Journal of Sound and Vibration (1999) **220**(2), 313–330
Article No. jsvi.1998.1935, available online at http://www.idealibrary.com on IDEAL

STOCHASTIC OPTIMAL PREVIEW CONTROL OF AN ACTIVE VEHICLE SUSPENSION

H.-S. Roh and Y. Park

Center for Noise and Vibration Control (NoViC), Department of Mechanical Engineering, Korea Advanced Institute of Science and Technology, Science Town, Taejon 305-701, *South Korea*

(*Received* 16 *April* 1998, *and in final form* 19 *August* 1998)

Preview control with an estimation scheme is investigated for an active vehicle suspension system with look-ahead sensors. Design of a preview compensator that may be called stochastic optimal, output feedback, preview regulator problem is reduced to the classical linear quadratic Gaussian problem by augmenting dynamics of the original system and previewed road inputs. The resulting solution is a combination of deterministic optimal preview controller and stochastic optimal estimator. The optimal estimator takes the form of a Kalman filter with an additional term of the estimate for the road input, which is given as the weighted preview sensor signal. The Kalman filter gain and the weight used for estimating state and road input, respectively, are designed so that performance degradation by measurement noise is minimized. Numerical examples of a quarter car model are given to verify the performance improvement achievable with the proposed preview control when the estimation from noisy measurement is considered.

1. INTRODUCTION

It is well-known that compromise between ride comfort and handling performance has to be made to design a passive suspension of a vehicle. To overcome this problem, many researchers have proposed to use active suspensions. Unlike passive systems which can only store or dissipate energy, active suspensions can continuously change the energy flow to or from the system when required. Furthermore, characteristics of active suspensions can adapt to instantaneous changes in driving conditions detected by sensors. As a result, active suspensions can improve both ride comfort and handling performance to satisfactory levels.

It has been first proposed by Bender [1] that performance of active suspension can be further improved if knowledge of the road surface in front of the actively controlled axles, i.e., preview information is used in the control strategy. With preview information, one can, for example, prepare the vehicle for a future road input and pass through abrupt road obstacles without severe impacts.

There are two possible ways to obtain preview information. One is to use a "look-ahead" sensor and the other is to estimate road profile from the response

0022–460X/99/070313 + 18 $30.00/0

of the front wheel by assuming that road inputs at the rear wheels are the same as those at the front wheels except for time delays. In this paper, the look-ahead preview control will be considered.

In vehicle suspension control, one cannot measure all the state variables for practical reasons and therefore, information on them is incomplete. All the measurement signals including the preview sensor signal, are assumed to be contaminated by sensor noises. To realize preview control in the vehicle suspension, one needs an optimal filter which will filter out the sensor noises from the preview sensor signals as well as a state estimator which will minimize estimation errors due to sensor noises. Thus, to obtain an optimal preview controller for a system with incomplete and noisy measurements one needs to solve a stochastic optimal preview regulator problem with incomplete and noisy measurements, or simply the stochastic optimal, output feedback, preview regulator problem.

Yoshimura and Edokoro [2] changed the problem into the LQG form by augmenting the dynamics of the original system and the road inputs. In their formulation the correlation between the road inputs is considered only in the estimation scheme but not in the control scheme. Their controller does not utilize the preview information estimated in the previous steps, which is said to be characteristic of preview control. Hac [3] also solved the problem by the variational approach. He derived the optimal control and estimation schemes independently and then showed that a separation principle is satisfied with his solution to prove its optimality. However, he did not consider the optimal estimation of the road input from preview sensor signals; rather, he used the delayed raw preview sensor signals as the estimated road input. Therefore, his solution is not optimal if the preview sensor signal is corrupted by measurement noise.

Louam *et al*. [4] derived the optimal preview regulator assuming availability of exact information on the state and the road inputs. They transformed the preview regulator problem into the LQR problem by introducing a state vector, whose dynamics is a combination of those of the original system and previewed road input. The resulting control input was given as the feedback input driven by the augmented state vector, which is different from the structure of the conventional preview controller consisting of feedback and feedforward parts.

In this paper, the solution for the stochastic optimal, output feedback, preview regulator problem is derived assuming incomplete and noisy measurements of the states and the road input. The problem based on the LQG framework is solved by augmenting dynamics of previewed road input with the original system dynamics. The resulting compensator is divided into the controller and the estimator part. While the controller is identical to the Louam's work, the estimator is unique in its optimality under the noisy measurement condition. The proposed compensator has high order dynamics compared with the original system, which makes it impractical to implement in real applications. The compensator structure is transformed into a combination of the feedback part with the same order of the original system and the feedforward part based on the measured road inputs only.

This novel transformation shows a relation between the conventional preview control and the present result explicitly.

The contents of the paper will be organized as follows: in the next section, an active suspension design of a quarter car into a discrete time, stochastic optimal, output feedback, preview regulator problem. In section 3, an augmented state vector is introduced to transform the problem into a well known LQG problem and then its optimal solution is derived. Numerical examples are given to verify performance of the resulting suspension in section 4. Finally in section 5, conclusions are drawn.

2. PROBLEM FORMULATION

2.1. SYSTEM MODEL

A two-degrees-of-freedom vehicle model considered in this paper is shown in Figure 1. Assuming that the characteristics of all passive suspension elements are linear, the model can be described by the following equations of motion:

$$m_s\ddot{z}_s = f, \qquad m_u\ddot{z}_u = -f - k_t(z_u - z_0), \tag{1, 2}$$

where f denotes the upward suspension force, which is given by

$$f = u - k_s(z_s - z_u) - b(\dot{z}_s - \dot{z}_u). \tag{3}$$

When the following state and disturbance vectors are introduced,

$$\mathbf{x}(t) = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}, \qquad w(t) = \dot{z}_0(t),$$

where

$$x_1 = z_s - z_u, \qquad x_2 = z_u - z_0, \qquad x_3 = \dot{z}_s, \qquad x_4 = \dot{z}_u,$$

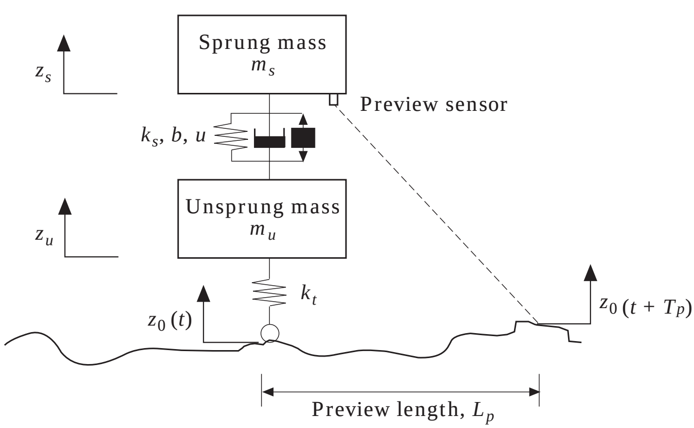

Figure 1. A quarter car model with active suspension.

equations (1) and (2) can be written as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{E}w, \tag{4}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -k_s/m_s & 0 & -b/m_s & b/m_s \\ k_s/m_u & -k_t/m_u & b/m_u & -b/m_u \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1/m_s \\ -1/m_u \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

Note that the rate of change in the road profile is an external disturbance in this model. This road input depends on two factors; (a) road roughness which is a function of the road roughness parameter G_r, and (b) vehicle velocity v. [5] Statistical properties of the road input are defined as

$$E[w(t)] = 0, \qquad \mathrm{cov}\,[w(t_1), w(t_2)] = \tilde{W}\delta(t_1 - t_2) = \pi R_r \delta(t_1 - t_2), \tag{5}$$

where $E[\cdot]$ is the mean value of $[\cdot]$, $\mathrm{cov}\,[x_1, x_2]$ is the covariance of x_1 and x_2, $\delta(\cdot)$ the Dirac delta function and R_r $(=2\pi G_r v)$ is the road roughness parameter.

