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# FUZZY MODAL FINITE ELEMENT ANALYSIS OF STRUCTURES WITH IMPRECISE MATERIAL PROPERTIES

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This paper extends the fuzzy set theory to a dynamic finite element analysis of engineering systems which have uncertainties in material properties. A general algorithm which resolves the uncertain eigenvalue problem by using a reanalysis approach is considered. This algorithm is applied to the study of the modal behaviour of structures presenting uncertain material properties. Some indexes which determine the more sensitive eigenvalue to several uncertainty sources are also put forward. Finally, a plate structure as numerical path-test is analysed. The results of such a calculation determine the sensitivity of the modal behaviour to multiple simultaneous material parameters.



## 1. INTRODUCTION

The accurate description of a system is a reciprocal function of its complexity. The number of components, the relationships between them and the difficulty in defining their specifications come into play. The need to study or manage such systems means that vague, imprecise, uncertain data must be taken into account.

Uncertainty has so far been tackled by the probability notion. However, this does not solve the problem generated by imprecise or vague knowledge. This has only been taken into consideration with the notion of fuzzy sets. This notion has arisen from the idea of partial membership of a class, a category with ill-defined boundaries, the gradual move in the passage from one situation to another, a generalisation of the classical theory of a set allowing the intermediate situation between everything and nothing. The developments of this concept supply the means to represent and handle inaccurately described, vague or imprecise knowledge. They also establish an interface between the data described symbolically and numerically. The fuzzy logic makes it possible to work with such knowledge. As to the theory of possibilities, it forms the approved framework to treat concepts of non-probabilistic uncertainty, and to exploit, in a same formalism, imprecision and uncertainty.

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Intuitively, probabilities are more similar to a degree of frequency or plausibility when possibilities are associated to our perception of a degree of feasibility or easiness of realisation.

Many factors are the cause of uncertainty or imprecision in structural analysis. They are related either to exogenous factors, such as boundary conditions or applied loads, or to endogenous factors, such as mechanical or geometric characteristics. These uncertainties have necessarily some repercussions on the structural behaviour. Generally speaking, it can be assumed that each structure reacts to different types of uncertainty.

There is no suitable technique available for the analysis of all types of imprecision in structural analysis. The stochastic finite element method can be used to handle uncertain parameters which are described by probability distributions. The stochastic finite element method was developed in the 1980s to account for uncertainties in the system parameters, geometry and external actions. The uncertain quantities were modelled as random variables with known characteristics [1–8].

Many papers have discussed the application of the fuzzy set theory to structural design [9, 10] and in particular in structural optimisation [11–13], in random vibration with application to aseismic structures [14] and in finite element analysis of engineering systems containing vague information, such as boundary conditions, implying prescribed displacements in static analysis [15, 16]. Some methodologies have been proposed for uncertainty concerning mechanical parameters [17–20]. However, some of these methodologies use optimization algorithms, by definition iterative, and are therefore numerically expensive for large systems. Others use the extension principle and are therefore difficult to program for larger finite element models.

This paper extends the fuzzy set theory to a dynamic finite element analysis of engineering systems which have uncertainties in material properties.

This methodology starts with the basic concepts of fuzzy. A general algorithm to resolve the uncertain eigenvalue problem using a reanalysis method is proposed. This algorithm, which can be used for several problems, is applied here to the study of the modal behaviour of structures presenting uncertain material parameters. Some indexes which determine the more sensitive eigenvalue for several uncertainty sources are also indicated. Finally, a plate structure is analysed as a numerical path-test.

## 2. TAKING UNCERTAINTY IN FINITE ELEMENT ANALYSIS INTO ACCOUNT

### 2.1. FUZZY FINITE ELEMENT FORMULATION

The case of uncertainties in material properties is discussed here. According to the fuzzy set theory, the two elastic parameters: Young's modulus $E$ and Poisson's ratio $\nu$, and the density $\rho$, are represented as fuzzy numbers. The information about the values of these parameters in a particular situation is imprecise, vague and unclear. The membership functions are expressed by the equations proposed by Valliapan and Pham [21]. These functions are then discretized by different intervals which are linked to an $\alpha$-cut (Figure 1) ranging from 1 to 0.

