Journal of Sound and Vibration (1999) **220**(3), 564–570
Article No. jsvi.1998.1944, available online at http://www.idealibrary.com on IDEAL

SPRING'S EFFECTIVE MASS IN SPRING MASS SYSTEM FREE VIBRATION

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(*Received* 28 *May* 1998, *and in final form* 13 *August* 1998)

1. INTRODUCTION

When the mass of the spring M_s is not neglected in the free vibration of the spring with one end fixed and a lumped mass M attached at the other end, the spring's effective mass αM_s might be added to the lumped mass M for the simplification to the case of the vibration of a system with one degree of freedom. The following conclusion based on the approximate method developed by Lord Rayleigh [1] is well known. The spring's effective mass αM_s is found to be one-third the mass of the spring M_s. Adding this to the lumped mass M, the revised natural circular frequency $\omega_{1/3}$ is [2–4]

$$\omega_{1/3} = \sqrt{k/[M + (1/3)M_s]}, \tag{1}$$

where k is the spring constant.

The free longitudinal vibration of a rod with one end fixed and a lumped mass M attached at the other end simulates the free vibration of the spring with the same boundary condition. These simulations are described in Timoshenko's literature [2].

Indeed, it is described in reference [2] that the ratio α of the spring's effective mass to the spring mass is one-third. However, the ratio η of the spring mass M_s to lumped mass M becomes large, when the ratio α of spring effective mass to spring mass would be greater than one-third. In this paper, the α–η relation between the spring effective mass to spring mass ratio α in the free vibration of the spring with one end fixed and a lumped mass attached at the other end and the spring mass to lumped mass ratio η is examined numerically by the free vibration of the fixed-lumped mass bar which is the simulation of the present problem.

2. LONGITUDINAL VIBRATION OF A BAR CARRYING A MASS

The differential equation of the longitudinal vibration of a bar is

$$\partial^2 u/\partial x^2 = (1/a^2)\, \partial^2 u/\partial t^2, \tag{2}$$

where x denotes the co-ordinate and t denotes the time, a is the acoustic velocity in the bar, $a = \sqrt{E/\rho}$, E denotes the modulus of elasticity and ρ the density. The differential equation of motion for a typical element of the bar may be written as

$$m_r \ddot{u}\, \mathrm{d}x - r u''\, \mathrm{d}x = 0, \tag{3}$$

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where the dots and primes signify differentiation of the displacement u with respect to t and x, respectively. The term $m_r = \rho A$ represents the mass of the bar per unit length, and the quantity $r = EA$ is its axial rigidity. When the bar vibrates in its ith natural mode, it has the harmonic motion

$$u_i = X_i(A_i \cos \omega_i t + B_i \sin \omega_i t). \tag{4}$$

Substitution of equation (4) into equation (3) and rearrangement of terms produces

$$rX_i'' + m_r \omega_i^2 X_i = 0,$$

for which the solution has the form

$$X_i = C_i \cos(\omega_i x/a) + D_i \sin(\omega_i x/a).$$

Let us consider the free longitudinal vibration of the prismatic bar with one end fixed and a lumped mass M attached at the other end. The boundary condition for the bar may be written as

$$\left.\begin{aligned} u|_{x=0} &= 0 \\ S|_{x=\ell} = ru'|_{x=\ell} &= -M\ddot{u}|_{x=\ell} \end{aligned}\right\}, \tag{5}$$

where S is the axial force. The frequency equation for the case under consideration is as follows

$$\xi_i \tan \xi_i = \eta, \tag{6}$$

where

$$\xi_i = \omega_i \ell/a, \qquad \eta = m_r \ell/M.$$

The fundamental mode of vibration is usually of greatest interest: for various values of the mass ratio η the corresponding value of ξ_1 (for the first mode) are given in references [2, 3].