Relative displacement of the suspension and the sprung mass acceleration are assumed to be measured by a relative displacement sensor and an accelerometer under the noisy environments as follows:

$$y_1 = z_s - z_u + \epsilon_1, \qquad y_2 = \ddot{z}_s + \epsilon_2, \tag{6}$$

where $\epsilon_i s$ $(i = 1, 2)$ are measurement noises existing in the corresponding measurements y_i. Using the notations defined above, equation (6) can be expressed in terms of state, input and measurement noise vectors as follows:

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x} + \mathbf{D}u + \boldsymbol{\epsilon}, \tag{7}$$

where

$$\mathbf{y}(t) = \begin{Bmatrix} y_1(t) \\ y_2(t) \end{Bmatrix}, \qquad \boldsymbol{\epsilon}(t) = \begin{Bmatrix} \epsilon_1(t) \\ \epsilon_2(t) \end{Bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -k_s/m_s & 0 & -b/m_s & b/m_s \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 0 \\ 1/m_s \end{bmatrix}.$$

One further assumes that the rate of change in the road profile is measured by a preview sensor with a preview distance of L_p,

$$y_w(t) = w(t + T_p) + \epsilon_w(t + T_p), \tag{8}$$

where ϵ_w is a measurement noise for the preview sensor and T_p $(= L_p/v)$ is a preview time. For a fixed preview distance, a preview time decreases with an increasing vehicle speed.

The measurement noises $\boldsymbol{\epsilon}$ and ϵ_w are assumed to be uncorrelated with the road input, $w(t)$,

$$\mathrm{cov}\,[\boldsymbol{\epsilon}(t_1)w(t_2)] = \mathbf{0}, \qquad \mathrm{cov}\,[\epsilon_w(t_1)w(t_2)] = 0. \tag{9}$$

Finally, the initial state $\mathbf{x}(0)$ and measurement noises $\boldsymbol{\epsilon}$ and ϵ_w are assumed to satisfy the following conditions:

$$\begin{aligned} E[\mathbf{x}(0)] &= \bar{\mathbf{x}}_0; \qquad \mathrm{cov}\,[\mathbf{x}(0), \mathbf{x}(0)] = \tilde{\mathbf{P}}_0 \geqslant \mathbf{0}, \\ E[\boldsymbol{\epsilon}(t)] &= \mathbf{0}; \qquad \mathrm{cov}\,[\boldsymbol{\epsilon}(t_1), \boldsymbol{\epsilon}(t_2)] = \tilde{\boldsymbol{\Xi}}(t_1)\delta(t_1 - t_2) > \mathbf{0}, \\ E[\epsilon_w(t)] &= 0; \qquad \mathrm{cov}\,[\epsilon_w(t_1), \epsilon_w(t_2)] = \tilde{\Xi}_w(t_1)\delta(t_1 - t_2) > 0, \end{aligned} \tag{10}$$

2.2. PROBLEM OF VEHICLE SUSPENSION DESIGN

The purpose of the active suspension is to reduce the required suspension working space and the maximum acceleration of the sprung mass, without increasing the dynamic tire force variation too much. Finally, for practical reasons, the magnitude of control force, u, is limited. These objectives can be achieved by finding the optimal input u that minimizes the following performance index:

$$J = \tfrac{1}{2}E\left[\int_0^{T_f} \{\rho_1 \ddot{z}_s^2 + \rho_2(z_s - z_u)^2 + \rho_3(z_u - z_0)^2 + \rho_4 u^2\}\,\mathrm{d}t\right], \tag{11}$$

where $\rho_i s$ $(i = 1, \ldots, 4)$ are weighting constants determined by designers. By expressing the sprung mass acceleration and the suspension deflection in terms of state, input and disturbance vectors, the performance index is written as the following quadratic form:

$$\begin{aligned} J = \tfrac{1}{2}E\Bigg[&\int_0^{T_f} \{\mathbf{x}^{\mathrm{T}}\mathbf{Q}_1\mathbf{x} + 2\mathbf{x}^{\mathrm{T}}\mathbf{N}u + u^{\mathrm{T}}\tilde{R}u + 2x^{\mathrm{T}}\mathbf{Q}_{12}w + w^{\mathrm{T}}Q_2 w\}\,\mathrm{d}t \\ &+ \mathbf{x}^{\mathrm{T}}(T_f)\mathbf{S}_1\mathbf{x}(T_f) + 2\mathbf{x}^{\mathrm{T}}(T_f)\mathbf{S}_{12}w(T_f) + w^{\mathrm{T}}(T_f)S_2 w(\mathbf{T}_f)\Bigg], \end{aligned} \tag{12}$$

where

$$\begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^{\mathrm{T}} & Q_2 \end{bmatrix} \geqslant \mathbf{0}, \qquad \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_{12} \\ \mathbf{S}_{12}^{\mathrm{T}} & S_2 \end{bmatrix} \geqslant \mathbf{0},$$

$$\begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^{\mathrm{T}} & Q_2 \end{bmatrix} - \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix} \tilde{R}^{-1}[\mathbf{N}^{\mathrm{T}} \quad 0] \geqslant \mathbf{0}, \qquad \tilde{R} > 0. \tag{13}$$

2.3. DISCRETIZATION

In this paper, a discrete time domain approach is employed. Thus, all the relevant equations and the performance index are converted into discrete equivalents. With a sampling time T_s, the discrete time representation of state and measurement equations (4), (7) and (8) will take the following form:

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k) + \mathbf{F}w(k), \qquad \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}u(k) + \boldsymbol{\epsilon}(k),$$
$$y_w(k) = w(k+N_p) + \epsilon_w(k+N_p), \tag{14}$$

where $\mathbf{G}, \mathbf{H}, \mathbf{F}$ are the discrete equivalents of $\mathbf{A}, \mathbf{B}, \mathbf{E}$ and $N_p = T_p/T_s$. Discrete equivalents of the statistical properties of the system in equations (5), (9) and (10) are given as

$$E[w(k)] = 0; \qquad \mathrm{cov}\,[w(k_1), w(k_2)] = W(k_1)\delta(k_1 - k_2),$$
$$\mathrm{cov}\,[\boldsymbol{\epsilon}(k_1)w(k_2)] = \mathbf{0}, \qquad \mathrm{cov}\,[\epsilon_w(k_1)w(k_2)] = 0,$$
$$E[\mathbf{x}(0)] = \bar{\mathbf{x}}_0; \qquad \mathrm{cov}\,[\mathbf{x}(0), \mathbf{x}(0)] = \mathbf{P}_0,$$
$$E[\boldsymbol{\epsilon}(0)] = \mathbf{0}; \qquad \mathrm{cov}\,[\boldsymbol{\epsilon}(k_1), \boldsymbol{\epsilon}(k_2)] = \boldsymbol{\Xi}(k_1)\delta(k_1 - k_2),$$
$$E[\epsilon_w(0)] = 0; \qquad \mathrm{cov}\,[\boldsymbol{\epsilon}_w(k_1), \boldsymbol{\epsilon}_w(k_2)] = \Xi_w(k_1)\delta(k_1 - k_2). \tag{15}$$

where $W(k) \approx \tilde{W}(t)/T_s$, $\Xi(k) = \tilde{\Xi}(t)/T_s$, $\Xi_w(k) = \tilde{\Xi}_w(t)/T_s$. Approximation made in obtaining discrete equivalents of the covariance matrix of the external disturbance is valid only when T_s is much smaller than the dominant time constant of the system. [6] Finally, performance index (12) can be transformed into the following discrete equivalent by using the discretization method given in [6].