The confidence level is now at $\alpha$. Each of the three fuzzy numbers $\tilde{E}$, $\tilde{v}$ and $\tilde{\rho}$ is then represented by an interval of confidence associated to this degree of confidence $\alpha$: $\tilde{E}^{\alpha} = [E_L^{\alpha}; E_R^{\alpha}]$, $\tilde{v}^{\alpha} = [v_L^{\alpha}; v_R^{\alpha}]$ and $\tilde{\rho}^{\alpha} = [\rho_L^{\alpha}; \rho_R^{\alpha}]$, where the subscripts $L$ and $R$ stand for the left and the right, respectively.

In the deterministic finite element analysis, the eigenvalue equation is written as:

$$[\mathbf{K}]\{\mathbf{\Phi}\}_i = \lambda_i[\mathbf{M}]\{\mathbf{\Phi}\}_i, \quad i = 1, \ldots, N\text{mode} \tag{1}$$

with $\{\mathbf{\Phi}\}_i^T[\mathbf{M}]\{\mathbf{\Phi}\}_i = [\mathbf{I}]$, for example.

The elementary stiffness matrix formulation is:

$$[\mathbf{K}^e] = \int_{V^e} [\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]\,\mathrm{d}V^e, \tag{2}$$

where $[\mathbf{B}]$ is the strain–displacement matrix and $[\mathbf{D}]$ is the stress–strain or constitutive matrix. In the case of plane–stress idealization with an isotropic material this constitutive matrix is:

$$[\mathbf{D}] = \frac{\mathrm{E}}{1 - v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1 - v)/2 \end{bmatrix}. \tag{3}$$

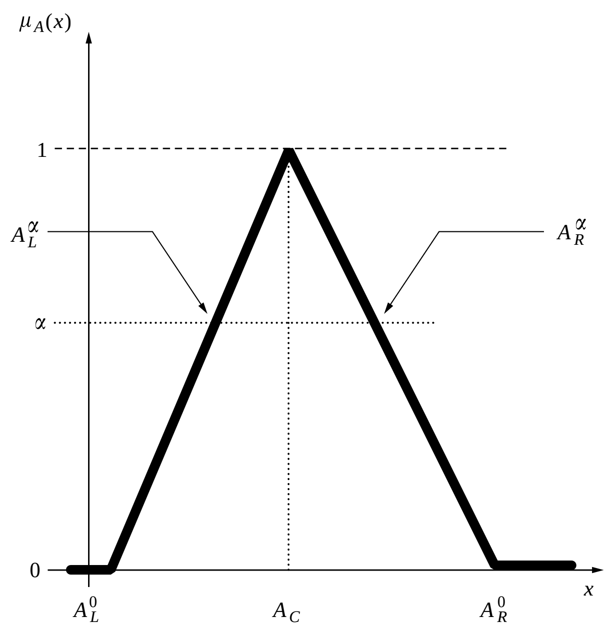


Figure 1. A Triangular Fuzzy Number (TFN) and an $\alpha$-cut $\alpha$.

The elementary mass matrix formulation is:

$$[\mathbf{M}^e] = \int_{V^e} \rho [\mathbf{H}]^T [\mathbf{H}] \, \mathrm{d}V^e, \tag{4}$$

where $[\mathbf{H}]$ is the displacement interpolation matrix and $\rho$ is the density.

After the assembly stage and in agreement with the arithmetic of fuzzy numbers [22], the lower and upper boundaries of the global stiffness matrix can be obtained by fuzzifying $E$ and $v$, therefore $[\mathbf{D}]$, at an $\alpha$-level cut as:

$$[\mathbf{K}]^\alpha = [[\mathbf{K}]^\alpha_L; [\mathbf{K}]^\alpha_R] = \int_V [\mathbf{B}^T][\mathbf{D}]^\alpha_L; [\mathbf{D}]^\alpha_R][\mathbf{B}] \, \mathrm{d}V. \tag{5}$$

