Journal of Sound and Vibration (1999) **220**(2), 225–249
Article No. jsvi.1998.1921, available online at http://www.idealibrary.com on IDEAL

NATURAL FREQUENCIES OF TRANSVERSE VIBRATIONS OF NON-UNIFORM CIRCULAR AND ANNULAR PLATES

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(*Received* 3 *September* 1997, *and in final form* 3 *August* 1998)

This paper presents a method for the free transverse vibration analysis of thin, elastic, isotropic, uniform and non-uniform circular and annular plates. The circumferential mode numbers ($n = 0$) and ($n = 1$) are dealt with in this paper. The method is a hybrid of plate theory and finite element analysis. The plate is subdivided into one circular and many annular finite elements. Two new finite elements were developed, the first type being a circular plate and the second an annular plate, the displacement functions of the finite element model are the classical solution shape functions of plate theory. Mass and stiffness matrices are determined by precise analytical integration. The free vibrations of uniform circular and annular plates are studied by this method as well as non-uniform plates. The results obtained reveal that the frequencies calculated by this method are in good agreement with those obtained by other authors. This method combines the advantages of the standard finite element analysis and the high-accuracy formulation provided by the use of displacement functions derived from plate theory instead of the usual low-order polynomials. The present method is remarkable for the fact that it enables us to determine with equal precision both low and high natural frequencies.

1. INTRODUCTION

The analysis of circular and annular plates have been of practical and academic interest for more than a century. The study of vibrations has been reviewed extensively by Leissa [1–7] and others [8–11]. There are now several theories available dealing with plates [12, 13]. More specifically, several methods have been developed for the analysis of the vibrations in thin circular plates. Among these were Galerkin's method, the Rayleigh–Ritz method, the transfer matrix method

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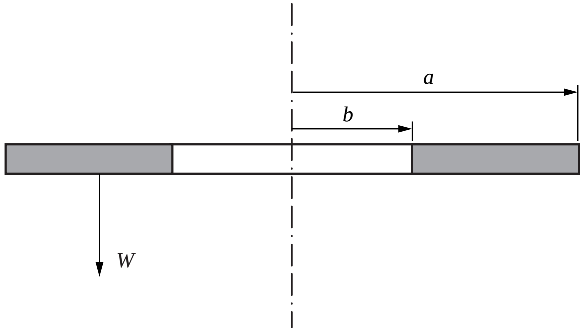

Figure 1. Geometry of the mean surface of a circular plate.

and the finite element method. All of these methods have their advantages and disadvantages. The best test of any method is probably its general content and the capacity to predict, with precision, both the high and low frequencies of vibration. The finite element method appears to be ideally suited to the analysis of complex structures. Numerous general computer programmes are available for industrial use in the linear and non-linear analysis, where the displacement functions of the finite elements used are assumed to be polynomial. To be able to predict, with precision, both the high and the low frequencies, requires the use of a great many elements in the classical finite element method. In order to achieve this, the present paper presents a new finite element for the analysis of elastic, thin, isotropic and radially non-uniform circular and annular plates (Figure 1). The plates may have any combination of boundary conditions (clamped, free and simply supported). The finite element method was employed, but it is a hybrid, a combination of the finite element method and classical plate theory. In this part of the study, we develop two new finite elements, the first type being a circular plate and the second an annular plate. This choice allowed us to use the complete equilibrium equations to determine the displacement functions and, further, the mass and stiffness matrices. This study is confined to circumferential modes $n = 0$ and $n = 1$. The analysis in the case of circumferential mode $n \geqslant 2$ has been developed in reference [28]. This method has been applied with satisfactory results to the linear and non-linear dynamic analyses of closed cylindrical shells [14–21], open cylindrical shells [22–25], conical shells [26] and spherical shells [27]. This method proves to be more accurate than the usual finite element method.

2. METHOD OF ANALYSIS

2.1. DETERMINATION OF THE DISPLACEMENT FUNCTIONS

Sanders' equation for thin circular plate, in terms of transversal displacement W for isotropic material is given by [29]

$$\frac{\partial^4 W}{\partial r^4} + 2\frac{\partial^4 W}{r^2\,\partial r^2\,\partial\theta^2} + \frac{\partial^4 W}{r^4\,\partial\theta^4} + \frac{\partial^3 W}{r\,\partial r^3} - 2\frac{\partial^3 W}{r^3\,\partial r\,\partial\theta^2} - \frac{\partial^2 W}{r^2\,\partial r^2} + 4\frac{\partial^2 W}{r^4\,\partial r\,\partial\theta} + 2\frac{\partial^4 W}{r^3\,\partial r} = 0 \tag{1}$$

and the deformation vector is given by

$$\{\varepsilon\} = \left\{ \begin{array}{c} -\dfrac{\partial^2 W}{\partial r^2} \\ -\dfrac{\partial W}{r\,\partial r} - \dfrac{\partial^2 W}{r^2\,\partial \theta^2} \\ -2\dfrac{\partial}{\partial r}\left(\dfrac{\partial W}{r\,\partial \theta}\right) \end{array} \right\}. \tag{2}$$

The corresponding stresses for isotropic material may be related to the strains by the elasticity matrix $[P]$:

$$\{\sigma\} = [P]\{\varepsilon\} = \begin{bmatrix} K & vK & 0 \\ vK & K & 0 \\ 0 & 0 & \dfrac{1-v}{2}K \end{bmatrix} \{\varepsilon\}, \tag{3}$$

where $K = Et^3/12(1 - v^2)$; E is the Young's modulus, t the thickness of the plate and v the Poisson's ratio.

The two finite elements developed in this paper are shown in Figures 2(a) and (b). The first one being an element of the circular plate type [Figure 2(a)] defined by one circular node j and the second an element of the annular plate type [Figure 2(b)] defined by two circular nodes i and j. Each node has two degrees of freedom: the transversal displacement W and the rotation $\mathrm{d}W/\mathrm{d}r$.

For motions associated with the nth circumferential mode number, we may write

$$W(r, \theta) = w_n(r)\cos(n\theta), \tag{4}$$

where n is the circumferential mode number, w_n is the magnitude of the deflections and depends on r only.

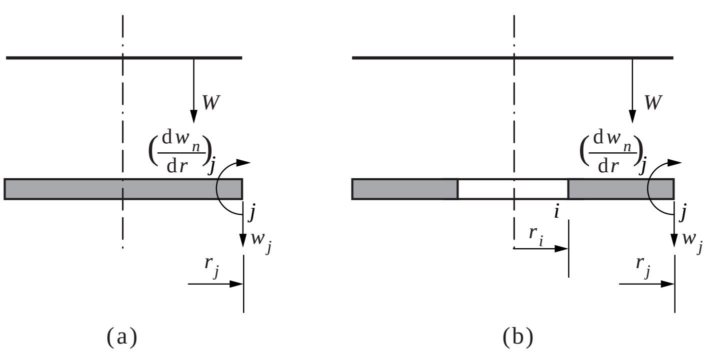

Figure 2. Displacements and degrees of freedom: (a) finite element of the circular plate type, (b) finite element of the annular plate type.