3. THE ANALOGY TO THE SPRING CONCENTRATED MASS SYSTEM

The free vibration of the spring with one end fixed and a lumped mass M attached at the other end where the spring mass is $M_s = \beta\ell$ is simulated by the free longitudinal vibration of the fixed-lumped mass prismatic rod. The spring constant k of the former corresponds to the axial rigidity of the rod EA/ℓ of the latter,

$$k = EA/\ell.$$

The spring mass $M_s = \beta\ell$ to lumped mass M ratio η corresponds to the rod mass $m_r\ell$ to block mass M ratio η (see Figure 1),

$$\beta\ell/M = m_r\ell/M = \eta.$$

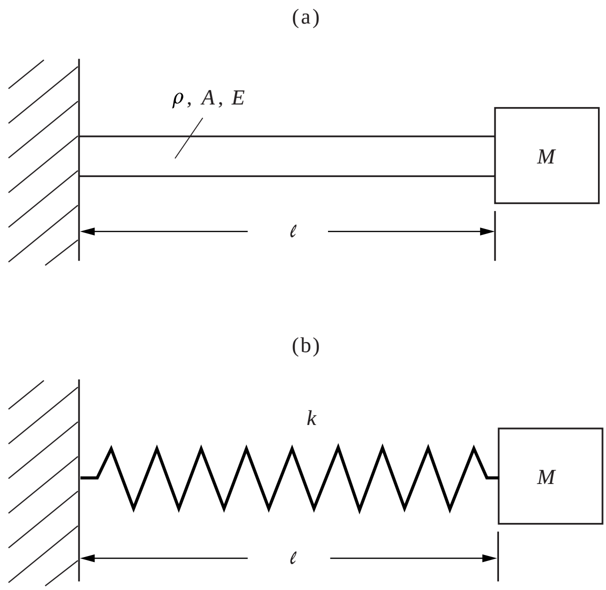

Figure 1. The free vibration of a bar with lumped mass at the end simulates the vibration of the spring with mass at the end. (a) The fixed-lumped mass prismatic rod. (b) The spring, one end fixed and a lumped mass attached at the other end.

If the mass of the bar $m_r\ell$ is small compared to that of the attached mass M, the values of η and ξ_1 will both be small and equation (6) can be simplified by taking $\tan\xi_1 \cong \xi_1$. Then one has

$$\xi_1 = \omega_1\ell/a \cong \sqrt{m_r\ell/M} = \sqrt{\eta}. \tag{7}$$

Hence,

$$\omega_1 \cong \sqrt{EA/M\ell}.$$

For the corresponding fixed-lumped mass spring system,

$$\omega_1 = \sqrt{k/M}. \tag{8}$$

This solution corresponds to the case in which the spring mass is zero.

A better approximation will be obtained by substituting

$$\tan\xi_1 \cong \xi_1 + \xi_1^3/3 \tag{9}$$

into equation (6). Then

$$\xi_1(\xi_1 + \xi_1^3/3) = \eta$$

or

$$\xi_1 = \sqrt{\eta/(1 + \xi_1^2/3)}. \tag{10}$$

Substituting the first approximation (7) for ξ_1 into the right side of this equation, one has

$$\xi_1 = \sqrt{\eta/(1 + \eta/3)}.$$

This solution corresponds to the relation

$$\omega_1 = \sqrt{k/[M + (1/3)M_s]}. \tag{11}$$

It can be concluded that a better approximation is obtained by adding one-third of the spring mass M_s to the lumped mass M. This is coincident with the well known approximate solution obtained by using Rayleigh's method.

4. NUMERICAL SOLUTION BY USING NEWTON'S METHOD

The frequency equation for the fundamental mode

$$\xi_1 \tan \xi_1 = \eta \tag{12}$$

is solved numerically by using Newton's method. The relations of ξ_1 and η are shown in Figure 2. Both the scales of ξ_1 and η are logarithmic. The limiting values of ξ_1 are as follows:

$$\text{for } \eta \to 0, \qquad \xi_1 \to 0 \text{ and for } \eta \to \infty, \qquad \xi_1 \to \pi/2.$$

For the application to obtain the frequency of the fixed-lump mass spring system vibration, the relation of the spring's effective mass αM_s to spring mass M_s ratio

