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FORCED VIBRATIONS OF SIMPLY SUPPORTED ANISOTROPIC RECTANGULAR PLATES

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1. INTRODUCTION

There is a limited amount of information on forced vibrations of isotropic plates or slabs [1–4]. The situation is more critical in the case of anisotropic plates [5].

The present study deals with the determination of the transverse, dynamic response of simply supported, rectangular plates of generalized anisotropy, subjected to a uniformly distributed $p_0 \cos \omega t$-type excitation.

The governing functional is formulated using Lekhnitskii's approach and notation [5] and the problem is solved using the Rayleigh–Ritz method by means of a double Fourier series which constitutes the exact solution of the mathematical system when the plate is isotropic or orthotropic.*

2. APPROXIMATE SOLUTION

In the case of forced vibrations the problem under investigation is governed by the functional.

$$J=\frac{1}{2}\int_{A_p}\left\{D_{11}\left(\frac{\partial^2 W'}{\partial x'^2}\right)^2+2D_{12}\frac{\partial^2 W'}{\partial x'^2}\frac{\partial^2 W'}{\partial y'^2}+D_{22}\left(\frac{\partial^2 W'}{\partial y'^2}\right)^2+4\left(\frac{\partial^2 W'}{\partial x'\,\partial y'}\right)\right.$$
$$\left.\times\left[D_{16}\left(\frac{\partial^2 W'}{\partial x'^2}\right)+D_{26}\left(\frac{\partial^2 W'}{\partial y'^2}\right)\right]+4D_{66}\left(\frac{\partial^2 W'}{\partial x'\,\partial y'}\right)^2\right\}\mathrm{d}x'\,\mathrm{d}y'$$
$$-\frac{\rho h\omega^2}{2}\int_{A_p}W'^2\,\mathrm{d}x'\,\mathrm{d}y'-p_0\int_{A_p}W'\,\mathrm{d}x'\,\mathrm{d}y', \qquad (1)$$

* Obviously: when the orthotropic material characteristics are parallel to the plate edges.

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where W' is the true displacement amplitude of the plate and A_p is the total area of the plate plan form. In equation (1), the D_{ij} are the well known flexural rigidities of the (anisotropic) plate, which, for an isotropic plate, take the simplest form

$$D_{11} = D_{22} = \frac{Eh^3}{12(1-\nu^2)}, \qquad D_{12} = \nu D_{11}, \qquad D_{66} = \frac{(1-\nu)}{2} D_{11}, \qquad D_{16} = D_{26} = 0, \tag{2}$$

If the length of the sides of the rectangular plate are a and b in the x and y directions, respectively, equation (1) can be cast in a non-dimensional form by introducing

$$W = W'/a. \qquad x = x'/a, \qquad y = y'/b. \tag{3}$$

One gets, for the functional for the whole system of Figure 1

$$J_{nd} = \frac{2J}{rD_{11}} = \int_{A_p} \left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \frac{2}{r^2}\frac{D_{12}}{D_{11}}\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + \frac{1}{r^4}\frac{D_{22}}{D_{11}}\left(\frac{\partial^2 W}{\partial y^2}\right)^2 + \frac{4}{r}\left(\frac{\partial^2 W}{\partial x\,\partial y}\right) + \left(\frac{D_{16}}{D_{11}}\frac{\partial^2 W}{\partial x^2} + \frac{1}{r^2}\frac{D_{26}}{D_{11}}\frac{\partial^2 W}{\partial y^2}\right) + \frac{4}{r^2}\frac{D_{26}}{D_{11}}\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2 \right] \mathrm{d}x\,\mathrm{d}y$$

$$- \Omega^2 \int_{A_p} W^2\,\mathrm{d}x\,\mathrm{d}y - 2P_0 \int_{A_p} W\,\mathrm{d}x\,\mathrm{d}y, \tag{4}$$

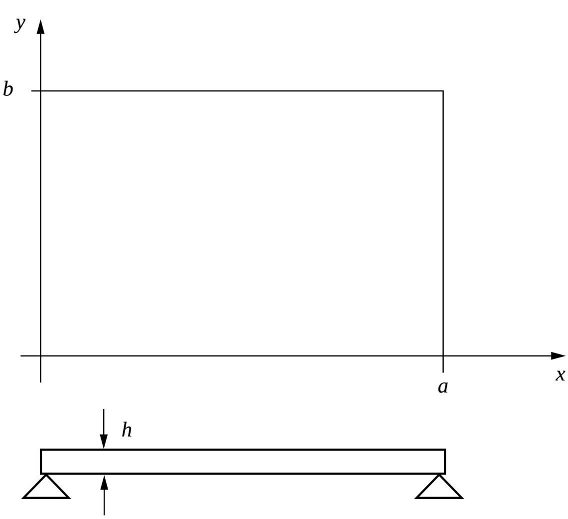

Figure 1. Vibrating system under consideration.