The displacement w_1 for the circumferential mode number $n = 1$ and the displacement w_0 for the circumferential mode number $n = 0$ are given by

(1) annular plate element:

$$w_{1a} = \{y, y^3, y^{-1}, y \ln y\}\{C_1, C_2, C_3, C_4\}^{\mathrm{T}} = [R_{1a}]\{C\}, \tag{5}$$

$$w_{0a} = \{1, y^2, \ln y, y^2 \ln y\}\{C_1, C_2, C_3, C_4\}^{\mathrm{T}} = [R_{0a}]\{C\}; \tag{6}$$

(2) circular plate element:

$$w_{1c} = \{y, y^3\}\{C_1, C_2\}^{\mathrm{T}} = [R_{1c}]\{C\}, \tag{7}$$

$$w_{0c} = \{1, y^2\}\{C_1, C_2\}^{\mathrm{T}} = [R_{0c}]\{C\}, \tag{8}$$

where $y = r/r_j$ and r_j is the outside radius of the plate finite element.

For the case of the annular plate element, the coefficients C_1 to C_4 are the only free constants, which must be determined from the four boundary conditions, two at each node of the finite element. For the circular plate element, the coefficients C_1 and C_2 are determined from the two boundary conditions at the node of the element.

We now express the nodal displacements vectors as follows.

(1) Annular plate element

($n = 1$):

$$\begin{Bmatrix} \delta_i \\ \delta_i \end{Bmatrix} = \left\{ w_{1i}, \left(\frac{\mathrm{d}w_1}{\mathrm{d}r}\right)_i, w_{1j}, \left(\frac{\mathrm{d}w_1}{\mathrm{d}r}\right)_j \right\}^{\mathrm{T}}$$

$$= \begin{bmatrix} y_0 & y_0^3 & y_0^{-1} & y_0 \ln y_0 \\ 1/r_j & 3y_0^2/r_j & -y_0^{-2}/r_j & (\ln y_0 + 1)/r_j \\ 1 & 1 & 1 & 0 \\ 1/r_j & 3/r_j & -1/r_j & 1/r_j \end{bmatrix} \{C\} = [A_{1a}]\{C\}, \tag{9}$$

($n = 0$):

$$\begin{Bmatrix} \delta_i \\ \delta_i \end{Bmatrix} = \left\{ w_{0i}, \left(\frac{\mathrm{d}w_0}{\mathrm{d}r}\right)_i, w_{0j}, \left(\frac{\mathrm{d}w_0}{\mathrm{d}r}\right)_j \right\}^{\mathrm{T}}$$

$$= \begin{bmatrix} 1 & y_0^2 & \ln y_0 & y_0^2 \ln y_0 \\ 0 & 2y_0/r_j & y_0^{-1}/r_j & y_0(2 \ln y_0 + 1)/r_j \\ 1 & 1 & 0 & 0 \\ 0 & 2/r_j & 1/r_j & 1/r_j \end{bmatrix} \{C\} = [A_{0a}]\{C\}. \tag{10}$$

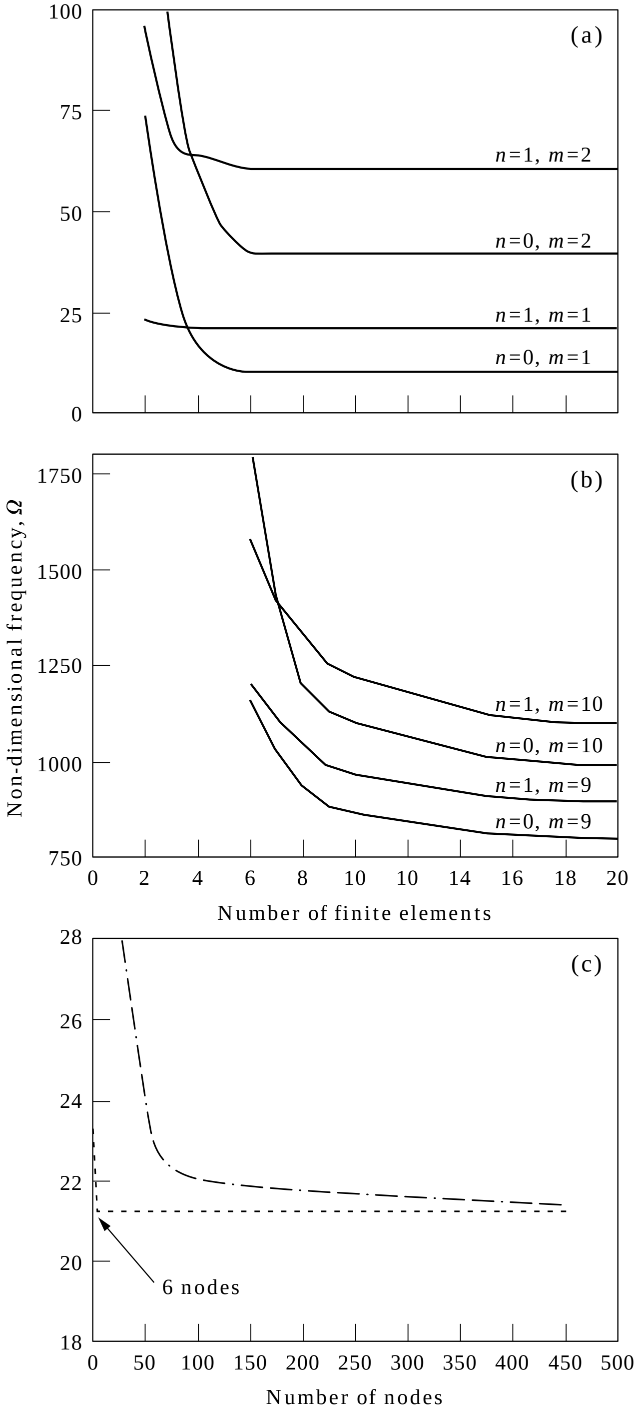

Figure 3. (a, b) Non-dimensional natural frequency Ω of a clamped circular plate as a function of the number of finite elements; (c) a comparison of non-dimensional natural frequency Ω of a clamped circular plate as a function of number of nodes ($n = 1, m = 1$): · · · · · , present method; – · – · , NASTRAN [40].

(2) circular plate element

($n = 1$):

$$\{\delta_i\} = \left\{ w_{1i}, \left(\frac{\mathrm{d}w_1}{\mathrm{d}r}\right)_i \right\}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 \\ 1 & 3/r_j \end{bmatrix} \{C\} = [A_{1c}]\{C\}, \tag{11}$$

($n = 0$):

$$\{\delta_i\} = \left\{ w_{0i}, \left(\frac{\mathrm{d}w_0}{\mathrm{d}r}\right)_i \right\}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 \\ 0 & 2/r_j \end{bmatrix} \{C\} = [A_{0c}]\{C\}, \tag{12}$$

where $y_0 = r_i/r_j$.

Multiplying equations (9)–(12) by their corresponding matrix $[A^{-1}]$ and substituting into equations (5)–(8), we obtain the displacement functions as functions of the nodal displacements.

