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VIBRATION OF SYMMETRICALLY LAMINATED THICK SUPER ELLIPTICAL PLATES

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This paper reports a free vibration analysis of thick plates with rounded corners subject to a free, simply-supported or clamped boundary condition. The plate perimeter is defined by a super elliptic function with a power defining the shape ranging from an ellipse to a rectangle. To incorporate transverse shear deformation, the Reddy third-order plate theory is employed. The energy integrals incorporating shear deformation and rotary inertia are formulated and the p-Ritz procedures are used to derive the governing eigenvalue equation. Numerical examples for plates with different shapes and boundary conditions are solved and their frequency parameters, where possible, are compared with known results. Parametric studies are carried out to show the sensitivities of frequency parameters by varying the geometry, fibre stacking sequence, and boundary condition.

1. INTRODUCTION

The extensive use of fiber-reinforced composites as primary structural components in aerospace, civil, electronic, and many other engineering disciplines has motivated research on the free vibration of laminated plates. Almost all previous research has focused on rectangular laminated plates and none has considered rectangular laminates with rounded corners even though this plate geometry has practical importance in various engineering applications, such as printed circuit boards. The rounded corners are advantageous in helping to diffuse and dilute stress concentrations at the otherwise sharp corners. The shape of rectangular laminated plates with rounded corners can be described by a super elliptical function. Varying the super elliptic power in the super elliptical function can generate a plate shape ranging from a square or rectangular to a circle or ellipse.

The free vibration characteristics of super elliptical plates, including elliptical and circular plates, can be found in many earlier works [1–6]. Most previous works considered the free vibration of circular or elliptical plates in polar or elliptic co-ordinates which are naturally unsuitable for laminated plates with fibrous directions coinciding with the Cartesian co-ordinate system. Wang *et al*. [7] presented a complete investigation of free vibration and buckling analyses of thin super elliptical plates using the *p*-Ritz method and the classical thin plate theory. This work was further expanded by Lim and Liew [8] and Lim *et al*. [9] to isotropic perforated and composite laminated super elliptical plates, respectively. To examine the effects of transverse shear deformation, Liew *et al*. [10] extended their previous works [7–9] to isotropic thick super elliptical plates by incorporating Reddy's higher-order plate theory [11] in the *p*-Ritz method for free vibration solutions.

Although the classical thin plate theory provides an easy way to analyze the thin composite laminates [9], this theory has many drawbacks because of the Kirchhoff assumptions which lead to zero transverse shear strains and zero transverse normal strain. As laminated composite panels are often weaker in shear mode, the transverse shear strain must be taken into account. The first-order shear deformation theory for composite laminates proposed by Yang *et al*. [12] gained its popularity because it provides an easy way to incorporate the effects of transverse shear. In this theory, shear correction factors are used to compensate for the assumption made of zero transverse shear strain on the top and bottom surfaces of the laminated plate. However, for laminated composite panels, the shear correction factor depends on various factors and is unknown for arbitrarily composite laminates. The requirement for shear correction factors in the first-order shear deformation theory has made it less attractive for many applications.

In an effort to circumvent the problems of shear correction factors, various second and higher-order shear deformation theories have been developed. The most popular one was the higher-order shear deformation theory proposed by Reddy [11]. The displacement field of Reddy's higher-order shear deformation theory accommodates parabolic variation of transverse shear strains and vanishing transverse shear stresses on the top and bottom of a general laminate. Therefore, no shear correction factor is required in this theory. The theory has been shown to provide reasonably accurate free vibration solutions for moderately thick laminates [13, 14].

This paper examines the free vibration behavior of moderately thick symmetric laminates of super elliptical planform. This investigation forms a natural extension of the work of Liew *et al*. [10] from the isotropic case to a laminated panel. Because transverse shear deformation plays an important role in the analysis of composite laminates, Reddy's higher-order plate theory has been used to formulate the energy integral functional so that no shear correction factor is needed. The *p*-Ritz procedure is used to minimize this energy integral functional to arrive at the governing eigenvalue equation. To illustrate the method, several numerical examples of super elliptical symmetrically laminated plates with different plate geometries and boundary conditions are solved. Parametric studies are also carried