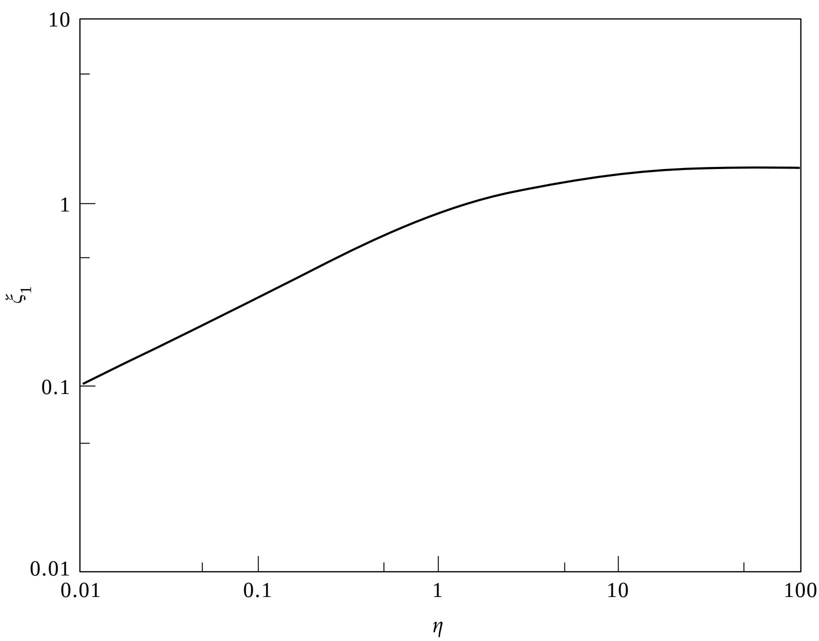

Figure 2. The root ξ_1 of the characteristic equation versus the rod mass to lumped mass ratio η.

α and the M_s to M ratio η is examined as follows. Since the circular frequency ω_1 is obtained by adding α of the bar or spring to the lumped mass, then

$$\omega_1 = \sqrt{EA/[M(1+\alpha\eta)\ell]} \tag{13}$$

corresponds to

$$\omega_1 = \sqrt{k/[M(1+\alpha\eta)]}, \tag{14}$$

and

$$\xi_1 = \omega_1 \ell / a.$$

Then

$$\xi_1 a/\ell = \sqrt{EA/[M(1+\alpha\eta)\ell]}.$$

Therefore

$$\begin{aligned} 1+\alpha\eta &= (1/\xi_1^2)(\ell^2/a^2)(EA/M\ell) \\ &= (1/\xi_1^2)(\rho A\ell/M) \\ &= (1/\xi_1^2)(m_r\ell/M) \\ &= (1/\xi_1^2)\eta. \end{aligned}$$

one obtains

$$\alpha = (1/\xi_1^2) - (1/\eta). \tag{15}$$

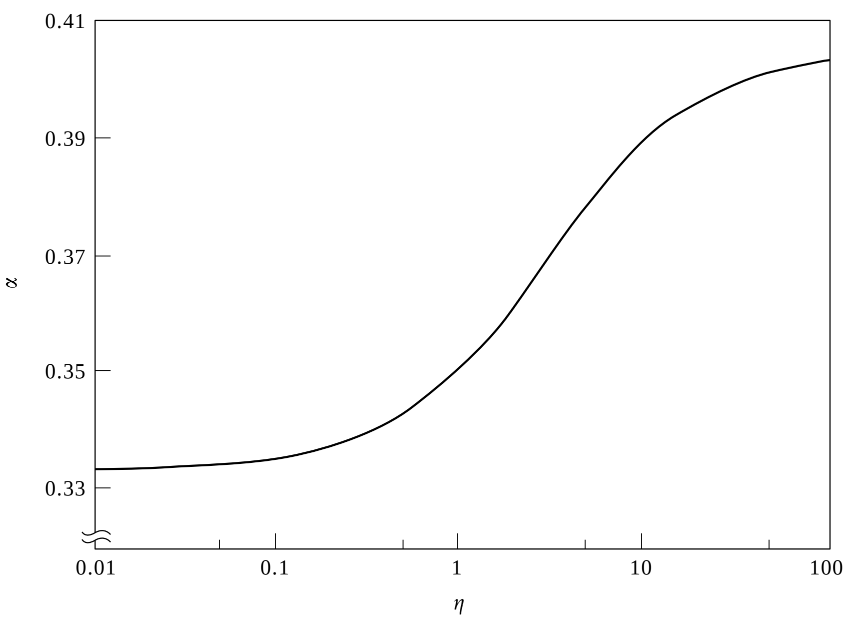

Figure 3. The spring's effective mass to lumped mass ratio α versus the spring mass to lumped mass ratio η.