In the same way, the lower and upper boundaries of the global mass matrix can be obtained by fuzzifying $\rho$ at an $\alpha$-level cut as:

$$[\mathbf{M}]^\alpha = [[\mathbf{M}]^\alpha_L; [\mathbf{M}]^\alpha_R] = \int_V [\rho^\alpha_L; \rho^\alpha_R][\mathbf{H}]^T [\mathbf{H}] \, \mathrm{d}V. \tag{6}$$

At an $\alpha$-level cut the eigenvalue equation becomes:

$$[\mathbf{K}]^\alpha \{\mathbf{\Phi}\}^\alpha_i = \lambda^\alpha_i [\mathbf{M}]^\alpha \{\mathbf{\Phi}\}^\alpha_i, \quad i = 1, \ldots, N\text{mode} \tag{7}$$

with $\lambda^\alpha_i = [\lambda^\alpha_{iL}; \lambda^\alpha_{iR}]$ and $\{\mathbf{\Phi}\}^\alpha_i = \{\mathbf{\Phi}\}^\alpha_{iL}; \{\mathbf{\Phi}\}^\alpha_{iR}]$.

### 2.2. MODAL ANALYSIS

A fuzzy eigenvector problem can be regarded as a special case of a linear fuzzy equations' system. Although the subject has been extensively studied, practical methods are available only in special cases where the number of equations is very small. A numerical algorithm which is available to solve large fuzzy eigenvector problems is presented in reference [23], but it is less effective in terms of results and expensive in calculation time.

A specific algorithm has been developed which solves the eigenvalue equation (7) using a perturbation method. The different steps of this algorithm are as follows.

For the first $\alpha$-level cut $\alpha = 1$ (see Figure 2):

$$E^1_L = E^1_R = E_C, \, v^1_L = v^1_R = v_C \text{ and } \rho^1_L = \rho^1_R = \rho_C.$$

So

$$[\mathbf{K}]^1_L = [\mathbf{K}]^1_R = [\mathbf{K}]_C \text{ and } [\mathbf{M}]^1_L = [\mathbf{M}]^1_R = [\mathbf{M}]_C.$$

The following deterministic system is again found:

$$[\mathbf{K}]_C \{\mathbf{\Phi}\}_{iC} = \lambda_{iC} [\mathbf{M}]_C \{\mathbf{\Phi}\}_{iC}, \quad i = 1, \ldots, N\text{mode}, \tag{8}$$

with $\{\mathbf{\Phi}\}^T_{iC} [\mathbf{M}]_C \{\mathbf{\Phi}\}_{iC} = [\mathbf{I}]$, which has to be solved to obtain the eigensolution $(\lambda_{iC}, \{\mathbf{\Phi}\}_{iC})_{i=1,N\text{mode}}$.

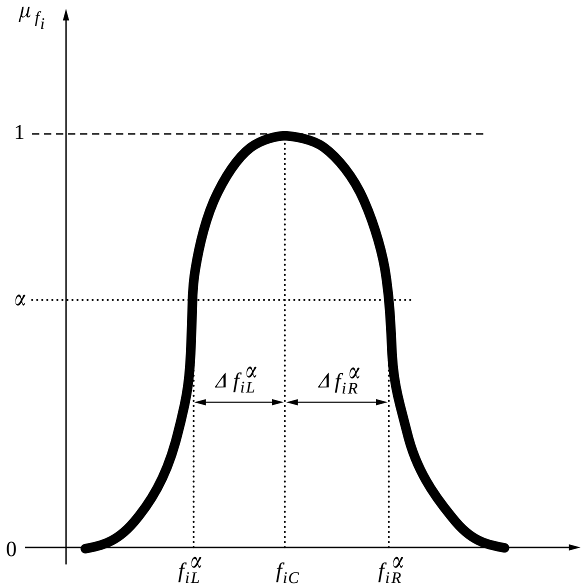


Figure 2. The $i$th fuzzy frequency.