$$J = \tfrac{1}{2}E\left[\begin{pmatrix}\mathbf{x}(n)\\ w(n)\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{S}_{xx}(n) & \mathbf{S}_{xw}(n)\\ \mathbf{S}_{xw}^{\mathrm{T}}(n) & S_{ww}(n)\end{bmatrix}\begin{pmatrix}\mathbf{x}(n)\\ w(n)\end{pmatrix}\right.$$
$$\left. + \sum_{i=0}^{n-1}\left\{\begin{pmatrix}\mathbf{x}(i)\\ w(i)\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{Q}_{xx} & \mathbf{Q}_{xw}\\ \mathbf{Q}_{xw}^{\mathrm{T}} & Q_{ww}\end{bmatrix}\begin{pmatrix}\mathbf{x}(i)\\ w(i)\end{pmatrix} + 2\begin{pmatrix}\mathbf{x}(i)\\ w(i)\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{M}_x\\ M_w\end{bmatrix}u(i) + u^{\mathrm{T}}(i)Ru(i)\right\}\right], \tag{16}$$

where

$$\mathbf{S} = \begin{bmatrix}\mathbf{S}_{xx}(n) & \mathbf{S}_{xw}(n)\\ \mathbf{S}_{xw}^{\mathrm{T}}(n) & S_{ww}(n)\end{bmatrix} \geqslant \mathbf{0}, \qquad \mathbf{Q} = \begin{bmatrix}\mathbf{Q}_{xx} & \mathbf{Q}_{xw}\\ \mathbf{Q}_{xw}^{\mathrm{T}} & Q_{ww}\end{bmatrix} \geqslant \mathbf{0},$$

$$\mathbf{Q} - \begin{bmatrix}\mathbf{M}_x\\ M_w\end{bmatrix}R^{-1}[\mathbf{M}_x^{\mathrm{T}} \quad M_w^{\mathrm{T}}] \geqslant \mathbf{0}; \qquad R > 0, \qquad n = T_f/T_s. \tag{17}$$

3. STOCHASTIC PREVIEW OPTIMAL OUTPUT FEEDBACK REGULATOR PROBLEM

Consider the following augmented state and output vectors

$$\mathbf{x}_a(k) = \begin{Bmatrix} \mathbf{x}(k) \\ w(k) \\ \vdots \\ w(k+N_p) \end{Bmatrix}, \qquad \mathbf{y}_a(k) = \begin{Bmatrix} \mathbf{y}(k) \\ y_w(k) \end{Bmatrix}. \tag{18}$$

Then, state and measurement equations will be changed as

$$\mathbf{x}_a(k+1) = \mathbf{G}_a\mathbf{x}_a(k) + \mathbf{H}_a u(k) + \mathbf{w}_a(k), \qquad \mathbf{y}_a(k) = \mathbf{C}_a\mathbf{x}_a(k) + \mathbf{H}_a u(k) + \boldsymbol{\epsilon}_a(k), \tag{19}$$

where

$$\mathbf{G}_a = \begin{bmatrix} \mathbf{G} & \mathbf{F} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & 0 & 1 & & 0 \\ & & & \ddots & \\ & & & & 1 \\ \mathbf{0} & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \mathbf{H}_a = \begin{bmatrix} \mathbf{H} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$w_a(k) = \begin{Bmatrix} \mathbf{0} \\ 0 \\ \vdots \\ 0 \\ w(k+N_p+1) \end{Bmatrix}$$

$$\mathbf{C}_a = \begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad \mathbf{D}_a = \begin{bmatrix} \mathbf{D} \\ 0 \end{bmatrix}, \qquad \boldsymbol{\epsilon}_a(k) = \begin{Bmatrix} \boldsymbol{\epsilon}(k) \\ \boldsymbol{\epsilon}_w(k+N_p) \end{Bmatrix}. \tag{20}$$

The initial condition $(\mathbf{x}_a)_0$ and noise vectors $\mathbf{w}_a$, $\boldsymbol{\epsilon}_a$ of the augmented system satisfy the following conditions:

$$E[(\mathbf{x}_a)_0] = \begin{Bmatrix} \bar{\mathbf{x}}_0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \qquad \operatorname{cov}[(\mathbf{x}_a)_0, (\mathbf{x}_a)_0] = (\mathbf{P}_a)_0 = \begin{bmatrix} \mathbf{P}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & 0 & \cdots & 0 \end{bmatrix} \geqslant \mathbf{0},$$

$$\operatorname{cov}[\mathbf{w}_a(k_1), \mathbf{w}_a(k_2)] = \mathbf{W}_a(k_1)\delta(k_1 - k_2) =$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & 0 & 0 \\ \mathbf{0} & 0 & 0 & \cdots & W(k_1 + N_p + 1) \end{bmatrix} \delta(k_1 - k_2) \geqslant \mathbf{0},$$

$$\operatorname{cov}[\boldsymbol{\epsilon}_a(k_1), \boldsymbol{\epsilon}_a(k_2)] = \boldsymbol{\Xi}_a(k_1)\delta(k_1 - k_2) = \begin{bmatrix} \boldsymbol{\Xi}(k_1) & \mathbf{0} \\ \mathbf{0} & \Xi_w(k_1) \end{bmatrix} \delta(k_1 - k_2) > \mathbf{0}. \quad (21)$$

The performance index in equation (16) is expressed in terms of the augmented state vector

$$J = \tfrac{1}{2} E\left[\mathbf{x}_a^{\mathrm{T}}(n)\mathbf{S}_a(n)\mathbf{x}_a(n) + \sum_{i=0}^{n-1} \{ \mathbf{x}_a^{\mathrm{T}}(i)\mathbf{Q}_a\mathbf{x}_a(i) + 2\mathbf{x}_a^{\mathrm{T}}(i)\mathbf{M}_a u(i) + u^{\mathrm{T}}(i)Ru(i) \} \right], \quad (22)$$

where

$$\mathbf{S}_a(n) = \begin{bmatrix} \mathbf{S}_{xx}(n) & \mathbf{S}_{xw}(n) & \cdots & \mathbf{0} \\ \mathbf{S}_{xw}^{\mathrm{T}}(n) & S_{ww}(n) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & 0 & \cdots & 0 \end{bmatrix} \geqslant \mathbf{0},$$

$$\mathbf{Q}_a = \begin{bmatrix} \mathbf{Q}_{xx} & \mathbf{Q}_{xw} & \cdots & \mathbf{0} \\ \mathbf{Q}_{xw}^{\mathrm{T}} & Q_{ww} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & 0 & \cdots & 0 \end{bmatrix} \geqslant \mathbf{0},$$

$$\mathbf{M}_a = \begin{bmatrix} \mathbf{M}_x \\ M_w \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathbf{Q}_a - \mathbf{M}_a R^{-1} \mathbf{M}_a^{\mathrm{T}} \geqslant \mathbf{0}. \quad (23)$$

One notes that by using the augmented state vector, the stochastic preview optimal output feedback regulator problem reduces to a classical LQG problem. It may be easily proved that the separation principle holds for the LQG problem and that its solution is the combination of a deterministic optimal state feedback regulator and a stochastic optimal estimator. [7] The optimal solution to the LQG problem is reproduced in *Theorem* 1 in Appendix A.