(1) Annular plate element:

$$W_{1a} = \cos\theta [R_{1a}][A_{1a}^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N_{1a}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}, \tag{13}$$

$$W_{0c} = [R_{0a}][A_{0a}^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N_{0a}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}. \tag{14}$$

(2) Circular plate element:

$$W_{1a} = \cos\theta [R_{1c}][A_{1c}^{-1}]\{\delta_i\} = [N_{1c}]\{\delta_i\}, \tag{15}$$

$$W_{0c} = [R_{0c}][A_{0c}^{-1}]\{\delta_i\} = [N_{0c}]\{\delta_i\}. \tag{16}$$

2.2. STRAIN VECTORS

The strains are related to the displacements through equation (3). Accordingly, by expressing the strain vector in terms of the nodal displacements, we obtain for

TABLE 1

Non-dimensional natural frequencies of a clamped circular plate; $v = 0{\cdot}3$, $n = 0$

m	Present method	Leissa [1]	Irie *et al.* [31]	Laura *et al.* [37]
1	10·216	10·216	10·216	10·327
2	39·771	39·771	39·771	—
3	89·108	89·104	89·104	—
4	158·20	158·183	158·184	—
5	247·08	247·005	—	—
6	355·80	355·568	—	—
7	484·45	483·872	—	—
8	633·22	631·914	—	—
9	802·21	799·702	—	—
10	992·21	987·216	—	—

TABLE 2

Non-dimensional natural frequencies of a clamped circular plate; $\nu = 0{\cdot}3$, $n = 1$

m	Present method	Leissa [1]	Irie *et al*. [31]
1	21·261	21·26	21·260
2	60·838	60·82	60·829
3	120·13	120·08	120·079
4	199·23	199·06	199·053
5	298·24	297·77	—
6	417·32	416·20	—
7	556·66	554·37	—
8	716·53	712·30	—
9	897·25	889·95	—
10	1099·2	1087·4	—

the annular and circular plate elements (circumferential modes $n = 0$ and $n = 1$), the following expressions.

(1) Annular plate element:

($n = 1$):

$$\{\varepsilon\}_{1a} = \cos\theta \begin{bmatrix} 0 & -6y/r_j^2 & -2y^{-3}/r_j^2 & -y^{-1}/r_j^2 \\ 0 & -2y/r_j^2 & 2y^{-3}/r_j^2 & -y^{-1}/r_j^2 \\ 0 & 4y/r_j^2 & -4y^{-3}/r_j^2 & +2y^{-1}/r_j^2 \end{bmatrix} [A_{1a}^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}$$

$$= \cos\theta [Q_{1a}][A_{1a}^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B_{1a}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}, \tag{17}$$

($n = 0$):

$$\{\varepsilon\}_{0a} = \begin{bmatrix} 0 & -2/r_j^2 & y^{-2}/r_j^2 & -(3 + 2\ln y)/r_j^2 \\ 0 & -2/r_j^2 & -y^{-2}/r_j^2 & -(1 + 2\ln y)/r_j^2 \\ 0 & 0 & 0 & 0 \end{bmatrix} [A_{0a}^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}$$

$$= [Q_{0a}][A_{0a}^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B_{0a}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}. \tag{18}$$

(2) Circular plate element:

($n = 1$):

$$\{\varepsilon\}_{1c} = \begin{bmatrix} 0 & -6y/r_j^2 \\ 0 & -2y/r_j^2 \\ 0 & 4y/r_j^2 \end{bmatrix} [A_{1c}^{-1}]\{\delta_i\} = \cos\theta [Q_{1c}][A_{1c}^{-1}]\{\delta_i\} = [B_{1c}]\{\delta_i\}, \tag{19}$$

TABLE 3

Non-dimensional natural frequencies of a simply-supported circular plate; $v = 0{\cdot}3$, $n = 0$

m	Present method	Leissa and Narita [32]	Irie *et al.* [31]	Laura *et al.* [37]
1	4·935	4·935	4·934	4·947
2	29·720	29·720	29·720	—
3	74·158	74·156	74·156	—
4	138·33	138·32	138·318	—
5	222·27	222·22	—	—
6	326·02	325·85	—	—
7	449·68	449·22	—	—
8	593·40	592·33	—	—
9	757·41	755·18	—	—
10	942·05	937·77	—	—

($n = 0$):

$$\{\varepsilon\}_{0c} = \begin{bmatrix} 0 & -2/r_j^2 \\ 0 & -2/r_j^2 \\ 0 & 0 \end{bmatrix} [A_{0c}^{-1}]\{\delta_i\} = [Q_{0c}][A_{0c}^{-1}]\{\delta_i\} = [B_{0c}]\{\delta_i\}. \tag{20}$$

2.3. MASS MATRICES

Following the framework of the finite element method [30], the mass matrix may be expressed as

$$[m] = \rho t \int_0^{2\pi} \int_{r_i}^{r_j} [N]^{\mathrm{T}}[N] r \, \mathrm{d}r \, \mathrm{d}\theta, \tag{21}$$

TABLE 4

Non-dimensional natural frequencies of a simply-supported circular plate; $v = 0{\cdot}3$, $n = 1$

m	Present method	Leissa and Narita [32]	Irie *et al.* [31]
1	13·898	13·898	13·898
2	48·484	48·479	48·479
3	102·81	102·77	102·733
4	176·93	176·80	176·801
5	270·95	270·57	—
6	384·98	384·07	—
7	519·23	517·31	—
8	673·93	670·29	—
9	849·40	843·01	—
10	1046·0	1035·47	—

TABLE 5

Non-dimensional natural frequencies of a free circular plate; $\nu = 0{\cdot}33$, $n = 0$

m	Present method	Leissa [1]	Itao and Crandall [33]
1	—	—	—
2	9·068	9·084	9·068
3	38·507	38·55	38·507
4	87·816	87·80	87·813
5	156·90	157·0	156·88
6	245·77	245·9	245·70
7	354·48	354·6	354·25
8	483·12	483·1	482·55
9	631·87	631·0	630·59
10	800·97	798·6	798·37

where ρ is the density of the plate, t its thickness and the matrix $[N]$ is obtained from equations (13) to (16) respectively for different cases. The matrix $[m]$ was obtained analytically by carrying out the necessary matrix operations and integration over r and θ in equation (21).

We give here the results of these analytical calculations (all matrices symmetric).

(1) Annular plate element:

($n = 1$):

$$[m_{1a}] = \rho t[A_{1a}^{-1}]^{\mathrm{T}}[S_{1a}][A_{1a}^{-1}], \tag{22}$$

where the elements of matrix $[S_{1a}]$ are given by

$$S_{1a}(1,1) = \frac{\pi r_j^2}{4}(1 - y_0^4), \qquad S_{1a}(1,2) = \frac{\pi r_j^2}{6}(1 - y_0^6), \qquad S_{1a}(1,3) = \frac{\pi r_j^2}{2}(1 - y_0^2),$$

$$S_{1a}(1,4) = \frac{\pi r_j^2}{4}\left(-\frac{1}{4} - y_0^4 \ln y_0 + \frac{y_0^4}{4}\right),$$

$$S_{1a}(2,2) = \frac{\pi r_j^2}{8}(1 - y_0^8), \qquad S_{1a}(2,3) = \frac{\pi r_j^2}{4}(1 - y_0^4),$$

$$S_{1a}(2,4) = \frac{\pi r_j^2}{6}\left(-\frac{1}{6} - y_0^6 \ln y_0 + \frac{y_0^4}{6}\right),$$

$$S_{1a}(3,3) = -\pi r_j^2 \ln y_0, \qquad S_{1a}(3,4) = \frac{\pi r_j^2}{2}\left(-\frac{1}{2} - y_0^2 \ln y_0 + \frac{y_0^4}{2}\right),$$

$$S_{1a}(4,4) = \pi r_j^2\left[\frac{1}{32} - \frac{y_0^4}{4}\left(\ln^2 y_0 - \frac{1}{2}\ln y_0 + \frac{1}{8}\right)\right]; \tag{23}$$