where, as usual,

$$\Omega^2 = \frac{\rho h \omega^2 a^4}{D_{11}} \tag{5}$$

is the non-dimensional frequency coefficient;

$$P_0 = \frac{p_0 a^3}{D_{11}} \tag{6}$$

is the non-dimensional load and

$$r = \frac{b}{a} \tag{7}$$

is the aspect ratio of the plate.

The plate response is now expressed in the classical (non-dimensional) form by means of a double Fourier series:

$$W_a(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} b_{mn} \sin(m\pi x) \sin(n\pi y). \tag{8}$$

One gets as a final expression for the functional the sum

$$\begin{aligned} J_{nd} = \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{l=1}^{N} \sum_{k=1}^{M} b_{kl} b_{mn} \Bigg\{ \pi^4 \left[m^2 k^2 + 2 \frac{D_{12}}{D_{11}} \frac{k^2 n^2}{r^2} + \frac{D_{22}}{D_{11}} \frac{n^2 l^2}{r^4} \right] A_{SS} \\ - 4\pi^4 \left[\frac{k^2 mn}{r} \frac{D_{16}}{D_{11}} + \frac{l^2 mn}{r^3} \frac{D_{26}}{D_{11}} \right] A_{SC} + \pi^4 \left[\frac{4 D_{66}}{D_{11}} \frac{klmn}{r^2} \right] A_{CC} - \Omega^2 A_{SS} \Bigg\} \\ - 2 P_0 \frac{4}{\pi^2} \sum_{n=0}^{(N-1)/2} \sum_{m=0}^{(M-1)/2} b_{(2n+1)(2m+1)} / [(2m+1)(2n+1)], \end{aligned} \tag{9}$$

where

$$A_{SS} = \int_{A_p} \sin(k\pi x) \sin(l\pi y) \sin(m\pi x) \sin(n\pi y)\, \mathrm{d}x\, \mathrm{d}y, \tag{10}$$

$$A_{SC} = \int_{A_p} \sin(k\pi x) \cos(m\pi x) \sin(l\pi y) \cos(n\pi y)\, \mathrm{d}x\, \mathrm{d}y, \tag{11}$$

$$A_{CC} = \int_{A_p} \cos(k\pi x) \cos(l\pi y) \cos(m\pi x) \cos(n\pi y)\, \mathrm{d}x\, \mathrm{d}y, \tag{12}$$

and (as shown) the last double sum in equation (9) must be taken only for odd values of the sub-indexes of b_{kl}.

When dealing with the situation of forced vibrations, using the minimization condition in expression (9),

$$\frac{\partial J}{\partial b_{mn}} = 0, \quad m, n = 1, \ldots, M, N, \tag{13}$$

one gets a linear system of non-homogeneous equations in the b_{mn}'s.

Once these are determined one is able to calculate approximate values of the displacement amplitude

$$W_a(x, y) \Big/ \left(\frac{p_0 a^4}{D_{11}}\right) = \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{b_{mn}}{P_0} \sin(m\pi x) \sin(n\pi y). \tag{14}$$

and of the stress resultants. In the case of bending moments, their amplitudes are given by the expressions

$$\frac{M_x}{p_0 a^2} = \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{b_{mn}}{P_0} \left\{ \sin(m\pi x) \sin(n\pi y) \left[m^2\pi^2 + \frac{D_{12}}{D_{11}} \frac{n^2\pi^2}{r^2} \right] - \cos(m\pi x) \cos(n\pi y) \left[2 \frac{D_{16}}{D_{11}} \frac{mn\pi^2}{r} \right] \right\}, \tag{15}$$

$$\frac{M_y}{p_0 a^2} = \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{b_{mn}}{P_0} \left\{ \sin(m\pi x) \sin(n\pi y) \left[\frac{D_{12}}{D_{11}} m^2\pi^2 + \frac{D_{22}}{D_{11}} \frac{n^2\pi^2}{r^2} \right] - \cos(m\pi x) \cos(n\pi y) \left[2 \frac{D_{26}}{D_{11}} \frac{mn\pi^2}{r} \right] \right\}. \tag{16}$$

3. NUMERICAL RESULTS

All calculations were performed taking $D_{12}/D_{11} = 0{\cdot}3$; $D_{22}/D_{11} = 0{\cdot}5 = D_{66}/D_{11}$; and $D_{16}/D_{11} = 1/3 = D_{26}/D_{11}$.