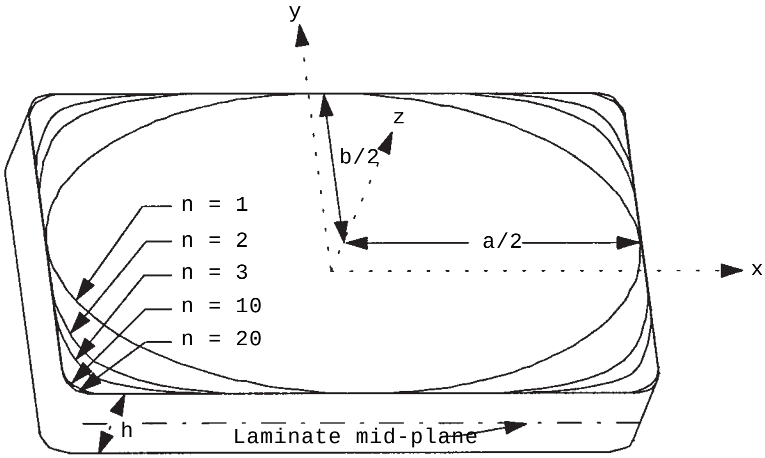


Figure 1. Geometric definitions of laminated super elliptical plates.

out to examine the effects of plate geometry, boundary conditions, super elliptical power, aspect ratio, length-to-thickness ratio, and fibre stacking sequences on the vibration frequency parameters.

2. MATHEMATICAL FORMULATION

The thick super elliptical laminated plate and associated reference Cartesian co-ordinate system are shown in Figure 1. The dimensions of the laminated plate

TABLE 1

Convergence of the frequency parameter, $\lambda = \omega ab\sqrt{\rho h/D_0}$, *for the super elliptical plate with* $a/b = 2$, $a/h = 5$, $n = 10$, *and stacking sequence* $[30/-30]_S$

Boundary condition	p	Mode sequence number 1	2	3	4	5	6	7	8
Free	7	3·1476	4·1104	6·1582	6·5514	6·8263	7·9043	8·4316	8·5328
	9	3·1463	4·1087	6·1509	6·5466	6·8194	7·9043	8·4292	8·5103
	11	3·1460	4·1083	6·1494	6·5458	6·8185	7·9042	8·4287	8·5081
	13	3·1460	4·1083	6·1492	6·5456	6·8183	7·9042	8·4286	8·5076
	15	3·1460	4·1083	6·1491	6·5456	6·8183	7·9042	8·4286	8·5075
Simply-supported	7	4·7687	7·0959	7·9043	8·4316	9·1080	10·0296	10·1051	11·1485
	9	4·7286	7·0227	7·9043	8·4292	9·1075	9·8742	10·0286	10·0681
	11	4·7256	7·0172	7·9042	8·4287	9·1074	9·8029	10·0284	10·0632
	13	4·7247	7·0156	7·9042	8·4286	9·1073	9·7956	10·0283	10·0616
	15	4·7244	7·0151	7·9042	8·4286	9·1073	9·7947	10·0283	10·0614
Clamped	7	6·1453	8·3177	11·4036	13·1248	15·3118	17·5963	18·5385	19·4088
	9	6·1019	8·2274	11·1160	11·3441	13·3422	14·3497	16·2542	17·5993
	11	6·0992	8·2211	10·9825	11·3373	13·1520	14·0344	15·8044	17·4993
	13	6·0983	8·2194	10·9710	11·3358	13·1401	14·0117	15·7639	17·1540
	15	6·0980	8·2190	10·9698	11·3355	13·1380	14·0097	15·7597	17·1133

TABLE 2

Comparison of the frequency parameters, $\lambda_1 = \omega(a/\pi)^2\sqrt{\rho h/D_0}$, for the thin, super elliptical, isotropic, plate with $a/b = 2$

Source	Mode sequence number					
	1	2	3	4	5	6
$n = 1$, simply-supported plate						
Reference [2]	5·358	–	–	–	–	–
Reference [7]	5·355	9·582	15·533	18·704	23·298	25·439
Present	5·355	9·579	15·529	18·701	23·290	25·429
$n = 1$, clamped plate						
Reference [3]	11·097	16·005	22·684	28·317	31·203	35·681
Reference [7]	11·100	16·008	22·689	28·327	31·205	35·683
Present	11·094	16·005	22·681	28·304	31·197	35·671
$n = 10$, simply-supported plate						
Reference [7]	4·986	7·969	12·955	16·989	19·953	20·003
Present	4·985	7·967	12·966	16·983	19·958	19·987
$n = 10$, clamped plate						
Reference [7]	9·951	12·897	18·132	25·743	25·926	28·805
Present	9·962	12·904	18·154	25·701	25·930	28·809

are assumed to be a, b, and h in the x, y, and z directions. The periphery of the super ellipse is defined by the super elliptical function

$$\left(\frac{2x}{a}\right)^{2n} + \left(\frac{2y}{b}\right)^{2n} = 1, \tag{1}$$

in which n is the power of super ellipse. The shape becomes an ellipse if the super elliptical power n is 1. Interestingly, if the power n is continually increased, the plate becomes a rectangle with four rounded corners. Higher values of n lead to a smaller corner radius. The plate becomes a rectangle as n approaches infinity.