($n = 0$):

$$[m_{0a}] = \rho t [A_{0a}^{-1}]^{\mathrm{T}} [S_{0a}] [A_{0a}^{-1}], \tag{24}$$

where the elements of matrix $[S_{0a}]$ are given by

$$S_{0a}(1, 1) = \pi r_j^2 (1 - y_0^2), \qquad S_{0a}(1, 2) = \frac{\pi r_j^2}{2}(1 - y_0^4),$$

$$S_{0a}(1, 3) = 2\pi r_j^2 \left[-\frac{1}{4} - \frac{y_0^2}{2}\left(\ln y_0 - \frac{1}{2} \right) \right],$$

$$S_{0a}(1, 4) = 2\pi r_j^2 \left[-\frac{1}{16} - \frac{y_0^4}{4}\left(\ln y_0 - \frac{1}{4} \right) \right],$$

$$S_{0a}(2, 2) = \frac{\pi r_j^2}{3}(1 - y_0^6), \qquad S_{0a}(2, 3) = 2\pi r_j^2 \left[-\frac{1}{16} - \frac{y_0^4}{4}\left(\ln y_0 - \frac{1}{4} \right) \right],$$

$$S_{0a}(2, 4) = 2\pi r_j^2 \left[-\frac{1}{36} - \frac{y_0^2}{6}\left(\ln y_0 - \frac{1}{6} \right) \right],$$

$$S_{0a}(3, 3) = 2\pi r_j^2 \left[\frac{y_0^2}{2}\left(\ln y_0 - \ln^2 y_0 - \frac{1}{2} \right) + \frac{1}{4} \right],$$

$$S_{0a}(3, 4) = 2\pi r_j^2 \left[\frac{y_0^4}{4}\left(\frac{\ln y_0}{2} - \ln^2 y_0 - \frac{1}{8} \right) + \frac{1}{32} \right],$$

TABLE 6

Non-dimensional natural frequencies of a free circular plate; $\nu = 0{\cdot}3$, $n = 1$

m	Present method	Leissa [1]	Itao and Crandall [33]
1	—	—	—
2	20·514	20·41	20·513
3	59·868	59·74	59·859
4	119·06	118·88	119·01
5	198·10	196·67	197·92
6	297·08	296·46	296·59
7	416·12	414·86	415·01
8	555·43	553·00	553·17
9	715·27	710·92	711·07
10	895·94	888·58	888·72

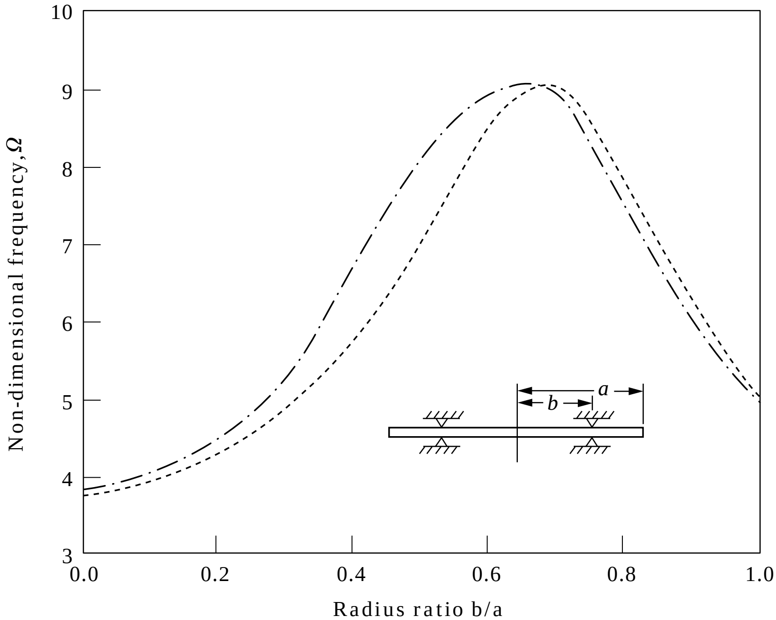

Figure 4. Non-dimensional natural frequency Ω of a circular plate simply-supported along an arbitrary circle ($n = 0, m = 1$): $\cdots\cdots$, present method; $- \cdot - \cdot$, Bodine [34].

$$S_{0a}(4, 4) = 2\pi r_j^2 \left[\frac{y_0^6}{6} \left(\frac{\ln y_0}{3} - \ln^2 y_0 - \frac{1}{18} \right) + \frac{1}{108} \right]. \tag{25}$$

(2) Circular plate element:

($n = 1$):

$$[m_{1c}] = \rho t [A_{1c}^{-1}]^{\mathrm{T}} [S_{1c}] [A_{1c}^{-1}], \tag{26}$$

where the matrix $[S_{1c}]$ is given by

$$S_{1c}(1, 1) = \frac{\pi r_j^2}{4}, \qquad S_{1c}(1, 2) = \frac{\pi r_j^2}{6}, \qquad S_{1c}(2, 2) = \frac{\pi r_j^2}{8}; \tag{27}$$

($n = 0$):

$$[m_{0c}] = \rho t [A_{0c}^{-1}]^{\mathrm{T}} [S_{0c}] [A_{0c}^{-1}], \tag{28}$$

where the matrix $[S_{0c}]$ is given by

$$S_{0c}(1, 1) = \pi r_j^2, \qquad S_{0c}(1, 2) = \frac{\pi r_j^2}{2}, \qquad S_{0c}(2, 2) = \frac{\pi r_j^2}{3}. \tag{29}$$

2.4. STIFFNESS MATRICES

Also, the stiffness matrix may be expressed as [30]

$$[k] = \int_0^{2\pi} \int_{r_i}^{r_j} [B]^{\mathrm{T}} [P] [B] r \, \mathrm{d}r \, \mathrm{d}\theta, \tag{30}$$

where $ds = r\,dr\,d\theta$, $[P]$ is the elasticity matrix given in relation (3) and the matrix $[B]$ is obtained from equations (17) to (20) respectively for different cases. The matrix $[k]$ was obtained analytically by carrying out the necessary matrix operations and integration over r and θ in equation (30).

The results of these analytical calculations are as follows (all matrices symmetric).

(1) Annular plate element:

($n = 1$):

$$[k_{1a}] = [A_{1a}^{-1}]^{\mathrm{T}}[G_{1a}][A_{1a}^{-1}], \tag{31}$$

where the matrix $[G_{1a}]$ is given by

$$G_{1a}(1, j) = 0, \quad \text{for } j = 1, \ldots, 4,$$

$$G_{1a}(2, 2) = \frac{4\pi K(3 + v)}{a^2}(1 - y_0^4), \qquad G_{1a}(2, 3) = 0,$$

$$G_{1a}(2, 4) = \frac{2\pi K(3 + v)}{a^2}(1 - y_0^2),$$

$$G_{1a}(3, 3) = -\frac{4\pi K(1 - v)}{a^2}(1 - y_0^{-4}), \qquad G_{1a}(3, 4) = \frac{2\pi K(1 - v)}{a^2}(1 - y_0^{-2}),$$

