Journal of Sound and Vibration (1999) **220**(1), 186–191
Article No. jsvi.1998.1909, available online at http://www.idealibrary.com on IDEAL

TRANSVERSE VIBRATION OF A UNIFORM CIRCULAR THICK BEAM WITH NON-CLASSICAL BOUNDARY CONDITIONS

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(*Received* 16 *September* 1996, *and in final form* 14 *August* 1998)

1. INTRODUCTION

This paper presents exact solutions of frequency equations and mode shapes, and also investigates the effect of axial loads on the natural frequencies of such a beam.

Bokaian [1, 2] studied uniform Bernoulli–Euler type beams subjected to compressive/tensile axial loads. In his work, the effect of axial loading on the natural frequencies and mode shapes of a uniform single-span beam with ten different end conditions was demonstrated. Abbas [3] solved, for the first time, the problem of free vibration of Timoshenko beams with elastically supported ends. The analysis, incorporated a unique finite element model developed by Thomas and Abbas [4]. This model satisfies all geometric and natural boundary conditions of an elastically restrained Timoshenko beam. Abbas presented natural frequencies of free vibrations of Timoshenko beams for six idealized end conditions. Maurizi *et al*. [5] considered free vibration of a uniform Timoshenko beam subjected to translational and rotational restraints at both ends. Recently, Abramovich [6] investigated the influence of compressive axial loads on the natural frequencies of Timoshenko type beams having various classical boundary conditions. To the authors' knowledge, no work has yet been seen for a uniform circular Timoshenko beam subjected to axial loads with non-classical boundary conditions.

2. EQUATIONS OF MOTION

The configuration of a uniform circular beam with symmetrical translational and rotational springs is shown in Figure 1. An inertial Cartesian frame $OXYZ$, with origin O at the left end of the beam is adopted. The Z-axis coincides with the beam center line when it is in its undeformed static equilibrium position. The Y-axis lies in the direction of the gravitational forces. Let u_x and u_y represent the two transverse displacements of the shaft along the X and Y directions, respectively, and ψ_x and ψ_y represent the angles of rotation due to bending in the two perpendicular planes of motion XZ and YZ, respectively. The linear deflection and angular rotation are assumed small.

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Considering motion in the YZ plane, the coupled non-dimensionalised equations of motion for the total deflection, u_y, and rotation, ψ_y, of a Timoshenko beam subjected to axial load, P may be written as

$$\frac{\partial^2 u_y}{\partial t^2} + \frac{\kappa G}{\rho l^2}\left(l\frac{\partial \psi_y}{\partial \zeta} - \frac{\partial^2 u_y}{\partial \zeta^2}\right) - \left(\frac{P}{\rho A l^2}\right)\frac{\partial^2 u_y}{\partial \zeta^2} = 0, \quad (1)$$

$$\frac{\partial^2 \psi_y}{\partial t^2} - \frac{E}{\rho l^2}\frac{\partial^2 \psi_y}{\partial \zeta^2} + \frac{\kappa A G}{\rho I l}\left(l\psi_y - \frac{\partial u_y}{\partial \zeta}\right) = 0, \quad (2)$$

where l is the length of the beam, ρ is the mass density, A is the cross-sectional area, I is the diametral second moment of area, E is the Young's modulus, G is the shear modulus, κ is the shear coefficient, and $\zeta = z/l$ is a non-dimensional space variable. The equations of motion for the XZ plane are similar except the subscript y is replaced by subscript x.

The boundary conditions at $\zeta = 0, 1$ in the YZ plane are

$$\kappa AG\left(\frac{1}{l}u_y'(0,t) - \psi_y(0,t)\right) = K_T u_y(0,t), \qquad EI\psi_y'(0,t) = -K_R\psi_y(0,t), \quad (3, 4)$$

$$\kappa AG\left(\frac{1}{l}u_y'(1,t) - \psi_y(1,t)\right) = -K_T u_y(1,t), \qquad EI\psi_y'(1,t) = K_R\psi_y(1,t), \quad (5, 6)$$

where K_T and K_R are the translational and rotational stiffness coefficients of the springs, respectively. Note that symmetrical translational and rotational stiffness