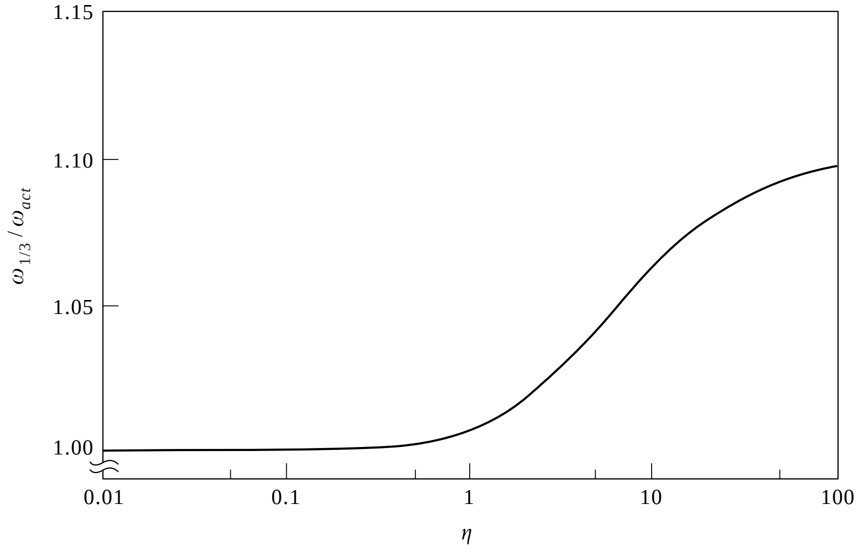

Figure 4. The $\omega_{1/3}/\omega_{act}$ ratio versus the spring mass to lumped mass ratio η.

The relation of the ratio α of the spring's effective mass αM_s to the spring mass M_s and the ratio η of the spring mass M_s to the lumped mass M is shown in Figure 3. The limiting values of α are as follows:

$$\text{for } \eta \to 0, \quad \alpha \to 1/3 \text{ and for } \eta \to \infty, \quad \alpha \to (2/\pi)^2.$$

The circular frequency ω_{act} is obtained

$$\omega_{act} = \sqrt{k/(M + \alpha M_s)}, \tag{16}$$

Table 1

η	ξ	α	$\omega_{1/3}/\omega_{act}$
0·01	0·0998	0·334	1·000
0·1	0·311	0·336	1·000
0·3	0·522	0·340	1·001
0·5	0·653	0·343	1·002
0·7	0·751	0·347	1·004
0·9	0·827	0·350	1·006
1	0·860	0·351	1·007
1·5	0·988	0·357	1·012
2	1·077	0·362	1·017
3	1·193	0·370	1·027
4	1·265	0·375	1·035
5	1·314	0·379	1·042
10	1·429	0·390	1·063
20	1·496	0·397	1·080
100	1·555	0·403	1·097
∞	$\pi/2$	$(2/\pi)^2$	$2\sqrt{3}/\pi$

and equation (1), i.e.,

$$\omega_{1/3} = \sqrt{k/[M + (1/3)M_s]}$$

is a well known approximate result. The relation of the $\omega_{1/3}/\omega_{act}$ ratio and the spring mass M_s to lumped mass M ratio η is shown in Figure 4. It is seen that $\omega_{1/3}$ is a good approximation of the frequency of the fixed-lumped mass spring system, when η is small. The limiting values of $\omega_{1/3}/\omega_{act}$ are as follows:

$$\text{for } \eta \to 0, \qquad \omega_{1/3}/\omega_{act} \to 1 \text{ and for } \eta \to \infty, \qquad \omega_{1/3}/\omega_{act} \to 2\sqrt{3}/\pi.$$

The relation between the root ξ_1, the effective spring's mass αM_s to spring mass M_s ratio α and the $\omega_{1/3}/\omega_{act}$ ratio and the spring mass M_s to lumped mass ratio η are shown in Table 1, obtained by numerical calculations.

5. CONCLUSIONS

The free longitudinal vibration of the fixed-lumped mass rod is examined numerically in order to estimate the spring's effective mass in the free vibration of the fixed-lumped mass spring system. The relation between the spring's effective mass to spring mass ratio and the spring mass to lumped mass ratio is examined.

ACKNOWLEDGMENT

The author would like to express his appreciation to Mr H. Niwa.

REFERENCES

1. LORD RAYLEIGH 1945 *The Theory of Sound*, Volume 1. New York: Dover; second edition.
2. S. TIMOSHENKO and D. H. YOUNG 1955 *Vibration Problems in Engineering*. New York: Van Nostrand; third edition.
3. W. WEAVER JR, S. P. TIMOSHENKO and D. H. YOUNG 1990 *Vibration Problems in Engineering*. New York: Wiley; fifth edition.
4. W. T. THOMSON 1993 *Theory of Vibration with Applications*. New York: Prentice-Hall; fifth edition.