$$G_{1a}(4, 4) = -\frac{4\pi K}{a^2}\ln y_0; \tag{32}$$

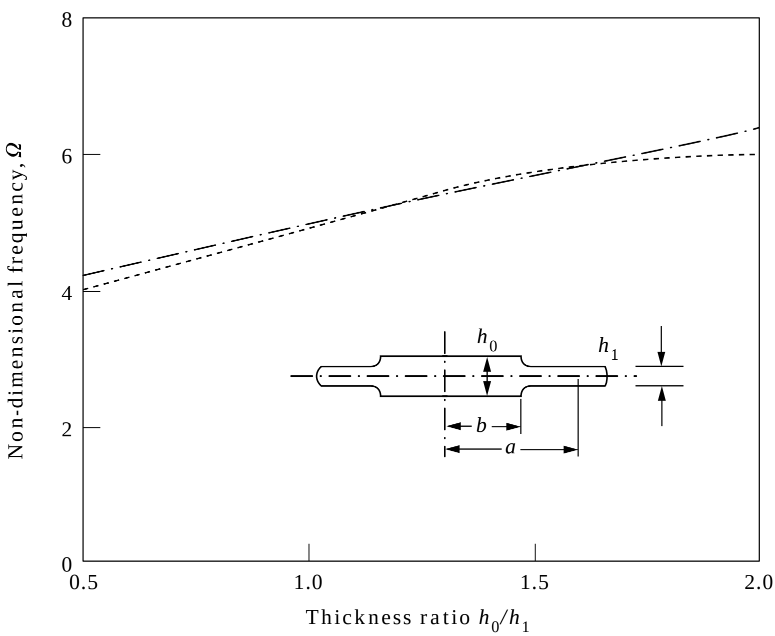

Figure 5. Non-dimensional natural frequency Ω of simply-supported circular plate with a discontinuity of thickness ($n = 0$, $m = 1$, $b/a = 0{\cdot}5$): · · · · · , present method; – · – · , Irie and Yamada [35].

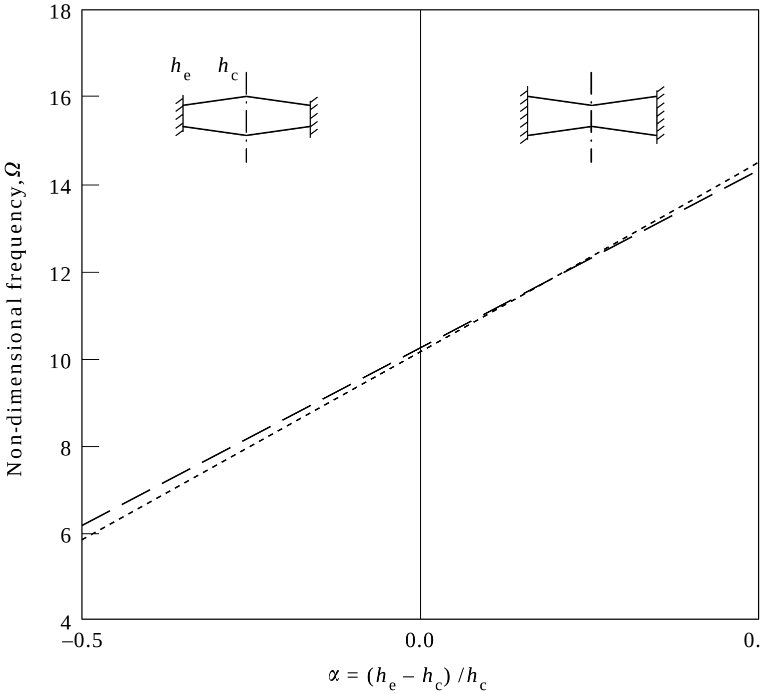

Figure 6. Non-dimensional natural frequency Ω of clamped circular plate with a linear variation of thickness ($n = 0, m = 1$). · · · · ·, present method; – · – ·, Sato and Shimizu [35].

($n = 0$):

$$[k_{0a}] = [A_{0a}^{-1}]^{\mathrm{T}}[G_{0a}][A_{0a}^{-1}], \tag{33}$$

where the matrix $[G_{0c}]$ is given by

$$G_{0a}(1, j) = 0, \quad \text{for } j = 1, \ldots, 4,$$

$$G_{0a}(2, 2) = \frac{8\pi K(1 + v)}{a^2}(1 - y_0^2), \qquad G_{0a}(2, 3) = 0,$$

$$G_{0a}(2, 4) = -\frac{4\pi K(1 + v)}{a^2}(2y_0^2 \ln y_0 + y_0^2 - 1),$$

$$G_{0a}(3, 3) = -\frac{2\pi K(1 - v)}{a^2}(1 - y_0^{-2}), \qquad G_{0a}(3, 4) = -\frac{4\pi K(1 - v)}{a^2}(1 - \ln y_0),$$

$$G_{0a}(4, 4) = \frac{4\pi K(1 + v)}{a^2}(y_0^2 - 2y_0^2 \ln y_0 - 2y_0^2 \ln^2 y_0 - 1) + \frac{2\pi K(5 + 3v)}{a^2}(1 - y_0^2). \tag{34}$$

(2) Circular plate element:

($n = 1$):

$$[k_{1c}] = [A_{1c}^{-1}]^{\mathrm{T}}[G_{1c}][A_{1c}^{-1}], \tag{35}$$

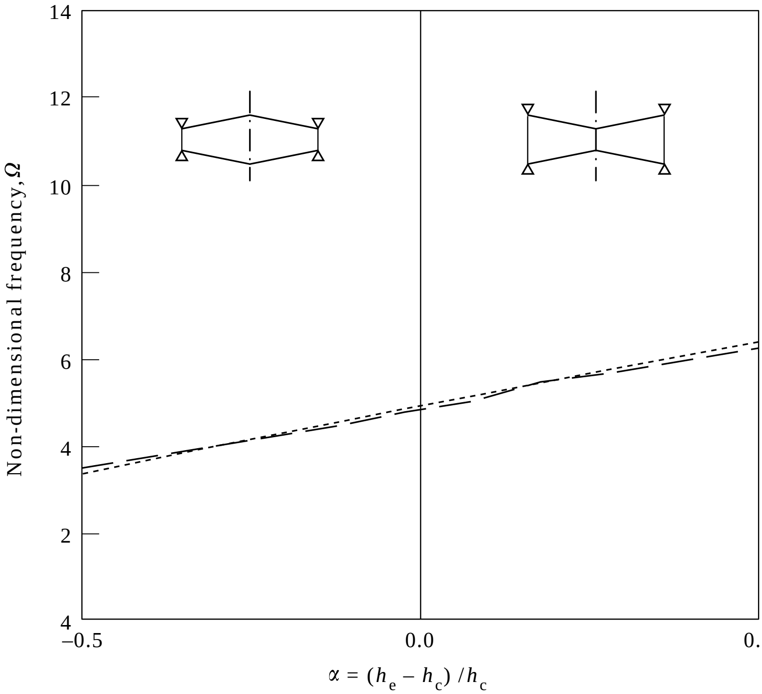

Figure 7. Non-dimensional natural frequency Ω of simply-supported circular plate with a linear variation of thickness ($n = 0, m = 1$): · · · · ·, present method; ———, Sato and Shimizu [35].

where the matrix $[G_{1c}]$ is given by

$$G_{1c}(1, 1) = G_{1c}(1, 2) = 0, \qquad G_{1c}(2, 2) = \frac{4\pi K(3 + \nu)}{a^2}; \tag{36}$$

($n = 0$):

$$[k_{0c}] = [A_{0c}^{-1}]^{\mathrm{T}}[G_{0c}][A_{0c}^{-1}], \tag{37}$$

TABLE 7

Influence of Poisson's ratio ν on the non-dimensional natural frequencies of a simply-supported circular plate

n	m	Present method $\Omega(0{\cdot}5)/\Omega(0)$*	Leissa and Narita [32] $\Omega(0{\cdot}5)/\Omega(0)$*
0	1	1·17308	1·17307
0	2	1·01982	1·01981
0	3	1·00742	1·00743
0	11	1·00520	1·00450
1	1	1·04825	1·04736

* $\Omega(0{\cdot}5)$, non-dimensional natural frequency calculated for $\nu = 0{\cdot}5$; $\Omega(0)$, non-dimensional natural frequency calculated for $\nu = 0$.