The optimal input given in equation (A.1) is in the form of an augmented state feedback input. It may seem to contradict the previous research results [1, 3, 9, 10], since the preview information does not make up the feedforward part in the proposed optimal input. In addition, the optimal input requires solutions of Riccati equations whose dimensions are much larger than the original system order. These contradictions can be resolved if one changes the solution in *Theorem* 1 into more convenient form, using notations in equations (18), (20) and (21).

Corollary 1. Considering notations in equations (18), (20) and (21), the optimal input in *Theorem* 1 reduces to the following form:

$$u^*(k) = -\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\}^{-1}[(\mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{G} + \mathbf{M}_x^{\mathrm{T}})\hat{\mathbf{x}}(k) + (\mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{F} + M_w^{\mathrm{T}})\hat{w}(k) + \mathbf{H}^{\mathrm{T}}\hat{\mathbf{r}}(k+1)], \quad (24)$$

where

$$(\mathbf{P}_c)_{xx}(k) = \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\hat{\mathbf{G}} + \hat{\mathbf{Q}}_{xx} - \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\}^{-1}\mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\hat{\mathbf{G}}, \quad (25)$$

$$\hat{\mathbf{r}}(k) = \hat{\mathbf{G}}^{\mathrm{T}}[\mathbf{I} - (\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\}^{-1}\mathbf{H}^{\mathrm{T}}]\hat{\mathbf{r}}(k+1) + [\hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\hat{\mathbf{F}} + \hat{\mathbf{Q}}_{xw} - \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H} \times \{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\}^{-1}\mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\hat{\mathbf{F}}]\hat{w}(k), \quad (26)$$

$$\hat{\mathbf{G}} = \mathbf{G} - \mathbf{H}R^{-1}\mathbf{M}_x^{\mathrm{T}}, \qquad \hat{\mathbf{Q}}_{xx} = \mathbf{Q}_{xx} - \mathbf{M}_x R^{-1}\mathbf{M}_x^{\mathrm{T}}, \qquad \hat{\mathbf{F}} = \mathbf{F} - \mathbf{H}R^{-1}M_w^{\mathrm{T}},$$

$$\hat{\mathbf{Q}}_{xw} = \mathbf{Q}_{xw} - \mathbf{M}_x R^{-1}\mathbf{M}_w^{\mathrm{T}}, \qquad (\mathbf{P}_c)_{xx}(n) = \mathbf{S}_{xx}(n), \qquad \hat{\mathbf{r}}(k + N_p + 1) = \mathbf{0}. \quad (27)$$

Estimates for state and preview information, $\hat{\mathbf{x}}(k)$ and $\hat{w}(k+j)$ $(j = 0, \ldots, N_p)$ are obtained from the outputs of the estimators whose structures are given as

$$\hat{\mathbf{x}}(k+1) = \mathbf{G}\hat{\mathbf{x}}(k) + \mathbf{H}u(k) + \mathbf{F}\hat{w}(k) + \mathbf{G}(\mathbf{P}_e)_{xx}(k)\mathbf{C}^{\mathrm{T}}[\mathbf{C}(\mathbf{P}_e)_{xx}(k)\mathbf{C}^{\mathrm{T}} + \mathbf{\Xi}(k)]^{-1}\{\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{x}}(k) - \mathbf{D}u(k)\}, \quad (28)$$

$$\hat{w}(k+j) = W(k+j)[W(k+j) + \Xi_w(k+j)]^{-1}y_w(k+j-N_p), \quad (29)$$

where

$$(\mathbf{P}_e)_{xx}(k+1) = \mathbf{G}(\mathbf{P}_e)_{xx}(k)\mathbf{G}^{\mathrm{T}} + \mathbf{F}[W(k) - W(k)\{W(k) + \Xi_w(k)\}^{-1}W(k)]\mathbf{F}^{\mathrm{T}} - \mathbf{G}(\mathbf{P}_e)_{xx}(k)\mathbf{C}^{\mathrm{T}}[\mathbf{C}(\mathbf{P}_e)_{xx}(k)\mathbf{C}^{\mathrm{T}} + \Xi(k)]^{-1}\mathbf{C}(\mathbf{P}_e)_{xx}(k)\mathbf{G}^{\mathrm{T}}, \quad (30)$$

$$\hat{\mathbf{x}}(0) = \bar{\mathbf{x}}_0, \qquad \hat{w}(0+j) = W(j)[W(j) + \Xi_w(j)]^{-1}y_w(j - N_p), \qquad (\mathbf{P}_e)_{xx}(0) = \mathbf{P}_0.$$

□

Proof. Details of proof are given in Appendix B.

The optimal solution in corollary 1 can be divided into 2 parts: control scheme and estimation scheme. As for control scheme, one finds from equation (24) that the optimal input does consist of a feedback and a feedforward part. The feedback input is exactly the same as the classical LQR input, and the feedforward input, made of available preview information, provides an anticipative action based on the predicted system response to the future road input. And one also notes that all the relevant equations have dimensions equal to the original system order. In fact, equations (24)–(26) are the discrete equivalents of Hac's results which is based on a variational approach. [3]

The estimation scheme consists of state and preview estimation parts. The estimator for the state variables in equation (28) have the same structure as the conventional Kalman–Bucy filter except for an additional term of the estimated road input. One notes in equation (29) that the estimate for preview information is given in the form of weighted preview sensor signal whose weighting is determined from statistical properties of the road input and the preview sensor noise. When the preview sensor noise is equal to zero, the estimate for the road input reduces to Hac's result which directly used the measured preview signal as the estimate of the road input. [3] It means that preview sensor signal will be totally trusted and used for the control scheme. As the covariance of the preview sensor noise becomes larger, the preview sensor signal will also contain erroneous information along with real road information. Compared with Hac's, the proposed estimator filters out the noise component of the preview sensor signal and improves control performance significantly.

Infinite time results ($n \to \infty$). Suppose all related parameters are time invariant, and n approaches infinity. In this case the performance index (22) may keep growing as n increases. So instead of J, its rate $J' = J/n$ is considered as a performance index. When n is finite, the optimal input which minimizes J also minimizes J'. It is assumed that $(\mathbf{G}, \mathbf{H})$ and $(\mathbf{G}, \mathbf{F})$ are stabilizable, and that $(\mathbf{G}, \hat{\mathbf{Q}}_{xx}^{1/2})$ and $(\mathbf{G}, \mathbf{C})$ are detectable.