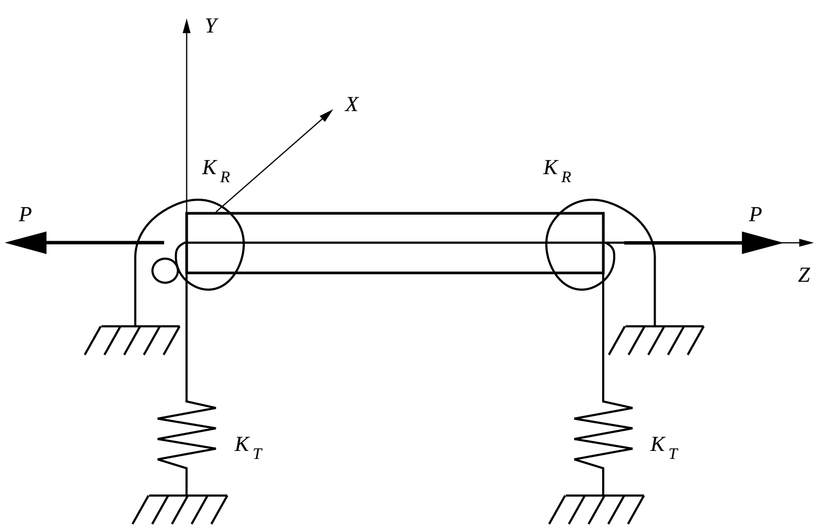

Figure 1. A uniform circular beam.

coefficients are assumed at either end. The notation ($'$) and ($\dot{}$) imply differentiation with respect to spatial and time variables, respectively.

3. FREQUENCY EQUATIONS AND MODE SHAPES

The following separable solutions of equation (1a) and (2a) are assumed:

$$u_y(\zeta, t) = U(\zeta)\, e^{i\omega t}, \qquad \psi_y(\zeta, t) = \Psi(\zeta)\, e^{i\omega t}. \tag{7, 8}$$

The coupled second order partial differential equations (1) and (2) may be expressed as two fourth order ordinary differential equations:

$$l_2 U^{(4)} + l_1 U^{(2)} + l_0 U = 0, \qquad l_2 \Psi^{(4)} + l_1 \Psi^{(2)} + l_0 \Psi = 0, \tag{9, 10}$$

where

$$l_0 = \frac{\rho I}{\kappa AG}\omega^4 - \omega^2, \qquad l_1 = \frac{I}{l^2 A}\left(1 + \frac{E}{\kappa G}\right)\omega^2 + \frac{P}{l^2 A}\left(\frac{I}{\kappa AG}\omega^2 - \frac{1}{\rho}\right),$$

$$l_2 = \frac{EI}{\rho A l^4}\left(1 + \frac{P}{\kappa AG}\right).$$

Solving equations (9) and (10), the normal modes for U and Ψ when $[l_1^2 - 4l_2 l_0]^{1/2} > l_1$ are obtained as

$$U(\zeta) = A_1 \cosh(s_1\zeta) + A_2 \sinh(s_1\zeta) + A_3 \cos(s_2\zeta) + A_4 \sin(s_2\zeta), \tag{11}$$

$$\Psi(\zeta) = A_1' \sinh(s_1\zeta) + A_2' \cosh(s_1\zeta) + A_3' \sin(s_2\zeta) + A_4' \cos(s_2\zeta), \tag{12}$$

in which

$$s_1 = \left[\frac{-l_1 + (l_1^2 - 4l_2 l_0)^{1/2}}{2l_2}\right]^{1/2}, \qquad s_2 = \left[\frac{l_1 + (l_1^2 - 4l_2 l_0)^{1/2}}{2l_2}\right]^{1/2}. \tag{13}$$

A_1 to A_4 and A_1' to A_4' are arbitrary constants, and only four are independent. Their relationship can be found by substituting equations (11) and (12) into equation (1). Similarly, the mode shape functions can be derived for the condition when $(l_1^2 - 4l_2 l_0)^{1/2} < l_1$.

