$$-H = \sigma_x \frac{\partial u}{\partial x} + \sigma_\theta \left(\frac{w}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \right) + \tau_{x\theta} \left(\frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \theta} \right) + \tau_{r\theta} \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{v}{r} \right) + \tau_{rx} \frac{\partial w}{\partial x}$$

$$-\tfrac{1}{2}\rho \left(\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right) - \tfrac{1}{2}\{\sigma\}^T [C]^{-1} \{\sigma\}$$

where ρ is the mass density and the matrix $[C]$ is the elastic stiffness matrix.

Using $\delta\Pi^* = 0$, and denoting

$$q = (ru, rv, rw)^T, \qquad p = (\tau_{rx} \quad \tau_{r\theta} \quad \sigma_r), \qquad F = (q, p),$$

the following relations can be obtained

$$\frac{\partial q}{\partial r} = \frac{\partial H}{\partial p}, \qquad \frac{\partial p}{\partial r} = -\frac{\partial H}{\partial q}. \qquad (2)$$

This is the classical Hamilton canonical equation [10].

For a clamped shell, the length of the shell is l, the radius of the outer and, inner surfaces are a and b, respectively. We introduce

$$u = u_x + \left(1 - \frac{x}{l} \right) \bar{u}^{(0)}_{(\theta,r,t)} + \frac{x}{l} \bar{u}^{(l)}_{(\theta,r,t)}, \qquad (3)$$

in which $\bar{u}^{(0)}_{(\theta,r,t)}$ and $\bar{u}^{(l)}_{(\theta,r,t)}$ are unknown coefficients at $x = 0, l$, respectively.

Expand the quantities into the following series system ($\zeta = m\pi/l$)

$$u_x = \sum_m \sum_n u_{x,mn}(r) \cos \zeta x \cos (n\theta) \, \mathrm{e}^{i\omega_{mn} t},$$

$$\tau_{rx} = \sum_m \sum_n \tau_{rx_{,mn}}(r) \cos \zeta x \cos (n\theta) \, \mathrm{e}^{i\omega_{mn} t},$$

$$v = \sum_m \sum_n v_{mn}(r) \sin \zeta x \sin (n\theta) \, \mathrm{e}^{i\omega_{mn} t},$$

$$\tau_{r\theta} = \sum_m \sum_n \tau_{r\theta_{,mn}}(r) \sin \zeta x \sin (n\theta) \, \mathrm{e}^{i\omega_{mn} t}, \tag{4}$$

$$w = \sum_m \sum_n w_{mn}(r) \sin \zeta x \cos (n\theta) \, \mathrm{e}^{i\omega_{mn} t},$$

$$\sigma_r = \sum_m \sum_n \sigma_{r,mn}(r) \sin \zeta x \cos (n\theta) \, \mathrm{e}^{i\omega_{mn} t},$$

$$x\bar{u}^{(0)}_{(\theta,r,t)} = \left(\frac{l}{2} + \frac{2l}{\pi^2} \sum_{m=1} \frac{\cos m\pi - 1}{m^2} \cos \zeta x \right) \sum_n \bar{u}^{(0)}_{(r)} \sin (n\theta) \, \mathrm{e}^{i\omega_{nm} t},$$

$$x\bar{u}^{(l)}_{(\theta,r,t)} = \left(\frac{l}{2} + \frac{2l}{\pi^2} \sum_{m=1} \frac{\cos m\pi - 1}{m^2} \cos \zeta x \right) \sum_n \bar{u}^{(l)}_{(r)} \sin (n\theta) \, \mathrm{e}^{i\omega_{mn} t}. \tag{5}$$