For $\alpha \notin [0; 1]$:

$$[\mathbf{K}]^{\alpha}\{\mathbf{\Phi}\}_i^{\alpha} = \lambda_i^{\alpha}[\mathbf{M}]^{\alpha}\{\mathbf{\Phi}\}_i^{\alpha}, \quad \mathrm{i} = 1, \cdots, N\mathrm{mode}. \tag{9}$$

Let

$$[\Delta\mathbf{K}]^{\alpha} = [[\mathbf{K}]_L^{\alpha} - [\mathbf{K}]_C; [\mathbf{K}]_R^{\alpha} - [\mathbf{K}]_C] \quad \text{and} \quad [\Delta\mathbf{M}]^{\alpha} = [[\mathbf{M}]_L^{\alpha} - [\mathbf{M}]_C; [\mathbf{M}]_R^{\alpha} - [\mathbf{M}]_C]. \tag{10}$$

The interval stiffness and mass matrices are then for a given $\alpha$:

$$\begin{cases} [\mathbf{K}]^{\alpha} = [\mathbf{K}]_C + [\Delta\mathbf{K}]^{\alpha} \\ [\mathbf{M}]^{\alpha} = [\mathbf{M}]_C + [\Delta\mathbf{M}]^{\alpha}. \end{cases}$$

If $[\Delta\mathbf{K}]^{\alpha}$ and $[\Delta\mathbf{M}]^{\alpha}$ are considered to be the perturbations of $[\mathbf{K}]_C$ and $[\mathbf{M}]_C$, respectively, the eigenvalue problem can be solved by the following sensitivity method [24]:

$$\lambda_1^{\alpha} = \lambda_{iC} + \Delta\lambda_i^{\alpha}, \quad \{\mathbf{\Phi}\}_i^{\alpha} = \{\mathbf{\Phi}\}_{iC} + \{\Delta\mathbf{\Phi}\}_i^{\alpha}. \tag{11}$$

From equation (9) one gets:

$$([\mathbf{K}]_C + [\Delta\mathbf{K}]^{\alpha})(\{\mathbf{\Phi}\}_{iC} + \{\Delta\mathbf{\Phi}\}_i^{\alpha}) = (\lambda_{iC} + \Delta\lambda_i^{\alpha})([\mathbf{M}]_C + [\Delta\mathbf{M}]^{\alpha})(\{\mathbf{\Phi}\}_{iC} + \{\Delta\mathbf{\Phi}\}_i^{\alpha}), \tag{12}$$

and finally:

$$
\begin{aligned}
&[\mathbf{K}]_C\{\boldsymbol{\Phi}\}_{iC} + [\Delta\mathbf{K}]^{\alpha}\{\boldsymbol{\Phi}\}_{iC} + [\mathbf{K}]_C\{\Delta\boldsymbol{\Phi}\}_i^{\alpha} + [\Delta\mathbf{K}]^{\alpha}\{\Delta\boldsymbol{\Phi}\}_i^{\alpha} \\
&\quad = \lambda_{iC}[\mathbf{M}]_C\{\boldsymbol{\Phi}\}_{iC} + \lambda_{iC}[\Delta\mathbf{M}]^{\alpha}\{\boldsymbol{\Phi}\}_{iC} \\
&\qquad + \lambda_{iC}[\mathbf{M}]_C\{\Delta\boldsymbol{\Phi}\}_i^{\alpha} + \lambda_{iC}[\Delta\mathbf{M}]^{\alpha}\{\Delta\boldsymbol{\Phi}\}_i^{\alpha} \\
&\qquad + \lambda_i^{\alpha}[\mathbf{M}]_C\{\boldsymbol{\Phi}\}_{iC} + \Delta\lambda_i^{\alpha}[\Delta\mathbf{M}]^{\alpha}\{\boldsymbol{\Phi}\}_{iC} \\
&\qquad + \Delta\lambda_i^{\alpha}[\mathbf{M}]_C\{\Delta\boldsymbol{\Phi}\}_i^{\alpha} + \Delta\lambda_i^{\alpha}[\Delta\mathbf{M}]^{\alpha}\{\Delta\boldsymbol{\Phi}\}_i^{\alpha}.
\end{aligned}
\tag{13}
$$