Corollary 2. The optimal input that minimizes $J'_\infty = \lim_{n\to\infty} J/n$ is given by

$$u^*(k) = -\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}^{\infty}\mathbf{H}\}^{-1}[(\mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}^{\infty}\mathbf{G} + \mathbf{M}_x^{\mathrm{T}})\hat{\mathbf{x}}(k) + (\mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}^{\infty}\mathbf{F} + M_w^{\mathrm{T}})\hat{w}(k) + \mathbf{H}^{\mathrm{T}}\hat{\mathbf{r}}(k+1)], \tag{31}$$

where $(\mathbf{P}_c)_{xx}^{\infty}$ is the steady state solution of equation (25) and

$$\hat{\mathbf{r}}(k) = \hat{\mathbf{G}}^{\mathrm{T}}[\mathbf{I} - (\mathbf{P}_c)_{xx}^{\infty}\mathbf{H}\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}^{\infty}\mathbf{H}\}^{-1}\mathbf{H}^{\mathrm{T}}]\hat{\mathbf{r}}(k+1) + [\hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}^{\infty}\hat{\mathbf{F}} + \hat{\mathbf{Q}}_{xw} - \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}^{\infty}\mathbf{H}\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}^{\infty}\mathbf{H}\}^{-1}\mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}^{\infty}\hat{\mathbf{F}}]\hat{w}(k), \tag{32}$$

with initial condition $\hat{\mathbf{r}}(k + N_p + 1) = \mathbf{0}$.

Estimates for $\hat{\mathbf{x}}(k)$ and $\hat{w}(k+j)$ $(j = 0, \ldots, N_p)$ are obtained as:

$$\hat{\mathbf{x}}(k+1) = \mathbf{G}\hat{\mathbf{x}}(k) + \mathbf{H}u(k) + \mathbf{F}\hat{w}(k) + \mathbf{G}(\mathbf{P}_e)_{xx}^{\infty}\mathbf{C}^{\mathrm{T}}[\mathbf{C}(\mathbf{P}_e)_{xx}^{\infty}\mathbf{C}^{\mathrm{T}} + \boldsymbol{\Xi}]^{-1}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}(k) - \mathbf{D}u(k)), \tag{33}$$

$$\hat{w}(k+j) = W[W + \Xi_w]^{-1} y_w(k+j-N_p), \tag{34}$$

where $(\mathbf{P}_e)_{xx}^{\infty}$ is the steady state solution of equation (30) and

$$\hat{\mathbf{x}}(0) = \bar{\mathbf{x}}_0, \qquad \hat{w}(0+j) = W[W + \Xi_w]^{-1} y_w(j - N_p).$$

Finally, the optimal performance index rate, J'_∞ is given as

$$J'_\infty = \lim_{n\to\infty} J/n = tr[\mathbf{P}_c^\infty \mathbf{W}_a] + tr[\mathbf{P}_e^\infty (\mathbf{K}_c^\infty)^{\mathrm{T}} [\mathbf{H}_a^{\mathrm{T}} \mathbf{P}_c^\infty \mathbf{H}_a + R]\mathbf{K}_c^\infty]$$

$$= tr[(\mathbf{P}_c)_{ww}^\infty W] + tr[(\mathbf{P}_e)_{xx}^\infty (\mathbf{H}^{\mathrm{T}} (\mathbf{P}_c)_{xx}^\infty \mathbf{G} + \mathbf{M}_x^{\mathrm{T}})^{\mathrm{T}} (\mathbf{H}^{\mathrm{T}} (\mathbf{P}_c)_{xx}^\infty \mathbf{H} + R)^{-1}$$

$$(\mathbf{H}^{\mathrm{T}} (\mathbf{P}_c)_{xx}^\infty \mathbf{G} + \mathbf{M}_x^{\mathrm{T}})]$$

$$+ \, tr[(\mathbf{P}_e)_{ww}^\infty (\mathbf{H}^{\mathrm{T}} (\mathbf{P}_c)_{xx}^\infty \mathbf{F} + M_w^{\mathrm{T}})^{\mathrm{T}} (\mathbf{H}^{\mathrm{T}} (\mathbf{P}_c)_{xx}^\infty \mathbf{H} + R)^{-1} (\mathbf{H}^{\mathrm{T}} (\mathbf{P}_c)_{xx}^\infty \mathbf{F} + M_w^{\mathrm{T}})]$$

$$+ \sum_{j=1}^{N_p} tr[(\mathbf{P}_e)_{ww}^\infty ((\mathbf{P}_c)_{xw_j}^\infty)^{\mathrm{T}} \mathbf{H} (\mathbf{H}^{\mathrm{T}} (\mathbf{P}_c)_{xx}^\infty \mathbf{H} + R)^{-1} \mathbf{H}^{\mathrm{T}} (\mathbf{P}_c)_{xw_j}^\infty], \tag{35}$$

where $(\mathbf{P}_c)_{xw_j}^\infty$ is the steady state solution of (A.12). □

When deriving the optimal solution, the specific value for the vehicle speed was assumed. Because a vehicle is supposed to operate under a wide range of speeds, it is important to know how the vehicle speed affects control and estimation schemes. Depending on the speed, the following two parameters will change: one is the covariance of road input $W(k)$ and the other is preview step N_p. The covariance $W(k)$ is proportional to vehicle speed and is related to the signal to noise ratio of the preview sensor. As $W(k)$ increases, the preview sensor signal is more trusted as seen in equation (29). And the state estimator will trust system modeling more than the remaining sensor signals in equation (30) because modelling uncertainty due to road input estimation error $F[W(k) - W(k)\{W(k) + \Xi_w(k)\}^{-1} W(k)]F^{\mathrm{T}}$ reduces with increase of $W(k)$.

The preview step N_p is inversely proportional to vehicle speed. By the initial condition $\hat{\mathbf{r}}(k + N_p + 1) = 0$, it determines the amount of road information to be used in feedforward control action. As vehicle speed increases, N_p decreases and so does the number of iterating equation (26) to get $\hat{\mathbf{r}}(k+1)$ from its initial condition.

In summary, if the vehicle speed varies, one needs to modify the structures of both the controller and the estimator. In order to implement preview control in real applications the scheduling technique to deal with these changes will be required.

4. NUMERICAL EXAMPLES

In this section, the results of the previous section are applied to the vehicle suspension described in section 2. The parameter values used in simulation are given as $m_s = 250$ kg, $m_u = 30$ kg, $k_s = 10\,000$ N/m, $k_u = 100\,000$ N/m, $b = 1000$ Ns/m. They correspond to a compact sedan. [3] For road roughness parameter G_r, a characteristic value $G_r = 5{\cdot}0 \times e^{-6}$ m cycle for a paved road will

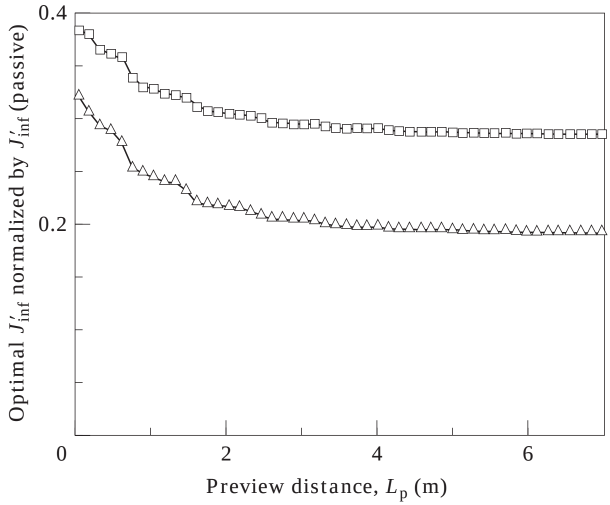

Figure 2. Optimal $J'_{\rm inf}$ for a varying preview distance, L_p. Key: —△—, state feedback preview; —□—, output feedback preview.

be used. [5] A vehicle is assumed to be running straight ahead with a speed, 15 m/s (= 54 km/h). Sampling time is set to 10 ms, considering that all the modal frequencies of the system is less than 10 Hz.