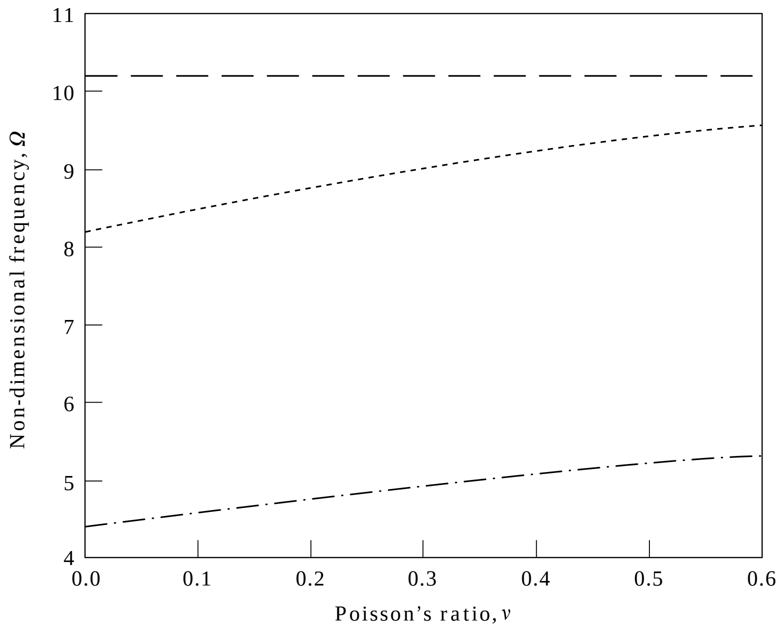

Figure 8. Non-dimensional natural frequency Ω of circular plate as a function of Poison's ratio ($n = 0, m = 1$): ———, clamped; · · · · ·, free; – · – ·, simply-supported.

where the matrix $[G_{0c}]$ is given by

$$G_{0c}(1, 1) = G_{0c}(1, 2) = 0, \qquad G_{0c}(2, 2) = \frac{8\pi K(1 + \nu)}{a^2}. \tag{38}$$

3. NUMERICAL RESULTS

The complete circular plate is divided into one circular finite element and a few annular finite elements. The position of the nodal points (nodal circle) may be chosen arbitrarily. With the mass and stiffness matrices known of each element, the global mass and stiffness matrices for the whole structure, $[M]$ and $[K]$ respectively, may be constructed by superposition in the usual manner. Each of these square matrices will be of order $2(N + 1)$ for the case of an annular plate and of order $2N$ for the case of a circular plate, where N is the total number of finite elements.

When the plates edges are constrained (such as simply-supported, clamped or free), the appropriate lines and columns in $[M]$ and $[K]$ are deleted to satisfy these constraints. Consequently, matrices $[M]$ and $[K]$ reduce to square matrices of order $2(N + 1) - J$, where J is the number of applied constraints. Thus, for a clamped-clamped annular plate, the number of constraints is $J = 4$. For a free circular plate, $J = 0$.

To ease comparison with previously published results, the natural frequencies calculated in this section are expressed in the non-dimensional form

$$\Omega = \omega a^2 \sqrt{\rho t / K},$$

where ω is the natural angular frequency (rad/s), a is the outside radius of the plate, t its thickness, ρ is the material density and K is the bending stiffness [see relation (3)].

3.1. CONVERGENCE OF THE METHOD

A first set of calculations was undertaken to determine the requisite number of finite elements for a precise determination of natural frequencies. Calculations were made for the same uniform clamped circular plate with the number of finite elements N varying from 2 to 20. The results for the circumferential modes $n = 0$ and 1 and the radial modes $m = 1$, 2, 9 and 10 are shown in Figure 3(a). We conclude that the convergence of the system demands six finite elements for the modes $m = 1$ and 2. For radial mode $m = 10$, twenty finite elements are sufficient for satisfactory results.

Figure 3(b) shows the non-dimensional natural frequency ($n = 1, m = 1$) computed by the present method (Hybrid Finite Elements) and compared to NASTRAN code [40] (Classical Finite Elements). In the NASTRAN solution,

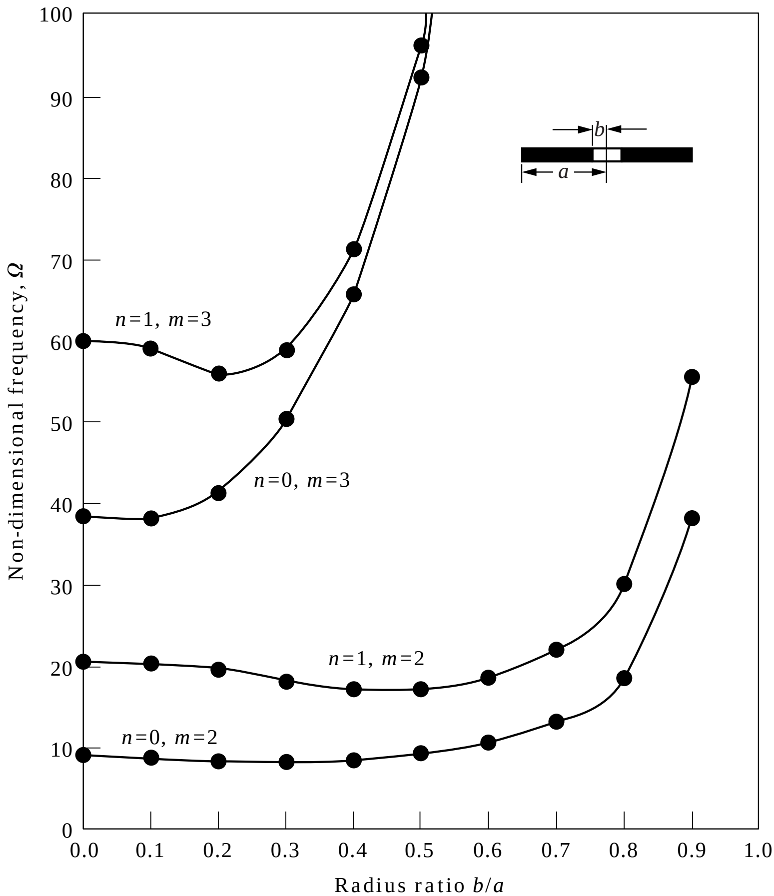

Figure 9. Non-dimensional natural frequency Ω of a free–free annular plate: ——, present method; ●, Leissa [1].

TABLE 8

Non-dimensional natural frequencies of annular plates for different boundary conditions; $m = 1$, $b/a = 0{\cdot}5$

Mode	Boundary conditions	Present method	Vera *et al*. [38]	Singh and Chakraverty [39]
$n = 0$	C–C	89·251	89·2500	89·25
	S–S	40·043	40·0431	40·01
	C–F	17·714	—	17·60
	C–S	63·973	63·9732	63·85
$n = 1$	C–C	90·230	90·2302	—
	S–S	41·797	41·7973	—
	C–S	65·486	65·4855	—

CQUAD4 finite elements were used to model the circular plate. As may be seen, the hybrid finite element method proves to be more accurate than the classical finite element method and demands only few finite elements to converge.