The second order terms are small and neglected. It is expected that their influence on the results will be insignificant for small variations. Handa [5] recommends variations of the order of 20% maximum. The crisp value and the fluctuating components can then be separated into different systems of equation:

$$
\begin{cases}
[\mathbf{K}]_C\{\boldsymbol{\Phi}\}_{iC} = \lambda_{iC}[\mathbf{M}]_C\{\boldsymbol{\Phi}\}_{iC}, \\
[\mathbf{K}]_C\{\Delta\boldsymbol{\Phi}\}_i^{\alpha} + [\Delta\mathbf{K}]^{\alpha}\{\boldsymbol{\Phi}\}_{iC} = \lambda_{iC}[\mathbf{M}]_C\{\Delta\boldsymbol{\Phi}\}_i^{\alpha} + \lambda_{iC}[\Delta\mathbf{M}]^{\alpha}\{\boldsymbol{\Phi}\}_{iC} + \Delta\lambda_i^{\alpha}[\mathbf{M}]_C\{\boldsymbol{\Phi}\}_{iC}
\end{cases}
\tag{14a, b}
$$

Equation (14a) gives the crisp values according to the standard deterministic finite element analysis. Regarding the eigenvectors as constant [24], equation (14b) gives:

$$
[\Delta\mathbf{K}]^{\alpha}\{\boldsymbol{\Phi}\}_{iC} = \lambda_{iC}[\Delta\mathbf{M}]^{\alpha}\{\boldsymbol{\Phi}\}_{iC} + \Delta\lambda_i^{\alpha}[\mathbf{M}]_C\{\boldsymbol{\Phi}\}_{iC}. \tag{15}
$$

Premultiplying equation (15) by $\{\boldsymbol{\Phi}\}_{iC}^T$, one obtains:

$$
\Delta\lambda_i^{\alpha} = \{\boldsymbol{\Phi}\}_{iC}^T([\Delta\mathbf{K}]^{\alpha} - \lambda_{iC}[\Delta\mathbf{M}]^{\alpha})\{\boldsymbol{\Phi}\}_{iC} = [\Delta\lambda_{iL}^{\alpha}; \Delta\lambda_{iR}^{\alpha}]. \tag{16}
$$

The following frequencies are obtained:

$$
\begin{aligned}
&\Delta f_i^{\alpha} = \frac{1}{4 \times \pi \times \sqrt{\lambda_{iC}}}(\Delta\lambda_i^{\alpha}) = [\Delta f_{iL}^{\alpha}; \Delta f_{iR}^{\alpha}], \quad \text{and} \\
&\quad f_i^{\alpha} = [f_{iL}^{\alpha}; f_{iR}^{\alpha}] = [f_{iC} + \Delta f_{iL}^{\alpha}; f_{iC} + \Delta f_{iR}^{\alpha}].
\end{aligned}
\tag{17a,b}
$$

Thus, the $i$th fuzzy frequency and its membership function can be generated (see Figure 2).

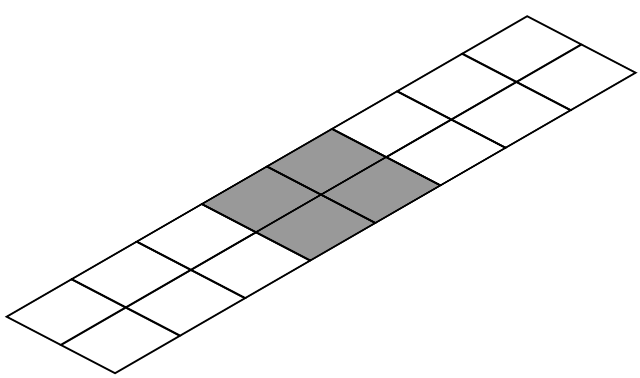

Figure 3. The structure studied.