For the performance index defined in equation (11), the following weighting factors were used: $\rho_1 = 300$, $\rho_2 = 2$, $\rho_3 = 2$, $\rho_4 = 1{\cdot}0e - 4$. It corresponds to the controller design whose main objective is to improve ride comfort. As for measurement variables, one assumes that the acceleration of the sprung mass and the suspension deflection are available. A preview sensor is assumed to measure the road input, 2·0 m ahead of the vehicle. Covariance matrices of measurement noises are given as

$$\tilde{\boldsymbol{\Xi}}(t) = \mu \begin{bmatrix} 4{\cdot}76e^{-4} & 0 \\ 0 & 2{\cdot}50 \cdot e^{-1} \end{bmatrix}, \qquad \tilde{\Xi}_w(t) = (5{\cdot}06e^{-4})\mu_w,$$

which are selected so that signal to noise ratios become 5 dB when μ and μ_w are equal to 1. Based on the parameters given above, a steady state, stochastic optimal, output feedback, preview regulator has been designed.

In Figure 2, the effect of a preview distance on performance of optimal preview controller is shown. The steady state, optimal performance index rate for an output feedback optimal preview controller is plotted with respect to the preview distance when μ and μ_w are equal to 1. For comparison purposes, the results for the case when complete and noise-free state and preview information are available, i.e., a state and preview feedback optimal preview controller are also included. Both of them are normalized by the performance index rate achievable with the passive suspension. By comparing two rates, one can observe performance deterioration due to measurement noises. As the preview distance increases, performance of both controllers improves. But when the preview distance is larger than 4·0 m, the performance improves very slowly and the optimal rates converge to certain values. As expected, the performance of output feedback optimal preview controller is worse than that of the state and preview feedback controllers due to estimation errors. One notes that the difference in performance of the two control schemes becomes more evident for a longer preview distance.

Figure 3 shows the influence of the covariance of preview sensor noise on the steady state performance of the output feedback optimal preview controller. In equation (29), an increase of the sensor noise covariance results in a decrease of the estimator gain by which the preview sensor signal is multiplied in its estimation procedure. With a decreasing gain, estimates for preview information will also decrease, which, in turns, make the feedforward part of the optimal input small for a given controller gain. In Figure 3 one finds that as μ_w increases from 0·01 to 100, the optimal performance index rate of the preview controller approaches that of the LQG controller which does not use preview information in control strategy. It means that if the preview sensor signal is too noisy, the feedforward part of preview control will have no effect and therefore, its feedback part will work just as the LQG controller.

By assuming that a preview controller has already been designed to optimize the performance index rate for a specific speed, 15 m/s, one can investigate how it will work when the actual speed of the vehicle is different from the nominal speed. In Figure 4(a), the performance index rates of a nominal preview and an optimal preview controller are plotted with respect to the vehicle speed, v. As pointed out in the previous section, vehicle speed affects both control and estimation schemes in terms of N_p and $W(k)$, respectively. The difference between actual and nominal values of vehicle speed will lead to performance deterioration of preview control. In the figure the performance index rate of the nominal preview control deteriorates from that of the optimal control as the vehicle speed deviates from its nominal value, 15 m/s. The sudden increase in the performance index rate of the nominal preview control occurs when the preview step N_p varies from one value to another with the vehicle speed. Compared with this, performance degradation by inaccurate $W(k)$ is negligible. Therefore one concludes that when

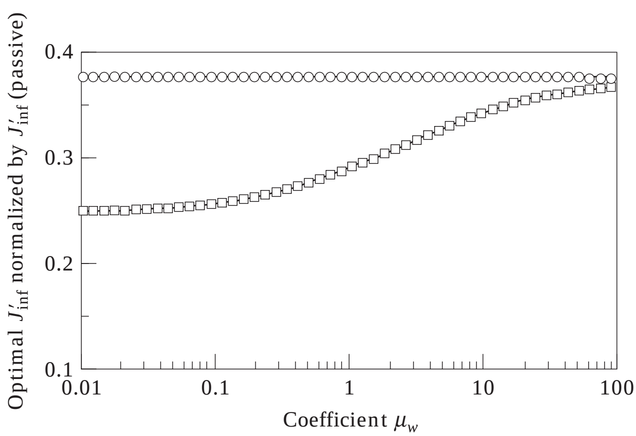

Figure 3. Optimal J'_{inf} for a varying covariance of a preview sensor noise. Key: —□—, output feedback preview; —○—, LQG.

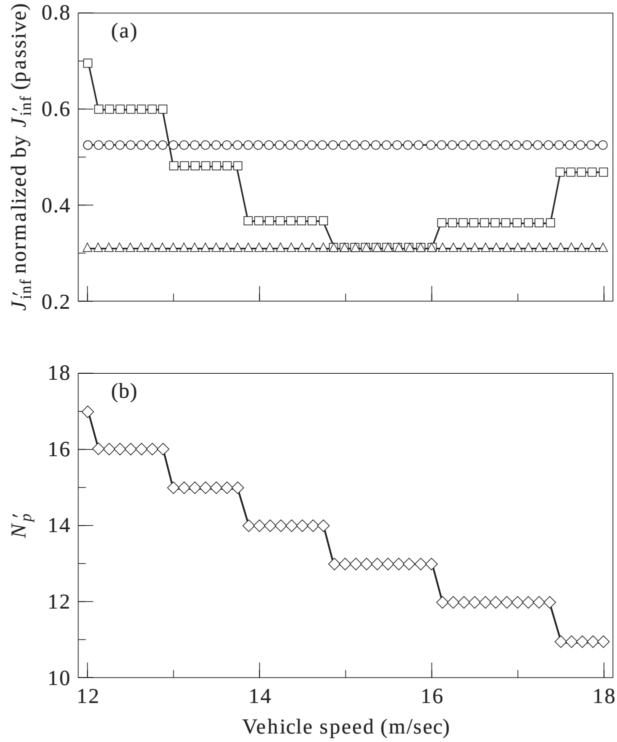

Figure 4. (a) J'_{inf} and (b) N_P for varying vehicle speed, v. Key: —○—, LQG control; —□—, nominal preview control; —△—, optimal preview control.

the velocity information is not exact, the vehicle will experience performance deterioration mainly due to inaccurate estimation of the preview step.

