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ON THE EXACT PERIODIC SOLUTION FOR $\ddot{x} + \mathrm{sign}\,(x) = 0$

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Lipscomb and Mickens [1] obtained a periodic solution to the antisymmetric, constant force oscillator equation [2]

$$\ddot{x} + \mathrm{sign}\,(x) = 0, \tag{1}$$

with initial conditions

$$x(0) = 0, \qquad \dot{x}(0) = A, \tag{2, 3}$$

where

$$\mathrm{sign}\,(x) = \begin{cases} +1, & \text{for } x > 0, \\ -1, & \text{for } x < 0, \end{cases} \tag{4}$$

and overdots denote differentation with respect to time, t.

Initially, they obtained the solution for $x(t)$ over one period.

$$x(t) = \begin{cases} -\dfrac{t}{2}(t - 2A) & \text{for } 0 \leqslant t \leqslant 2A, \\ \dfrac{t^2}{2} - 3At + 4A^2 & \text{for } 2A \leqslant t \leqslant 4A, \end{cases} \tag{5}$$

which is a piecewise continuous function composed of two parts separately defined over the intervals, and cannot be represented in terms of a single function. For values of t outside this interval, $x(t)$ can be determined from the periodicity relation

$$x(t + nT) = x(t), \quad T = 4A, \quad n = \text{integer}. \tag{6}$$

Here T is a period.

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The solution of the problem given by equations (5) and (6) is finally represented by Fourier series as

$$x(t) = \frac{16A^2}{\pi^3} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^3} \sin\left\{(2m+1)\frac{\pi t}{2A}\right\}, \tag{7}$$

which is called an exact solution. The purpose of this letter is to suggest a direct method for obtaining the above periodic solution of the equation of motion (1) with the initial conditions (2) and (3).

The restoring force function, sign (x) in the equation of motion (1) is an odd function, which can be expressed in terms of Fourier series over a period, T as

$$\text{sign}\,(x) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)} \sin\{(2m+1)\omega t\}, \tag{8}$$

where the angular frequency parameter,

$$\omega = \frac{2\pi}{T}. \tag{9}$$

The behaviour of oscillations will be the same for both positive and negative amplitudes. The sign (x) defined in equation (8) represents positive and negative line graph at unit distance periodically. It takes positive form at $t \to 0^+$. Hence, the periodic oscillations begin with positive amplitudes for which the constant, A, in equation (3) is greater than zero. If $A < 0$, then the periodic oscillations begin with negative amplitudes, and

$$\text{sign}\,(x) = -\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)} \sin\{(2m+1)\omega t\}.$$

Whether A is greater or less than zero, there is no change in the behaviour of oscillations for the problem considered. In the present study, it is assumed that the periodic oscillations begin with positive amplitudes and $A > 0$.

The following summation of series useful in the present analysis taken from the *Engineering Mathematics Handbook* [3] are

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}, \tag{10}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3} = \frac{\pi^3}{32}. \tag{11}$$

Substituting equation (8) into equation (1), and integrating with respect to t, one obtains

$$\dot{x} = \frac{4}{\pi\omega} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \cos\{(2m+1)\omega t\}. \tag{12}$$

The constant of integration in equation (12) is neglected to eliminate the secular term in the periodic solution. Using equations (3), (10) and (12), the unknown frequency parameter ω is obtained as

$$\omega = \frac{4}{\pi A} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{4}{\pi A} \frac{\pi^2}{8} = \frac{\pi}{2A}. \tag{13}$$

Integrating equation (12) with respect to t and applying the condition (2), one gets the final solution of reference [1] as

$$x = \frac{4}{\pi \omega^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^3} \sin\{(2m+1)\omega t\}. \tag{14}$$

Multiplying equation (1) with $2x$ and integrating, one can get the energy relation for the initial conditions (2) and (3),

$$(\dot{x})^2 + 2|x| = A^2. \tag{15}$$

It is verified that the periodic solution (14) satisfies the energy relation (15) for any specified time, t. At quarter period, $t = T/4 = \pi/2\omega$, and equations (11), (12) and (14) give

$$x = \frac{A^2}{2}, \quad \dot{x} = 0. \tag{16, 17}$$

Shifting the origin of reference to $\pi/2\omega$ and replacing A by $\sqrt{2B}$, one can get the periodic solution for equation (1) as

$$x = \frac{4}{\pi \omega^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3} \cos\{(2m+1)\omega t\}, \tag{18}$$

which satisfies the initial conditions

$$x(0) = B, \qquad \dot{x}(0) = 0, \tag{19, 20}$$

and has the frequency parameter $\omega = \pi/2\sqrt{2B}$. The energy relation for the equation of motion (1) corresponding to the conditions (19) and (20) is

$$(\dot{x})^2 + 2|x| = 2B. \tag{21}$$

It is also verified that the solution (18) satisfies (21) for any specified time, t. As stated in reference [1], the initial conditions (2) and (3) for equation (1) result in an odd solution (14), while the conditions (19) and (20) for equation (1) lead to an even solution (18). In summary, Lipscomb and Mickens [1] offered a piecewise polynomial solution for the equation of motion (1) which is a closed-form one. Later on, they imposed the periodicity condition to express the solution obtained in the Fourier series form. In this letter, we have derived directly the Fourier series solution by expressing the restoring force function, sign (x) in the appropriate form (8), which is directed essentially to validate the indirect procedure of reference [1]. It should be noted that the method proposed for the title problem directly yields the series solution which is not a closed-form one, and the method is not applicable to other non-linear problems.

REFERENCES

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