TABLE 1

*Low (L), Probable (P) and High (H) values for E, v and ρ*

| | $L$ | $P$ | $H$ |
|---|---|---|---|
| $E$ | $1{\cdot}89 \times 10^{11}$ N/m$^2$ | $2{\cdot}10 \times 10^{11}$ N/m$^2$ | $2{\cdot}31 \times 10^{11}$ N/m$^2$ |
| $v$ | 0·27 | 0·30 | 0·33 |
| $\rho$ | 7020 kg/m$^3$ | 7800 kg/m$^3$ | 8580 kg/m$^3$ |

## 3. MEASUREMENT OF THE IMPRECISION

In order to identify the fuzziest eigenvalues, that is to say those which propagate the most the uncertainty, different indicators can be used, for example: cardinality [25], uncertainty [26], non-specificity, ambiguity and fuzzy indexes [27]. The non-probabilistic entropy [28] and specificity [29] are the easiest to use.

Results can be analysed in two different ways. In the first case, the fuzziest of the set obtained can be determined by analysing the entropy for example. Alternatively, it may be rather interesting to identify the fuzzy number with the lowest specificity. In our study, this specificity can be interpreted as a measurement of the sensibility of the results. This point will be further developed. The following paragraph describes the main indexes.

### 3.1. FUZZY ENTROPY

The concept of a fuzzy entropy was introduced by De Luca and Termini [28] to measure the degree of fuzziness of a fuzzy set. If $A$ is a fuzzy set which is expressed as:

$$A = \sum_{i=1}^{n} \mu(x_i)/x_i, \tag{18}$$

then the fuzzy entropy of $A$ is defined by

$$E(A) = K \times \sum_{i=1}^{n} S(\mu(x_i)), \tag{19}$$

TABLE 2

*Comparison between the real and defuzzified frequencies*

| Mode | Real frequency (Hz) | Defuzzified frequency (Hz) |
|---|---|---|
| 7 | 29·734 | 29·741 |
| 8 | 72·613 | 72·645 |
| 9 | 82·504 | 82·512 |
| 10 | 151·689 | 151·711 |
| 11 | 162·670 | 162·709 |
| 12 | 243·153 | 243·222 |
| 13 | 270·231 | 270·302 |
| 14 | 352·488 | 352·583 |

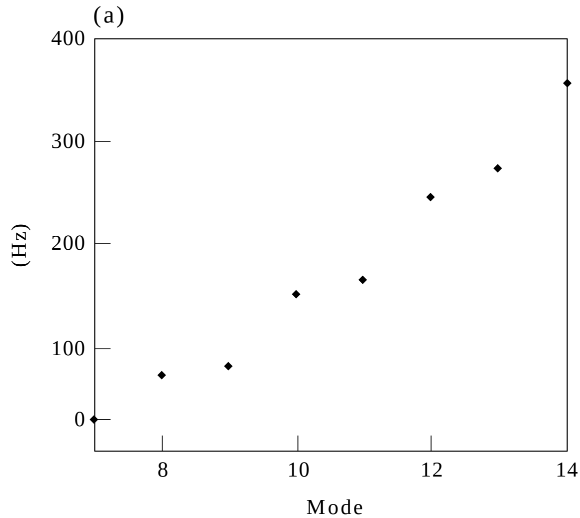


Figure 4a. Fuzzy frequencies for $E$, $\nu$ and $\rho$ fuzzy and $\alpha = 1{\cdot}0$.

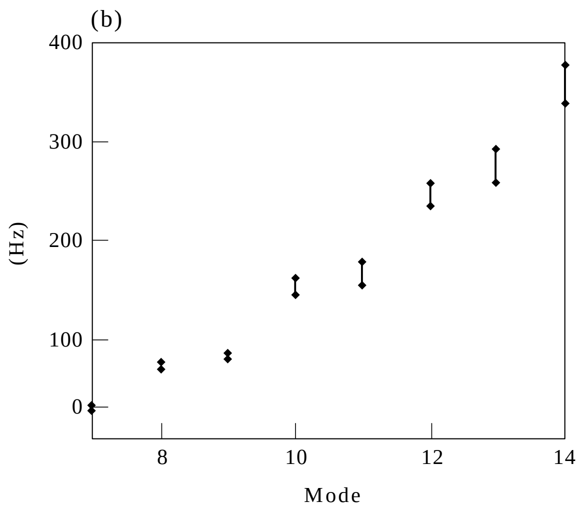


Figure 4b. Fuzzy frequencies for $E$, $\nu$ and $\rho$ fuzzy and $\alpha = 0{\cdot}0$.