Considering equations (3) and (4), it can be seen that on $x = 0, l$, $w = v = 0$. The remainding boundary conditions that must be satisfied are

$$u = 0, \qquad \text{on} \qquad x = 0, l. \tag{6}$$

Because σ_x, σ_θ and $\tau_{x\theta}$ are discontinuous variables at interfaces by first eliminating them from equations (2) then introducing equations (3), (4) and (5) into equation (2), and letting

$$C_1 = -C_{13}/C_{33}, \quad C_2 = C_{11} - C_{13}^2/C_{33}, \quad C_3 = C_{12} - C_{13}C_{23}/C_{33},$$

$$C_4 = C_{22} - C_{23}^2/C_{33}, \quad C_5 = -C_{23}/C_{33}, \quad C_6 = C_{66}, \quad C_7 = 1/C_{33},$$

$$C_8 = 1/C_{55}, \quad C_9 = 1/C_{44}$$

this yields for each combination of m and n:

$$\frac{\mathrm{d}}{\mathrm{d}r} \mathbf{F}_{mn}(r) = \mathbf{D}(r)\mathbf{F}_{mn}(r) + \mathbf{B}(r), \tag{7}$$

where

$$\mathbf{D}(r) = \begin{bmatrix} \mathbf{A}^T & \mathbf{M} \\ \mathbf{E} & -\mathbf{A} \end{bmatrix}, \tag{8}$$

$$\mathbf{A}^T = \begin{bmatrix} \dfrac{1}{r} & 0 & -\zeta \\ 0 & \dfrac{2}{r} & \dfrac{n}{r} \\ -C_1\zeta & \dfrac{C_5 n}{r} & \dfrac{C_5+1}{r} \end{bmatrix}, \qquad -\mathbf{A} = \begin{bmatrix} -\dfrac{1}{r} & 0 & C_1\zeta \\ 0 & -\dfrac{2}{r} & -\dfrac{C_5 n}{r} \\ \zeta & -\dfrac{n}{r} & -\dfrac{C_5+1}{r} \end{bmatrix},$$

$$\mathbf{M} = \mathbf{M}^T = \begin{bmatrix} C_8 r & 0 & 0 \\ 0 & C_9 r & 0 \\ 0 & 0 & C_7 r \end{bmatrix},$$

$$\mathbf{E} = \mathbf{E}^T = \begin{bmatrix} \dfrac{C_2}{r}\zeta^2 + \dfrac{C_6}{r^3}n^2 - \dfrac{\rho\omega^2}{r} & -\dfrac{C_3+C_6}{r^2}\zeta n & -\dfrac{C_3}{r^2}\zeta \\ -\dfrac{C_3+C_6}{r^2}\zeta n & \dfrac{C_6}{r}\zeta^2 + \dfrac{C_4}{r^3}n^2 - \dfrac{\rho\omega^2}{r} & \dfrac{C_4}{r^3}n \\ -\dfrac{C_3}{r^2}\zeta & \dfrac{C_4}{r^3}n & \dfrac{C_4}{r^3} - \dfrac{\rho\omega^2}{r} \end{bmatrix},$$

$$\mathbf{B}(r) = \left[\frac{r}{m^2\pi^2}\frac{\mathrm{d}A}{\mathrm{d}r} \quad 0 \quad \frac{C_1 A r}{m\pi l} \quad -\left(\frac{C_2}{l^2} + \frac{C_6 n^2}{m^2\pi^2 r^2}\right)A \quad \frac{(C_3+C_6)nA}{m\pi r l} \quad \frac{C_3 A}{m\pi l r} \right]^T,$$

$$\omega = \omega_{mn}, \quad A = -2(\cos m\pi - 1)(\bar{u}^{(l)}_{(r)} - \bar{u}^{(0)}_{(r)}).$$

Equation (7) is called a variable coefficient non-homogenous mixed state equation.

3. ANALYTICAL SOLUTION OF MIXED STATE EQUATION

Considering a p-plied laminated thick cylindrical shell made up of orthotropic layers. The length and the thickness of the shell are l and h $(=a-b)$ respectively. Divide the laminated shell into k thin plies. The thickness of each thin ply is $h_i = h/k$, and its middle radius is denoted by $c_1, c_2, \ldots, c_k$, respectively.

Because r has a little variation in the thin ply, one can substitute c_i $(i = 1, 2, \ldots, k)$ for the variable r in matrix (8), which will not result in significant errors. If each layer of the laminated shell is quite thin, one only needs to substitute the middle radius of each layer for r in the matrix (8). However, when some layers are thick, one may divide the layers into k and $k+1$ thin plies. If we find from calculation that the needful effective digits hardly change, it can be said that the results obtained with k thin plies are exact within the prescribed accuracy limits.