Moreover, the reader may consult references [16, 22] for comparison with available experimental data. In these references, it has been shown, by comparing the numerical results of our approach with the experimental results, that this hybrid finite element method embodies simultaneously the advantages of the finite element method and the precise formulation of classical shell and plate theories.

3.2. FREE VIBRATIONS OF UNIFORM CIRCULAR PLATES

We present here a comparison of the non-dimensional natural frequencies determined by this method with those obtained by other authors, both for different boundary conditions (plate clamped, simply-supported and free) and for different values of the circumferential modes number $n = 0$ and $n = 1$ and for the radial modes number $m = 1$ to 10.

TABLE 9

Non-dimensional natural frequencies of clamped–clamped annular plate, $n = 0$

			b/a		
m	0·1	0·3	0·5	0·7	0·9
1	27·281	45·346	89·251	248·43	1986·5
2	75·367	125·36	246·35	685·06	7055·1
3	148·22	246·17	483·25	1343·3	12 273
4	245·53	407·31	799·15	2220·9	12 907
5	367·31	608·86	1194·2	3318·7	13 427
6	513·64	850·93	1668·8	4638·3	15 778
7	684·63	1133·7	2223·1	6182·6	17 094
8	880·47	1457·6	2857·8	7946·5	20 248
9	1101·4	1823·0	3573·9	9945·5	23 404
10	1347·9	2230·5	4372·6	12 179	24 516

TABLE 10

Non-dimensional natural frequencies of clamped–clamped annular plate, $n = 1$

	b/a				
m	0·1	0·3	0·5	0·7	0·9
1	28·916	46·644	90·230	249·16	2258·2
2	78·637	127·38	247·74	686·04	6235·3
3	152·55	248·52	484·81	1344·3	11 828
4	250·65	409·86	800·82	2222·0	15 690
5	373·04	611·54	1196·0	3319·4	16 653
6	519·86	853·71	1670·5	4637·2	18 713
7	691·24	1136·6	2224·9	6176·6	19 414
8	887·42	1460·5	2859·7	7939·0	19 690
9	1108·7	1825·9	3575·9	9926·1	20 402
10	1355·4	2233·5	4374·6	12 140	21 280

As may be seen from Tables 1 to 6, the results obtained by the present method are in good agreement with those of Irie *et al*. [31], Leissa [1, 32], Itao and Crandall [33] and Laura *et al*. [37].

3.3. NATURAL FREQUENCIES OF A CIRCULAR PLATE SIMPLY-SUPPORTED ALONG AN ARBITRARY CIRCLE

The non-dimensional natural frequencies of a circular plate which is simply-supported along an arbitrary circle have been obtained by the present method and compared with those obtained by Bodine [34] for the circumferential mode number $n = 0$ and radial mode number $m = 1$. As may be seen, acceptable agreement has been obtained (Figure 4).

TABLE 11

Non-dimensional natural frequencies of simply-supported–simply-supported annular plate, $n = 0$

	b/a				
m	0·1	0·3	0·5	0·7	0·9
1	14·485	21·079	40·043	110·08	937·95
2	51·782	81·737	158·64	439·21	5276·6
3	112·99	182·54	356·09	987·69	12 247
4	198·47	323·61	632·53	1755·6	12 490
5	308·31	505·02	988·04	2743·5	12 840
6	442·59	726·86	1422·8	3952·3	13 980
7	601·42	989·31	1937·2	5383·5	16 989
8	784·97	1292·6	2531·6	7031·4	19 994
9	993·46	1637·2	3207·0	8912·3	20 459
10	1227·3	2023·6	3964·3	11 019	22 772

TABLE 12

Non-dimensional natural frequencies of simply-supported–simply-supported annular plate, $n = 1$

			b/a		
m	0·1	0·3	0·5	0·7	0·9
1	16·776	23·317	41·797	111·44	995·84
2	56·507	84·636	160·57	440·59	3959·4
3	119·33	185·65	358·06	989·00	8726·3
4	205·84	326·81	634·51	1756·8	15 426
5	316·35	508·27	990·03	2744·2	15 860
6	451·10	730·14	1424·8	3951·9	16 978
7	610·26	992·61	1939·2	5380·6	19 150
8	794·05	1295·9	2533·7	7031·5	19 635
9	1002·7	1640·5	3209·0	8906·3	20 004
10	1236·7	2026·9	3966·3	11 007	20 969

3.4. NATURAL FREQUENCIES OF NON-UNIFORM CIRCULAR PLATES

Two types of non-uniform circular plate have been studied. The first is a clamped circular plate with a discontinuity of thickness and the second is a plate with a linear variation of thickness in the radial direction.

3.4.1. *Circular plate with thickness discontinuity*

The plate is of uniform thickness h_0 as far as radius b and of thickness h_1 from radius b to exterior radius a (Figure 5).

The natural frequencies of this type of circular plate have been established by Irie and Yamada [35] for the circumferential mode $n = 0$ and radial mode $m = 1$, with a h_0/h_1 thickness ratio varying from 0·5 to 2. The results obtained by our method are in good agreement with those obtained by Irie and Yamada [35] (Figure 5).

TABLE 13

Non-dimensional natural frequencies of clamped–free annular plate, $n = 0$

			b/a		
m	0·1	0·3	0·5	0·7	0·9
1	10·159	11·424	17·714	43·143	347·55
2	39·521	51·745	93·847	251·67	1840·8
3	90·447	132·41	252·20	692·09	6310·8
4	164·32	253·14	488·97	1350·1	12 271
5	262·03	414·24	804·85	2227·7	12 900
6	383·96	615·75	1199·9	3325·6	13 040
7	530·32	857·80	1674·4	4645·2	14 114
8	701·31	1140·6	2228·7	6189·6	17 094
9	897·12	1464·4	2863·4	6703·8	19 087
10	1118·1	1829·7	3579·4	7953·5	20 070

TABLE 14

Non-dimensional natural frequencies of clamped–free annular plate, $n = 1$

	b/a				
m	0·1	0·3	0·5	0·7	0·9
1	21·195	19·540	22·015	45·333	355·19
2	60·062	59·760	97·376	253·72	2229·1
3	117·09	138·66	255·06	693·79	6282·4
4	192·62	258·51	491·58	1351·6	11 969
5	288·93	419·12	807·33	2229·1	15 690
6	408·52	620·33	1202·3	3326·4	16 670
7	552·68	862·18	1676·7	4644·2	18 351
8	721·87	1144·8	2231·0	6183·5	18 777
9	916·30	1468·5	2865·7	6439·8	19 453
10	1136·2	1833·8	3581·7	7945·7	19 704

Using the present method, it is possible to determine the natural frequencies for this type of plate for different thickness ratio h_0/h_1, and particularly for high radial modes.

3.4.2. *Circular plate with a linear thickness variation*

The non-dimensional natural frequencies are determined for the circumferential mode $n = 0$ and radial mode $m = 1$ for different values of $\alpha = (h_e - h_c)/h_c$ (Figures 6 and 7), where h_c is the thickness at the centre of the plate and h_e is the thickness at the outside edge of the plate.

The calculations have been carried out for two types of boundary conditions: a clamped plate (Figure 6) and a simply-supported plate (Figure 7). The results obtained are compared with those obtained by Sato and Shimizu [36] who used the transfer matrix method. Good agreement has been obtained.