where $K$ is a positive real constant and $S$, based on Shannon's function, can be written as:

$$S(x) = -x \ln(x) - (1 - x) \ln(1 - x). \tag{20}$$

The functional defined by equation (19) must satisfy three properties: $S(x) = 0$ if and only if $x = 0$ or 1; $S(x) = S_{max}$ if and only if $x = 0{\cdot}5$; $S(x^*) \geqslant S(x)$ where $x^*$ is the "sharpened" version of $y$ such that $x^* \geqslant x$ for $x \geqslant 0{\cdot}5$ and $x^* \leqslant x$ for $x \leqslant 0{\cdot}5$.

### 3.2. SPECIFICITY

The concept of specificity was developed by Yager *et al*. [29]. If $A$ is a fuzzy set which is expressed as

$$A = \sum_{i=1}^{n} \mu(x_i)/x_i,$$

then the specificity of $A$ is defined by

$$Sp(A) = \sum_{j=1}^{n} [\mu(x_j) - \mu(x_{j+1})]/j, \tag{21}$$

where $\mu(x_j)_{j=1,n}$ is the decreasing sequence as $\mu(x_1) \geqslant \mu(x_2) \geqslant \cdots \geqslant \mu(x_n)$.

### 3.3. DEFUZZIFICATION

The defuzzification is a process by which the fuzzy sets are converted into precise numerical results [30]. The defuzzified value of a fuzzy number is achieved by identifying its centre of gravity according to the real axis, namely:

$$A_D = \frac{A_C + \sum_{i=1}^{n} \alpha_i (A_L^{\alpha_i} + A_R^{\alpha_i})}{1 + 2\sum_{i=1}^{n} \alpha_i}, \tag{22}$$

where $A_D$ is the defuzzified value of $A$, $A_C$ is the crisp value of $A$, $\alpha_i$ is the $\alpha$-level cut, and $n$ is the number of $\alpha$-level cut.

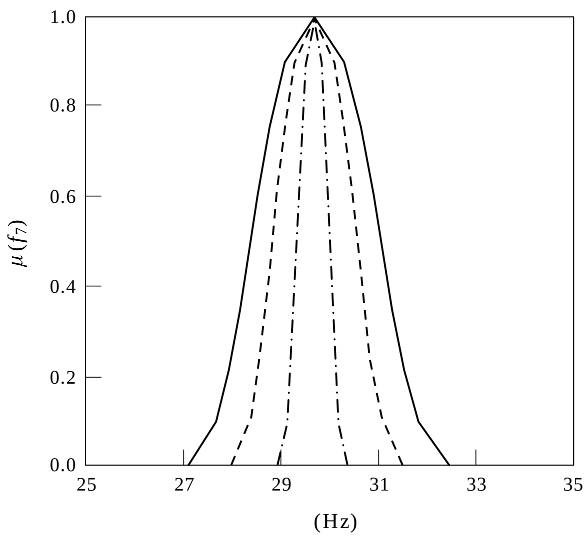


Figure 5. Influence of the structural parameters on the 7th fuzzy frequency: – · – ·, only $\rho$ fuzzy; —, $E$ and $\nu$ fuzzy; ——, $E$, $\nu$ and ρ fuzzy.