TABLE 1

Non-dimensional displacement and bending moments amplitudes at the center of an anisotropic rectangular plate of uniform thickness and aspect ratio $b/a = 1/2$. The first two frequency coefficients for free vibrations of the plate are $\Omega_1 = 39{\cdot}3618$; $\Omega_2 = 68{\cdot}5757$

Ω/Ω_1	W	M_x	M_y
0·0	0·0010108	0·018363	0·020055
0·1	0·0010214	0·018579	0·020278
0·2	0·0010546	0·019254	0·020973
0·3	0·0011149	0·020480	0·022235
0·4	0·0012113	0·022446	0·024255
0·5	0·0013618	0·025520	0·027410
0·6	0·0016034	0·030466	0·032477
0·7	0·0020237	0·039087	0·041295
0·8	0·0028866	0·056826	0·059407
0·9	0·0055137	0·110933	0·114574
$(\Omega - \Omega_1)/(\Omega_2 - \Omega_1)$			
0·1	0·0069186	0·145563	0·146589
0·2	0·0033693	0·072447	0·072052
0·3	0·0021922	0·048256	0·047346
0·4	0·0016081	0·036302	0·035095
0·5	0·0012611	0·029247	0·027826
0·6	0·0010325	0·024648	0·023047
0·7	0·0008716	0·021459	0·019693
0·8	0·0007530	0·019160	0·017230
0·9	0·0006627	0·017465	0·015365

TABLE 2

Non-dimensional displacement and bending moments amplitudes at the center of an anisotropic rectangular plate of uniform thickness and aspect ratio $b/a = 2/3$. The first two frequency coefficients for free vibrations of the plate are $\Omega_1 = 27{\cdot}2446$; $\Omega_2 = 54{\cdot}3345$

Ω/Ω_1	W	M_x	M_y
0·0	0·0021259	0·029759	0·024805
0·1	0·0021480	0·030102	0·025083
0·2	0·0022173	0·031176	0·025949
0·3	0·0023429	0·033124	0·027521
0·4	0·0025440	0·036244	0·030037
0·5	0·0028577	0·041117	0·033966
0·6	0·0033613	0·048949	0·040274
0·7	0·0042367	0·062582	0·051248
0·8	0·0060334	0·090594	0·073780
0·9	0·0115020	0·175939	0·142391
$(\Omega - \Omega_1)/(\Omega_2 - \Omega_1)$			
0·1	0·0106163	0·169593	0·135226
0·2	0·0051131	0·083734	0·066189
0·3	0·0032934	0·055407	0·043381
0·4	0·0023937	0·041459	0·032122
0·5	0·0018614	0·033260	0·025476
0·6	0·0015123	0·027939	0·021135
0·7	0·0012677	0·024266	0·018109
0·8	0·0010882	0·021632	0·015906
0·9	0·0009522	0·01970	0·014253

Table 3

Non-dimensional displacement and bending moments amplitudes at the center of an anisotropic square plate of uniform thickness $b/a = 1$. The first two frequency coefficients for free vibrations of the plate are $\Omega_1 = 18{\cdot}2417$; $\Omega_2 = 36{\cdot}8608$

Ω/Ω_1	W	M_x	M_y
0·0	0·0047255	0·053171	0·029462
0·1	0·0047747	0·053771	0·029799
0·2	0·0049286	0·055645	0·030852
0·3	0·0052078	0·059046	0·032762
0·4	0·0056544	0·064491	0·035822
0·5	0·0063515	0·072994	0·040600
0·6	0·0074702	0·086648	0·048276
0·7	0·0094151	0·110406	0·061635
0·8	0·0134062	0·159194	0·089075
0·9	0·0255519	0·307750	0·172645
$(\Omega - \Omega_1)/(\Omega_2 - \Omega_1)$			
0·1	0·0229441	0·285757	0·161305
0·2	0·0110410	0·140197	0·079426
0·3	0·0071058	0·092139	0·052407
0·4	0·0051609	0·068446	0·039098
0·5	0·0040108	0·054488	0·031271
0·6	0·0032570	0·045396	0·026185
0·7	0·0027293	0·039086	0·022667
0·8	0·0023425	0·034520	0·020136
0·9	0·0020496	0·031125	0·018271