Applying boundary conditions in equations (3)–(6) to equations (11) and (12), the frequency equation is hence obtained by expanding the determinant

$$\begin{vmatrix} R_1 & -R_2 & R_1 & -R_4 \\ R_5 & R_6 & R_7 & -R_8 \\ (R_2 Sh(s_1) + R_1 Ch(s_1)) & (R_2 Ch(s_1) + R_1 Sh(s_1)) & (R_1 Cs(s_2) - R_4 Sn(s_2)) & (R_4 Cs(s_2) + R_1 Sn(s_2)) \\ (R_5 Ch(s_1) - R_6 Sh(s_1)) & (R_5 Sh(s_1) - R_6 Ch(s_1)) & (R_7 Cs(s_2) - R_8 Sn(s_2)) & (R_7 Sn(s_2) + R_8 Cs(s_2)) \end{vmatrix} = 0, \tag{14}$$

where

$$R_1 = \frac{K_T}{\kappa AG}, \qquad R_2 = \left(\frac{s_1}{l} - c_1\right), \qquad R_3 = \left(\frac{s_2}{l} - c_2\right), \qquad R_4 = \left(\frac{s_2}{l} + c_2\right),$$

$$R_5 = c_1 s_1, \qquad R_6 = \frac{K_R c_1}{EI}, \qquad R_7 = c_2 s_2, \qquad R_8 = \frac{K_R c_2}{EI}.$$

Ch and *Sh* represent the hyperbolic cosine and sine functions, respectively, and *Cs* and *Sn* represent cosine and sine functions.

The mode shapes can be obtained as

$$U(\zeta) = D_1(m_1 \cosh(s_1\zeta) + m_2 \sinh(s_1\zeta) + m_3 \cos(s_2\zeta) + \sin(s_2\zeta)), \tag{15}$$

$$\Psi(\zeta) = D_2(m_1 c_1 \sinh(s_1\zeta) + m_2 c_1 \cosh(s_1\zeta) + m_3 c_2 \sin(s_2\zeta) - c_2 \cos(s_2\zeta)), \tag{16}$$

where

$$m_1 = \frac{(V_2W_3 - V_3W_2)}{(V_1W_2 - V_2W_1)}, \qquad m_2 = \left(\frac{R_1}{R_2}\right)\frac{(V_2W_3 - V_3W_2 + V_3W_1 - V_1W_2)}{(V_1W_2 - V_2W_1)} - \left(\frac{R_4}{R_2}\right),$$

$$m_3 = \frac{(V_3W_1 - V_1W_2)}{(V_1W_2 - V_2W_1)}$$

and

$$V_1 = (R_1R_6 + R_2R_5), \qquad V_2 = (R_1R_6 + R_2R_7), \qquad V_3 = -(R_2R_8 - R_4R_6),$$

$$W_1 = (R_1d_2 + R_2d_1), \qquad W_2 = (R_1d_2 + R_2d_3), \qquad W_3 = (R_2d_4 - R_4d_3),$$

$$d_1 = R_2Sh(s_1) + R_1Ch(s_1), \qquad d_2 = R_2Ch(s_1) + R_1Sh(s_1),$$

$$d_3 = R_1Cs(s_2) - R_4Sn(s_2), \qquad d_4 = R_4Cs(s_2) + R_1Sn(s_2). \tag{17}$$

4. NUMERICAL EXAMPLE

A typical shaft system is simulated and the data used in this investigation is: $l = 50$ in (1·27 m), $d = 4$ in (0·1016 m), $E = 30 \times 10^6$ lb/in^2 ($2{\cdot}07 \times 10^{11}$ Pa), $G = 11{\cdot}5 \times 10^6$ lb/in^2 ($0{\cdot}79 \times 10^{11}$ Pa), $\rho = 0{\cdot}283$ lb/in^3 ($7{\cdot}68 \times 10^3$ kg/m^3). The variation of the first four non-dimensional natural frequencies with dimensionless axial compressive load were calculated for $K_R = 0$ and $K_T = 1 \times 10^{15}$ N/m, and the results are shown in Figure 2. The frequencies are non-dimensionalized by the first natural frequency of the beam for zero axial loading, $P = 0$. The axial compressive load is non-dimensionalized by the buckling load. (The buckling load of Timoshenko's beam is obtained first, by finding solutions to the equations of motion (1) and (2) without the time dependence term, and then by imposing the boundary conditions. For the geometry considered, the buckling load is $P_b = 4{\cdot}96 \times 10^8$ N). It is observed from Figure 2 that the natural frequencies decrease as the compressive load is increased. The first natural frequency decreases to zero as the load tends to the buckling load. Similar variations on the first four natural frequencies were observed for $K_R = 0$ and $K_T = 1 \times 10^{15}$ N/m.