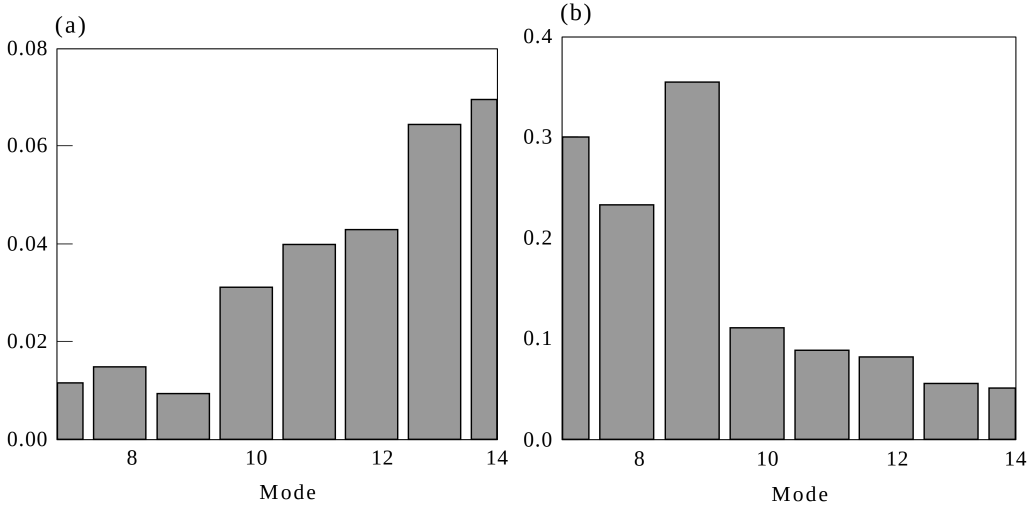


Figure 6. (a) Entropy of fuzzy frequencies. (b) Specificity of fuzzy frequencies.

## 4. EXAMPLE AND RESULTS

The adaptation of the fuzzy eigenvalues has been developed and applied to a test case in order to calculate the eigenvalues' sensitivities to multiple simultaneous material parameters. The structure studied is a plate, which is modelled by 16 shell elements, i.e., 162 dofs. The material properties are: Young's modulus $E = 2{\cdot}1 \times 10^{11}$ N/m$^2$, Poisson's ratio $\nu = 0{\cdot}3$, and material density $\rho = 7800$ kg/m$^3$. The geometric characteristics are: length $L = 0{\cdot}6$ m, width $l = 0{\cdot}15$ m, and thickness $h = 2{\cdot}$e-3 m.

The material parameters of the central area are voluntarily perturbed in order to evaluate one or several parameters' influence on the structure's dynamic behavior (see Figure 3). Its material properties are obtained in the form of low, probable and high values (see Table 1). The free vibration analysis of the plate determines eight flexible frequencies (see Table 2).

The defuzzified values were obtained from fuzzy frequencies due to simultaneous perturbations of $E$, $\nu$ and $\rho$. The fuzzy frequencies are represented in Figure 4.

The influence of the different structural parameters on the frequencies can also be visualized (see Figure 5).

Consequently one notices that the structure is, on the whole, almost insensitive to a variation of $\rho$: regardless of the uncertainty in $\rho$, the frequencies will be close to their deterministic values. On the other hand, uncertainty in $E$ or $\nu$ creates uncertainty on frequencies.

The modal behaviour in relation to a simultaneous perturbation of $E$, $\nu$ and $\rho$ comes closer to that due to a perturbation of $E$ and $\nu$ only. Moreover, the 9th frequency (double flexure) spreads less uncertainty than the others (see Figure 4). This result makes sense of the entropy and specificity calculus (see Figure 6).

It is then noticed that the more extensive the entropy (mode 14), the less certain one is of the frequency's value. Consequently there is less latitude on the values to give the studied parameters in order that the modal behaviour will be certain. Conversely, the more extensive the specificity (mode 9), the more confidence one can have in the frequency's value. There is then more flexibility regarding the values to give the studied parameters in order that the modal behaviour of the structure will be certain.

This result can be explained by the fact that for the 9th frequency, in the uncertain area, the kinetic energy is higher than the strain energy. The 9th frequency is therefore less sensitive to $E$ and $\nu$ than to $\rho$.

## 5. CONCLUSION

A methodology which determines simultaneous sensitivities has been put forward for the finite element analysis of fuzzy systems. The feasibility of this method is shown using a numerical example. By constructing membership functions for the imprecise quantities, the fuzzy calculus and integration techniques are used to derive the finite element equations. The resulting fuzzy system of equations is solved using the theory of interval equations and a sensitivity analysis.

The extension of the methodology to the analysis of complex engineering systems is under investigation. This methodology uses a new approach for the solution of structural problems involving imprecisely defined geometry, external loads, boundary conditions and material properties. The theory of fuzzy sets therefore has vast potential in the field of finite element analysis.

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