Provided the ply is thin enough, it is reasonable that the $\bar{u}^{(0)}_{(r)}$ and $\bar{u}^{(l)}_{(r)}$ are considered as linear functions in the thin ply. (Babuska *et al.* [11] gave a much more efficient method in the thickness direction.)

The solution of equation (7) is

$$\mathbf{F}(r) = \mathbf{G}(r-a)\mathbf{F}(a) + \mathbf{C}(r-a), \tag{9}$$

where

$$\mathbf{G}(r-a) = \exp[\mathbf{D}(r-a)], \qquad \mathbf{C}(r-a) = \int_a^r \exp[\mathbf{D}(r-\tau)\mathbf{B}(\tau)]\,\mathrm{d}\tau. \tag{10}$$

By virtue of the continuity conditions for displacements and stresses at interfaces, the mechanical quantities of the inner and outer surfaces for the entire laminated shell can be linked together to be of the form

$$\mathbf{F}(b) = \Pi\mathbf{F}(a) + \bar{\Pi}, \tag{11}$$

where

$$\Pi = \mathbf{G}(-h_k)\mathbf{G}(-h_{k-1})\mathbf{G}(-h_{k-2})\cdots\mathbf{G}(-h_2)\mathbf{G}(-h_1),$$

$$\bar{\Pi} = \mathbf{G}(-h_k)\sum_{j=1}^{k-1}\left[\prod_{i=k-1}^{j+1}\mathbf{G}(-h_i)\mathbf{C}(-h_j)\right] + \mathbf{C}(-h_k).$$

$\mathbf{F}(b)$ and $\mathbf{F}(a)$ in equation (11) are the mechanical quantities for the interior and outer surfaces of the laminated shell, respectively. In the calculation of natural frequencies, considering the boundary condition of the inner and outer surfaces of the shell, one has

$$\sigma_{r,mn}(a) = \tau_{rx,mn}(a) = \tau_{r\theta,mn}(a) = \sigma_{r,mn}(b) = \tau_{rx,mn}(b) = \tau_{r\theta,mn}(b) = 0. \tag{12}$$

In order to satisfy the boundary condition (6) of the shell with clamped edges, it is necessary to write the expression of $u_{x,mn}(r)$. In imitation of the deductive process of equation (11), the mechanical quantities in the ith thin ply of laminated shell can be expressed by $\mathbf{F}(a)$:

$$\mathbf{F}(r_i) = \Pi_i\mathbf{F}(a) + \bar{\Pi}_i. \tag{13}$$

The meaning of the Π_i and $\bar{\Pi}_i$ are similar to those in equation (11).

By selecting the first rows of equation (13), and considering boundary conditions (6), for each m and n, and letting $r = r_i$ in condition (6), then four equations about unknown coefficients are obtained. When $i = 1, 2, \ldots, k$, there are altogether $2(k+1)$ linear homogeneous algebraic equations. It makes coefficients determinant of those equations equal to zero, thus the frequency equation can be obtained.

TABLE 1

Frequency parameters Ω for single-ply and three-plied laminated shells

h/R_0	SAP5 Ω_1	Present (single-ply) Ω_1	Ω_2	Ω_3	SAP5 Ω_1	Present (three-plied) Ω_1	Ω_2	Ω_3
0·2	0·0435	0·0459	0·2075	1·6447	–	–	–	–
0·4	0·0896	0·0918	0·4076	1·7130	–	–	–	–
0·8	–	–	–	–	0·1975	0·2081	0·8034	1·6402
1·0	–	–	–	–	0·2559	0·2601	0·9272	1·7381

$\Omega = \omega h\sqrt{\rho_2/C_{11}^{(2)}}$.