Table 4

Non-dimensional displacement and bending moments amplitudes at the center of an anisotropic rectangular plate of uniform thickness and aspect ratio $b/a = 3/2$. The first two frequency coefficients for free vibrations of the plate are $\Omega_1 = 13{\cdot}8667$; $\Omega_2 = 23{\cdot}4429$

Ω/Ω_1	W	M_x	M_y
0·0	0·0080452	0·082097	0·032965
0·1	0·0081300	0·083014	0·033349
0·2	0·0083949	0·085882	0·034550
0·3	0·0088754	0·091086	0·036731
0·4	0·0096447	0·099421	0·040228
0·5	0·0108459	0·112443	0·045698
0·6	0·0127746	0·133368	0·054499
0·7	0·0161302	0·169800	0·069844
0·8	0·0230207	0·244666	0·101424
0·9	0·0440044	0·472800	0·197774
$(\Omega - \Omega_1)/(\Omega_2 - \Omega_1)$			
0·1	0·0595466	0·653648	0·278519
0·2	0·0290742	0·322315	0·138554
0·3	0·0189634	0·212452	0·092210
0·4	0·0139432	0·157969	0·069285
0·5	0·0109587	0·125642	0·055740
0·6	0·0089922	0·104402	0·046896
0·7	0·0076075	0·089509	0·040754
0·8	0·0065871	0·078601	0·036318
0·9	0·0058106	0·070372	0·033041

TABLE 5

Non-dimensional displacement and bending moments amplitudes at the center of an anisotropic rectangular plate of uniform thickness and aspect ratio $b/a = 2$. The first two frequency coefficients for free vibrations of the plate are $\Omega_1 = 12{\cdot}1997$; $\Omega_2 = 18{\cdot}0591$

Ω/Ω_1	W	M_x	M_y
0·0	0·0101820	0·100760	0·035141
0·1	0·0102906	0·101887	0·035551
0·2	0·0106301	0·105411	0·036834
0·3	0·0112463	0·111808	0·039166
0·4	0·0122335	0·122064	0·042910
0·5	0·0137763	0·138102	0·048777
0·6	0·0162565	0·163907	0·058237
0·7	0·0205773	0·208902	0·074775
0·8	0·0294627	0·301515	0·108903
0·9	0·0565633	0·584215	0·213321
$(\Omega - \Omega_1)/(\Omega_2 - \Omega_1)$			
0·1	0·1116861	1·172130	0·436745
0·2	0·0551243	0·581901	0·218522
0·3	0·0363122	0·385686	0·146076
0·4	0·0269493	0·288108	0·110138
0·5	0·0213693	0·230030	0·088831
0·6	0·0176829	0·191736	0·074867
0·7	0·0150811	0·164784	0·065125
0·8	0·0131598	0·144962	0·058052
0·9	0·0116952	0·129940	0·052793

TABLE 6

Comparison of values of displacement amplitudes and bending moments for a square anisotropic plate of uniform thickness as the number of terms in the Fourier Series approximating function are increased from $M = N = 10$ (100 terms) to $M = N = 30$ (900 terms)

Ω/Ω_1	Number of terms	W	M_x	M_y
0·0	100	0·00465656	0·0502690	0·0270319
	400	0·00472553	0·0531713	0·0294623
	900	0·00475115	0·0521469	0·0283676
0·3	100	0·00512418	0·0557287	0·0300198
	400	0·00520780	0·0590463	0·0327625
	900	0·00523907	0·0579400	0·0315688
0·5	100	0·00622778	0·0686270	0·0370819
	400	0·00635157	0·0729943	0·0406007
	900	0·00639849	0·0717203	0·0391866
0·7	100	0·00914662	0·1027888	0·0557987
	400	0·00941514	0·1104069	0·0616354
	900	0·00951965	0·1088684	0·0597339
0·9	100	0·02368048	0·2731322	0·1491894
	400	0·02555199	0·3077502	0·1726456
	900	0·02634536	0·3094042	0·1707183

Tables 1 to 5 depict non-dimensional displacement amplitudes and bending moments at the center of anisotropic simply supported rectangular plates of uniform thickness, for several values of the aspect ratios, taking $M = N = 20$. The calculations were performed using in all cases 80 bit floating point variables (IEEE-standard temporary reals) in order to ensure accurate results.

Table 6, on the other hand, shows displacement amplitudes and bending moments at the center of an anisotropic square plate as the number of terms is varied from $M = N = 10$ to $M = N = 30$. As this table seems to show, one gets results within 1–2% of real values when $M = N = 20$.

The present approach can be extended in a straightforward fashion to the case of plates of non-uniform thickness, presence of orifices, etc.

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