One also observes that the preview controller designed for the vehicle speed of 15 m/s can be used for the speeds ranging from 14 m/s–17 m/s without severe performance deterioration. A set of speeds may be selected so that the preview controllers based on these speeds can be used over the whole speed range. It will greatly reduce computational burden required to modify control and estimation schemes continuously for varying speed.

5. CONCLUSIONS

In this paper, the solution of the stochastic optimal, output feedback, preview regulator problem for active vehicle suspension control has been derived. By augmenting the dynamics of the original system and the previewed road inputs, the problem is shown to be equivalent to the classical linear quadratic Gaussian problem. The LQG solution of the augmented system can be written in the conventional preview compensator form with feedback and feedforward parts. It was found based on the structure of the proposed compensator that as the covariance of the preview sensor noise increases, the preview sensor signal is less trusted in the road input estimation while the remaining measurement signals are more trusted in the state estimation. The sensitivity of the control and estimation scheme with respect to speed variation was also investigated through numerical simulations of a quarter car model. It was shown that miscalculation of the preview step due to the inaccurate vehicle velocity information could be the major source of performance degradation of the preview control, while the miscalculated estimator gain due to it has minimal effect on the performance.

REFERENCES

1. E. K. Bender 1968 *Journal of Basic Engineering Series D* **90**, 213–221. Optimum linear preview control with application to vehicle suspension.
2. T. Yoshimura and K. Edokoro 1993 *Journal of Sound and Vibration* **166**, 507–519. An active suspension model for rail/vehicle systems with preview and stochastic optimal control.
3. A. Hac 1992 *Vehicle System Dynamics* **21**, 167–195. Optimal linear preview control of active vehicle suspension.
4. N. Louam, D. A. Wilson and R. S. Sharp 1988 *Vehicle System Dynamics* **17**, 317–336. Optimal control of a vehicle suspension incorporating the time delay between front and rear wheel inputs.
5. E. M. Elbeheiry and D. C. Karnopp 1996 *Journal of Sound and Vibration* **189**, 547–564. Optimal control of vehicle random vibration with constrained suspension deflection.
6. G. F. Franklin, J. D. Powell and M. L. Workman 1990 *Digital Control of Dynamic Systems*. Massachusetts: Addison-Wesley; second edition.
7. K. J. Åström 1970 *Introduction to Stochastic Control Theory*. San Diego: Academic Press.
8. M. Tomizuka 1976 *ASME Journal of Dynamic Systems, Measurement and Control* **98**, 309–315. Optimum linear preview control with application to vehicle suspension-revisited.
9. A. Hac and I. Youn 1993 *ASME Journal of Vibration and Acoustics* **115**, 498–508. Optimal design of active and semi-active suspensions including time delays and preview.
10. R. G. M. Huisman 1994 *PhD dissertation, Eindhoven University of Technology, Department of Mechanical Engineering*. A controller and observer for active suspensions with preview.

APPENDICES

A. *Theorem* 1

The optimal solution of the LQG problem is the same as the solution to the linear quadratic regulator(LQR) problem except that in the control law the state $\mathbf{x}_a(k)$ is replaced with its estimate $\hat{\mathbf{x}}_a(k)$. The optimal input, $u^*(k)$ is given by

$$u^*(k) = -\mathbf{K}_c(k)\hat{\mathbf{x}}_a(k); \qquad k \geqslant 0, \tag{A.1}$$

where $\mathbf{K}_c$ is the gain matrix for the optimal regulator such as

$$\mathbf{K}_c(k) = [R + \mathbf{H}_a^{\mathrm{T}}\mathbf{P}_c(k+1)\mathbf{H}_a]^{-1}\{\mathbf{H}_a^{\mathrm{T}}\mathbf{P}_c(k+1)\mathbf{G}_a + \mathbf{M}_a^{\mathrm{T}}\}, \tag{A.2}$$

and matrix $\mathbf{P}_c(k)$ satisfies the discrete Riccati equation

$$\begin{aligned}\mathbf{P}_c(k) = {} & \hat{\mathbf{G}}_a^{\mathrm{T}}\mathbf{P}_c(k+1)\hat{\mathbf{G}}_a + \hat{\mathbf{Q}}_a \\ & - \hat{\mathbf{G}}_a^{\mathrm{T}}\mathbf{P}_c(k+1)\mathbf{H}_a[R + \mathbf{H}_a^{\mathrm{T}}\mathbf{P}_c(k+1)\mathbf{H}_a]^{-1}\mathbf{H}_a^{\mathrm{T}}\mathbf{P}_c(k+1)\hat{\mathbf{G}}_a,\end{aligned} \tag{A.3}$$

with initial condition $\mathbf{P}_c(n) = \mathbf{S}_a(n)$ and $\hat{\mathbf{G}}_a = \mathbf{G}_a - \mathbf{H}_a R^{-1}\mathbf{M}_a^{\mathrm{T}}$, $\hat{\mathbf{Q}}_a = \mathbf{Q}_a - \mathbf{M}_a R^{-1}\mathbf{M}_a^{\mathrm{T}}$. $\hat{\mathbf{x}}_a(k)$ is the conditional mean of $\mathbf{x}_a(k)$ given $\mathbf{y}_a(j)$, $0 \leqslant j \leqslant k$; $\hat{\mathbf{x}}_a(k)$ can be obtained as the output of the optimal estimator described as

$$\hat{\mathbf{x}}_a(k+1) = \mathbf{G}_a\hat{\mathbf{x}}_a(k) + \mathbf{H}_a u(k) + \mathbf{K}_e(k)\{\mathbf{y}_a(k) - \hat{\mathbf{y}}_a(k)\},$$
$$\hat{\mathbf{y}}_a(k) = \mathbf{C}_a\hat{\mathbf{x}}_a(k) + \mathbf{D}_a u(k). \tag{A.4}$$

The optimal estimator gain $\mathbf{K}_e(k)$

$$\mathbf{K}_e(k) = \mathbf{G}_a\mathbf{P}_e(k)\mathbf{C}_a^{\mathrm{T}}[\mathbf{C}_a\mathbf{P}_e(k)\mathbf{C}_a^{\mathrm{T}} + \mathbf{\Xi}_a(k)]^{-1}, \tag{A.5}$$

is obtained from the discrete Riccati equation,

$$\mathbf{P}_e(k+1) = \mathbf{G}_a\mathbf{P}_e(k)\mathbf{G}_a^{\mathrm{T}} + \mathbf{W}_a(k) - \mathbf{G}_a\mathbf{P}_e(k)\mathbf{C}_a^{\mathrm{T}}[\mathbf{C}_a\mathbf{P}_e(k)\mathbf{C}_a^{\mathrm{T}} + \mathbf{\Xi}_a(k)]^{-1}\mathbf{C}_a\mathbf{P}_e(k)\mathbf{G}_a^{\mathrm{T}}, \tag{A.6}$$

with the initial condition $\mathbf{P}_e(0) = (\mathbf{P}_a)_0$. The minimal performance index achievable with the optimal control input (A.1) is given as

$$\min_u E[J] = (\bar{\mathbf{x}}_a)_0^{\mathrm{T}}\mathbf{P}_c(0)(\bar{\mathbf{x}}_a)_0 + tr[\mathbf{P}_c(0)\mathbf{P}_e(0)] + \sum_{i=0}^{n-1} tr[\mathbf{P}_c(i+1)\mathbf{W}_a] + \sum_{i=0}^{n-1} tr[\mathbf{P}_e(i)\mathbf{K}_c^{\mathrm{T}}(i)[\mathbf{H}_a^{\mathrm{T}}\mathbf{P}_c(i+1)\mathbf{H}_a + R]\mathbf{K}_c(i)]. \tag{A.7}$$