4. NUMERICAL EXAMPLE

Example. A three-plied closed laminated cylindrical shell is used. The materials of the first and third layers are identical. Each layer has the same elastic constants:

$$C_{12}/C_{11} = 0{\cdot}246269, \qquad C_{13}/C_{11} = 0{\cdot}0831715, \qquad C_{22}/C_{11} = 0{\cdot}543103,$$

$$C_{23}/C_{11} = 0{\cdot}115017, \qquad C_{33}/C_{11} = 0{\cdot}530172, \qquad C_{44}/C_{11} = 0{\cdot}266810,$$

$$C_{55}/C_{11} = 0{\cdot}159914, \qquad C_{66}/C_{11} = 0{\cdot}262931, \qquad C_{11}^{(1)}/C_{11}^{(2)} = 5,$$

where $C_{11}^{(1)}$ and $C_{11}^{(2)}$ denote C_{11} of the materials corresponding to the first and second layer, respectively. The densities for the outer and middle layers are denoted by ρ_1 and ρ_2, respectively. The laminated shell has the following geometry parameters:

$$h_1 = h_3 = 0{\cdot}1h, \qquad h_2 = 0{\cdot}8h, \qquad l = s = 2\pi R_0,$$

where l = the length of the shell, s = the arc length of the middle surface, R_0 = the radius of the middle surface, h_1, h_2 and h_3 are the thicknesses of the outer, middle and interior layers, resepctively.

When $m = n = 1$, the first three natural frequencies for the single-ply shell and the three-plied shell ($\rho_1/\rho_2 = 3$) are indicated in Table 1. The results for the three-dimensional FEM using SAP5 with 64 isoparametric elements (for 1/4 shell) with 16 nodes are also given in Table 1.

5. CONCLUSION

A three-dimensional solution for the free vibration problem is investigated. Exact frequencies are also given for thick laminated closed cylindrical shells with two clamped edges. The principle and method suggested here have clear physical concepts and can also handle more general boundary conditions. The present study satisfies the continuity conditions of stresses and displacements at the interfaces. Numerical results denote that the method adopted in this paper is an efficient one.

ACKNOWLEDGMENT

This project has been supported by the National Natural Science Foundation of China (Grant No. 19392305). The authors would like to express their thanks to both referees for their valuable comments.

REFERENCES

1. C. PRASAD and R. K. JAIN 1965 *Journal of the Acoustical Society of America* **38**, 1006–1009. Vibration of transversely isotropic cylindrical shells of finite length.
2. F. H. CHOU and J. D. ACHENBACH 1972 *Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineering* **98**, 813–822. Three-dimensional vibrations of orthotropic cylinders.
3. P. R. HEYLIGER and A. JILANI 1992 *International Journal of Solids and Structures* **29**, 2689–2708. The free vibrations of inhomogeneous elastic cylinders and spheres.
4. K. P. SOLDATOS and V. P. HADJIGEORGIOU 1990 *Journal of Sound and Vibration* **137**, 369–384. Three-dimensional solution of the free vibration problem of homogeneous isotropic cylindrical shells and panels.
5. K. P. SOLDATOS 1991 *Journal of Sound and Vibration* **151**, 149–152. An iterative solution of a Bessel equation based on vibration of orthotropic hollow cylinders.
6. T. D. HAWKES and K. P. SOLDATOS 1992 *American Institute of Aeronautics and Astronautics Journal* **30**, 1089–1098. Three-dimensional axisymmetric vibrations of orthotropic and cross-ply laminated hollow cylinders.
7. J. Q. YE and K. P. SOLDATOS 1994 *Composite Engineering* **4**, 429–444. Three-dimensional vibrations of laminated cylinders and cylindrical panels with a symmetric or an antisymmetric cross-ply lay-up.
8. K. P. SOLDATOS and J. Q. YE 1994 *Composite Structures* **29**, 131–143. Three-dimensional static, dynamic, thermoelastic and buckling analysis of homogeneous and laminated composite cylinders.
9. K. W. DING, L. M. TANG and J. R. FAN 1997 *Journal of Sound and Vibration* **206**, 435–441. Exact analysis for axisymmetric vibration and buckling of the thick laminated closed cylindrical shells in a Hamilton system.
10. L. M. TANG 1991 *Computational Structural Mechanics and Applications* (in Chinese) **8**, 343–350. Mixed formulation and Hamilton canonical equation of theory of elasticity.
11. I. BABUSKA, B. A. SZABO and R. L. ASTIS 1992 *International Journal for Numerical Methods in Engineering* **33**, 503–535. Hierarchic moduls for laminated composites.