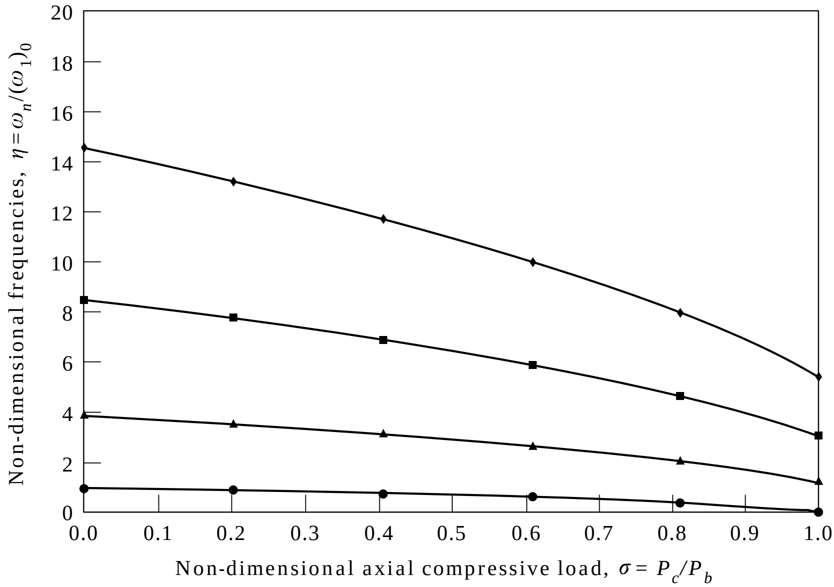

Figure 2. Variation of the first four natural frequencies with axial compressive load: ●, 1st mode; ▲, 2nd mode; ■, 3rd mode; ◆, 4th mode.

Shown in Figure 3 are variations of the non-dimensionalized natural frequencies with the dimensionless axial tensile load calculated for $K_R = 0$ and $K_T = 1 \times 10^{15}$ N/m. Here, the load is non-dimensionalized by the maximum tensile load used for the investigation (i.e. $P_{max} = 5 \times 10^9$ N). It is seen in Figure 3 that

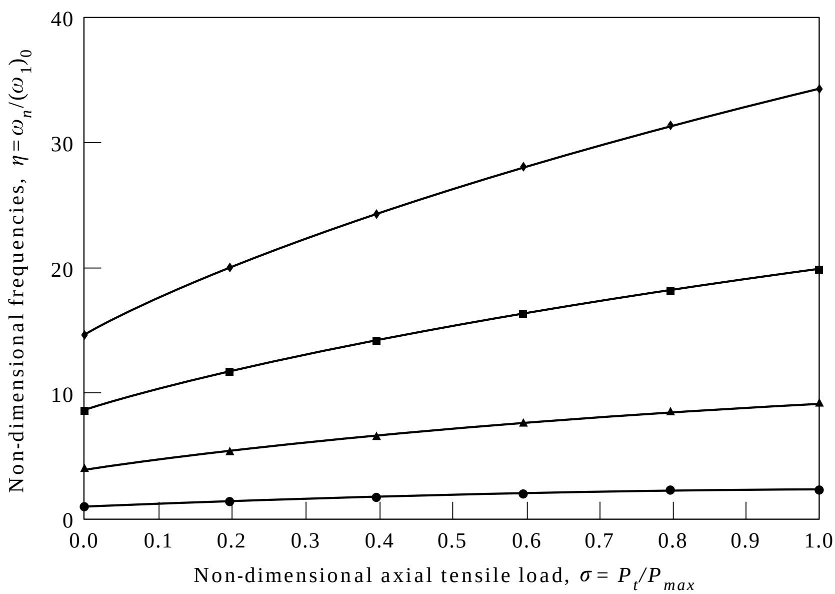

Figure 3. Variation of the first four natural frequencies with axial tensile load. Key as in Figure 2.

the frequencies increase as the axial tensile load is increased. The same behavior was again observed on the natural frequencies for $K_R = 0$ and $K_T = 1 \times 10^{15}$ N/m.

5. SUMMARY

A uniform circular shaft with translational and rotational support at its ends is analyzed analytically with the Timoshenko beam theory. The frequency determinant is derived using the boundary conditions. The transverse and shear mode shapes are also determined. A numerical example is presented to demonstrate the free vibration of the beam system. It is shown that the increase in the compressive axial load decreases the natural frequencies and first natural frequency becomes zero at the buckling load.

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