B. *Proof of corollary* 1

If one assumes that the solution to Riccati equation (A.3) takes the following form,

$$\mathbf{P}_c(k) = \begin{bmatrix} (\mathbf{P}_c)_{xx}(k) & (\mathbf{P}_c)_{xw_0}(k) & \cdots & (\mathbf{P}_c)_{xw_{N_p}}(k) \\ (\mathbf{P}_c)_{xw_0}^{\mathrm{T}}(k) & (P_c)_{w_0w_0}(k) & \cdots & (P_c)_{w_0w_{N_p}}(k) \\ \vdots & \vdots & \ddots & \vdots \\ (\mathbf{P}_c)_{xw_{N_p}}^{\mathrm{T}}(k) & (P_c)_{w_0w_{N_p}}^{\mathrm{T}}(k) & \cdots & (P_c)_{w_{N_p}w_{N_p}}(k) \end{bmatrix}, \tag{A.8}$$

then the optimal input (A.1) can be rewritten as:

$$u(k) = -\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\}^{-1} \times \Bigg[\{\mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)G + \mathbf{M}_x^{\mathrm{T}}\}\hat{x}(k) + \{\mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{F} + \mathbf{M}_w^{\mathrm{T}}\}\hat{w}(k) + \mathbf{H}^{\mathrm{T}}\sum_{j=1}^{N_p}(\mathbf{P}_c)_{xw_{j-1}}(k+1)\hat{w}(k+j)\Bigg]. \tag{A.9}$$

Substituting (20) and (A.8) into equation (A.3), one can easily prove that subblocks $(\mathbf{P}_c)_{xx}(k)$ and $(\mathbf{P}_c)_{xw_j}(k)$ ($j = 1, \ldots, N_p - 1$) of $\mathbf{P}_c(k)$ satisfy the following recursive equations:

$$(\mathbf{P}_c)_{xx}(k) = \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\hat{\mathbf{G}} + \hat{\mathbf{Q}}_{xx} - \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\}^{-1}\mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\hat{\mathbf{G}}, \tag{A.10}$$

$$(\mathbf{P}_c)_{xw_0}(k) = \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\hat{\mathbf{F}} + \hat{\mathbf{Q}}_{xw} - \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\}^{-1}\mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\hat{\mathbf{F}}, \tag{A.11}$$

$$(\mathbf{P}_c)_{xw_j}(k) = [\hat{\mathbf{G}}^{\mathrm{T}} - \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\}^{-1}\mathbf{H}^{\mathrm{T}}] \times (\mathbf{P}_c)_{xw_{j-1}}(k+1). \tag{A.12}$$

Using the equations given above, the summation in equation (A.9) can be written

$$\sum_{j=1}^{N_p} (\mathbf{P}_c)_{xw_{j-1}}(k+1)\hat{w}(k+j) = (\mathbf{P}_c)_{xw_0}(k+1)\hat{w}(k+1) + \sum_{j=2}^{N_p}\prod_{i=2}^{j}[\hat{\mathbf{G}}^{\mathrm{T}} - \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+i)\mathbf{H}\{R + \mathbf{H}^{\mathrm{T}} \times (\mathbf{P}_c)_{xx}(k+i)\mathbf{H}\}^{-1}\mathbf{H}^{\mathrm{T}}](\mathbf{P}_c)_{xw_0}(k+j)\hat{w}(k+j). \tag{A.13}$$

If one defines $\hat{\mathbf{r}}(k)$ as the vector satisfying the following recursive equation

$$\begin{aligned}\hat{\mathbf{r}}(k) &= [\hat{\mathbf{G}}^{\mathrm{T}} - \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\}^{-1}\mathbf{H}^{\mathrm{T}}]\hat{\mathbf{r}}(k+1) \\ &\quad + (\mathbf{P}_c)_{xw_0}(k+1)\hat{w}(k) \\ &= [\hat{\mathbf{G}}^{\mathrm{T}} - \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\{R + H^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\}^{-1}\mathbf{H}^{\mathrm{T}}]\hat{\mathbf{r}}(k+1) \\ &\quad + [\hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\hat{\mathbf{F}} + \hat{\mathbf{Q}}_{xw} \\ &\quad - \hat{\mathbf{G}}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\mathbf{H}\{R + \mathbf{H}^{\mathrm{T}}(\mathbf{P}_c)_{xx}(k+1)\}^{-1}\mathbf{H}^{\mathrm{T}} \\ &\quad \times (\mathbf{P}_c)_{xx}(k+1)\hat{\mathbf{F}}]\hat{w}(k),\end{aligned} \tag{A.14}$$

with an initial condition $\hat{\mathbf{r}}(k + N_p + 1) = \mathbf{0}$, the summation can be replaced with the single vector $\hat{\mathbf{r}}(k+1)$. This completes the proof of the controller part in the corollary.

In the meantime, if one assumes the similar block elements for the Riccati solution for the estimation $P_e(k)$ and substitute this into (A.6), one can easily show that the Riccati solution reduces to the following block diagonal matrix:

$$\mathbf{P}_e(k) = \begin{bmatrix} (\mathbf{P}_e)_{xx}(k) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & (P_e)_{w_0w_0}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & 0 & \cdots & (P_e)_{w_{N_p}w_{N_p}}(k) \end{bmatrix}. \tag{A.15}$$

Using this, the optimal estimator gain can be written as

$$\mathbf{K}_e(k) = \begin{bmatrix} \mathbf{G}(\mathbf{P}_e)_{xx}(k)\mathbf{C}^{\mathrm{T}}\{\mathbf{C}(\mathbf{P}_e)_{xx}(k)\mathbf{C}^{\mathrm{T}} + \boldsymbol{\Xi}(k)\}^{-1} & 0 \\ \vdots & \vdots \\ 0 & W(k)\{W(k) + \boldsymbol{\Xi}_w(k)\}^{-1} \\ 0 & 0 \end{bmatrix}. \tag{A.16}$$

Substituting this into the estimator dynamics in (A.4) produces

$$\begin{aligned} \hat{\mathbf{x}}(k+1) = \mathbf{G}\hat{\mathbf{x}}(k) + \mathbf{H}u(k) + \mathbf{F}\hat{w}_0(k) + \mathbf{G}(\mathbf{P}_e)_{xx}(k)\mathbf{C}^{\mathrm{T}}[\mathbf{C}(\mathbf{P}_e)_{xx}(k)\mathbf{C}^{\mathrm{T}} + \boldsymbol{\Xi}]^{-1} \\ \times (\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{x}}(k) - \mathbf{D}u(k)) \end{aligned}$$

$$\begin{aligned} \hat{w}_0(k+1 &= \hat{w}_1(k), \\ \vdots &= \vdots \\ \hat{w}_{N_p-1}(k+1) &= W[W + \Xi_w]^{-1}y_w(k), \\ \hat{w}_{N_p}(k+1) &= 0. \end{aligned} \tag{A.17}$$

Using notation in (18), one can obtain the estimator given in (28) and (29) and this completes the proof.