TABLE 15

Non-dimensional natural frequencies of clamped–simply-supported annular plate, $n = 0$

	b/a				
m	0·1	0·3	0·5	0·7	0·9
1	22·701	33·765	63·973	174·42	1391·9
2	65·640	104·22	202·07	558·19	5986·6
3	132·90	215·08	419·24	1161·4	12 270
4	224·46	366·22	715·42	1984·2	12 780
5	340·39	557·72	1090·7	3026·6	12 917
6	480·79	789·69	1545·4	4290·4	14 080
7	645·78	1062·3	2079·7	5779·3	17 094
8	835·54	1375·9	2694·3	7488·5	20 055
9	1050·3	1730·9	3390·0	9425·4	22 351
10	1290·5	2127·8	4167·8	11 603	23 603

TABLE 16

Non-dimensional natural frequencies of clamped–simply-supported annular plate, $n = 1$

	b/a				
m	0·1	0·3	0·5	0·7	0·9
1	25·283	35·906	65·486	175·52	1557·6
2	70·690	107·03	203·85	559·37	5040·6
3	139·43	218·11	421·11	1162·7	10 211
4	231·93	369·35	717·33	1985·4	15 689
5	348·50	560·91	1092·7	3027·8	16 432
6	489·33	792·92	1547·3	4290·5	17 937
7	654·64	1065·6	2081·6	5774·4	19 211
8	844·64	1379·2	2696·3	7480·7	19 646
9	1059·6	1734·2	3392·0	9411·4	20 105
10	1300·0	2131·2	4170·0	11 569	21 049

We note that frequencies vary linearly with α for both boundary conditions. The equations for these straight lines are as follows.

$$\text{Clamped plate:} \quad \Omega = 8{\cdot}6\alpha + 10{\cdot}2, \quad \text{for } n = 0, m = 1.$$

$$\text{Simply-supported:} \quad \Omega = 3\alpha + 4{\cdot}9, \quad \text{for } n = 0, m = 1.$$

This method can also give the natural frequencies for a circular plate of other non-uniform thickness and properties, whether the modes are high or low.

3.5. INFLUENCE OF POISSON'S RATIO ON THE NATURAL FREQUENCIES OF CIRCULAR PLATES

Analysis of the results of Table 7 shows that the effect of Poisson's ratio on the natural frequencies of circular plates is only important for low radial vibration modes. This effect is evident in Figure 8, which shows the natural frequencies of

TABLE 17

Non-dimensional natural frequencies of free–free annular plate, $n = 0$

	b/a				
m	0·1	0·3	0·5	0·7	0·9
1	—	—	—	—	—
2	8·7745	8·3535	9·3135	13·162	38·014
3	38·236	50·353	92·308	250·57	1233·4
4	89·028	130·48	249·39	687·09	1847·5
5	162·86	251·20	486·23	1345·2	5954·6
6	260·54	412·28	802·09	1390·3	12 267
7	382·45	613·78	1197·1	2222·5	12 937
8	528·80	855·81	1612·2	3320·0	13 001
9	699·76	1138·6	1671·6	4639·0	14 058
10	895·54	1462·3	2225·9	6182·2	17 016

TABLE 18

Non-dimensional natural frequencies of free–free annular plate, $n = 1$

	b/a				
m	0·1	0·3	0·5	0·7	0·9
1	—	—	—	—	—
2	20·406	18·292	17·198	21·914	55·250
3	59·072	58·784	96·266	253·13	1365·8
4	116·01	137·11	252·61	689·31	2249·1
5	191·46	256·83	489·09	1347·2	6293·7
6	287·71	417·36	804·77	1542·4	12 056
7	407·23	618·52	1199·7	2224·7	15 703
8	551·33	860·33	1674·1	3322·0	16 808
9	720·47	1142·9	1796·7	4639·7	18 744
10	914·84	1466·6	2228·3	6179·1	19 456

a circular plate for the circumferential mode number $n = 0$ and the first radial mode. We conclude that the effect is more pronounced for a simply-supported circular plate (17% difference between the frequency calculated with $\nu = 0$ and that calculates with $\nu = 0{\cdot}5$); for a free plate the error is 16%, while for a clamped plate Poisson's ratio has no effect. For a simply-supported annular plate the effect of the ratio is not very pronounced (3%), while it is negligible for other boundary conditions.

3.6. FREE VIBRATIONS OF ANNULAR PLATES

Figure 9 shows a comparison of results between the present method and those of Leissa [1] for a free–free annular plate. For other boundary conditions, Table 8 shows comparative results between the present method and those of Vera *et al.* [38] and Singh and Chakraverty [39]. As may be seen, very good agreement was found between the two results. The present method is remarkable for the fact that it enables us to determine with equal precision both low and high natural frequencies. The results obtained in the literature for annular plates are only for relatively low radial modes ($m = 1, 2, 3$). To extend the range of results, Tables 9–18 show part of the results obtained for $n = 0$, 1 and $m = 1$ to 10 for different boundary conditions and various dimensions of the annular plate. The annular plate was modelled with twenty finite elements.

4. CONCLUSIONS

A method based on the classical solution shape functions of plate theory and the finite element method has been formulated for the transverse vibration analysis of non-uniform circular and annular plates. The circumferential modes numbers ($n = 0$) and ($n = 1$) have been studied in this paper. Two new finite elements were developed, the first type being a circular plate and the second an annular plate. Mass and stiffness matrices were determined by analytical integration. The convergence of the proposed method was established and the natural frequencies were obtained for different circular and annular plates, both uniform and non-uniform. These were compared with the results of other investigations and generally good agreement was obtained.

This method offers many advantages, some of which are the following.

(1) Simple inclusion of thickness discontinuities, material property variations, differences in materials comprising the plate.
(2) Arbitrary boundary conditions: the problem can be resolved for a supported, clamped–free or clamped–clamped plate without changing the displacement functions in each case.
(3) High as well as low frequencies may be obtained with high accuracy as shown in the present study and in references [16, 20, 22, 26, 28].
(4) As shown in reference [16], this method is more precise than the usual finite element methods, but suffers from a lack of versatility; for instance, it cannot be used to analyse other than geometrically axially symmetric, non-uniform circular and annular plates and cylindrical and conical shells.
(5) This approach has been also applied with satisfactory results to the dynamic linear and non-linear analysis of cylindrical and conical shells containing a flowing fluid or partially filled with liquid [14–28].

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APPENDIX: NOMENCLATURE

a = outside radius of an annular or circular plate
b = inside radius of an annular plate
E = Young's modulus
J = number of boundary conditions
K = bending stiffness = $Et^3/12(1-\nu^2)$
ln = Napierian logarithm
m = radial mode number
n = circumferential mode number
r = radial coordinate
r_i = inside radius of annular finite element
r_j = outside radius of annular or circular finite element
t = thickness of the plate
W = transversal displacement
w_n = amplitude of W associated with the nth circumferential mode number
y = coordinate defined by $y = r/r_j$
y_0 = coordinate defined by $y_0 = r_i/r_j$
θ = circumferential coordinate
ν = Poisson's ratio
ρ = density of the material of the plate
ω = natural angular frequency
Ω = non-dimensional natural frequency = $\omega a^2(\rho t/K)^{1/2}$
$[A]$ = defined by equations (9)–(12)
$[k]$ = stiffness matrix
$[m]$ = mass matrix
$[P]$ = elasticity matrix, given in equation (3)
$\{C\}$ = arbitrary constants vector
$\{\varepsilon\}$ = deformation vector
$\{\sigma\}$ = stress vector.