The laminae are assumed to possess a plane of elastic symmetry parallel to the xy plane and are stacked symmetrically with respect to the middle surface of the laminate. The vibration frequencies of the super elliptical laminate subjected to a variety of boundary conditions, aspect ratios, length-to-thickness ratios, super elliptical powers, number of plies, and stacking angles are to be determined.

2.1. GOVERNING EQUATIONS

Let u, v, and w be the in-plane and out-of-plane displacement components of a general point of the thick super elliptical laminated plate. The displacement field of a laminated plate based on Reddy's higher-order shear deformation theory can be assumed as:

$$u(x, y, z, t) = u_0(x, y, t) + z\phi_x(x, y, t) - \frac{4z^3}{3h^2}\left(\phi_x(x, y, t) + \frac{\partial w(x, y, t)}{\partial x}\right),$$

$$v(x, y, z, t) = v_0(x, y, t) + z\phi_y(x, y, t) - \frac{4z^3}{3h^2}\left(\phi_y(x, y, t) + \frac{\partial w(x, y, t)}{\partial y}\right),$$

$$w(x, y, z, t) = w_0(x, y, t), \tag{2}$$

where u_0, v_0, w_0, ϕ_x, and ϕ_y are the displacement and rotation components of the mid-plane of the laminated plate in the Cartesian co-ordinate system.

In linear elastic analysis, the stress–strain relationship for the kth lamina in the Cartesian co-ordinate system is given by

$$[\sigma]_k = [\bar{Q}]_k[\xi]_k, \tag{3}$$

TABLE 3

Comparison of frequency parameters, $\lambda_2 = \omega a\sqrt{\rho/E}$, for the isotropic, super elliptical, thick plate with $a/b = 1$ and $h/a = 0{\cdot}3$

Source	Mode sequence number					
	1	2	3	4	5	6
$n = 1$, free plate						
Reference [10]	0·6192	0·6193	1·0298	1·3704	1·3704	2·1784
Present	0·6192	0·6193	1·0298	1·3704	1·3707	2·1784
$n = 10$, free plate						
Reference [10]	0·3898	0·5734	0·7151	0·9783	0·9783	1·6874
Present	0·3898	0·5734	0·7151	0·9780	0·9783	1·6874
$n = 1$, simply-supported plate						
Reference [10]	0·5785	1·5291	1·5291	2·6301	2·6301	2·9091
Present	0·5784	1·5291	1·5291	2·6301	2·6301	2·9091
$n = 10$, simply-supported plate						
Reference [10]	0·5532	1·3429	1·3429	2·0376	2·4412	2·5420
Present	0·5530	1·3429	1·3429	2·0377	2·4412	2·5420
$n = 1$, clamped plate						
Reference [10]	1·1216	2·1616	2·1616	3·2944	3·2948	3·6893
Present	1·1216	2·1615	2·1617	3·2948	3·2951	3·6897
$n = 10$, clamped plate						
Reference [10]	0·9862	1·8839	1·8839	2·6440	3·1176	3·1467
Present	0·9863	1·8845	1·8845	2·6455	3·1187	3·1478

Table 4

Comparison of frequency parameters, $\lambda_3 = \omega a^2\sqrt{\rho h/D_0}$, for the clamped, thin, laminated, circular plate of E-glass/epoxy with stacking sequence $[(\theta/-\theta)_4]_S$

Source	θ	Mode sequence number 1	2	3	4	5	6	7	8
Reference [15]	0	32·859	60·062	75·686	99·436	109·43	139·76	149·39	154·08
Present		32·860	60·057	75·688	99·426	109·41	139·76	149·37	154·05
Reference [15]	15	32·871	61·208	74·842	101·29	110·72	137·67	151·62	156·87
Present		32·871	61·203	74·843	101·28	110·71	137·66	151·59	156·85
Reference [15]	30	32·893	64·192	72·438	106·10	113·09	132·43	157·76	162·28
Present		32·893	64·186	72·438	106·09	113·07	132·42	157·73	162·26
Reference [15]	45	32·904	67·000	69·908	109·88	113·80	128·85	162·76	164·33
Present		32·904	66·996	69·904	109·87	113·78	128·84	162·74	164·30

in which, $[\sigma]_k$, $[\xi]_k$ and $[\bar{Q}]_k$ are stress, strain, and moduli of the kth lamina in the reference Cartesian co-ordinate. Here, $(\bar{Q}_{ij})_k$ are obtained from the transform matrix with fibre angle θ_k and the stiffness constants, $(Q_{ij})_k$, which are related to the material properties, E_1, E_2, v_{12}, v_{21}, G_{12}, G_{13}, G_{23}, of each ply.

Neglecting the effect of transverse normal stress σ_z, for a laminated plate consisting of N orthotropic laminae, the total strain energy for the entire laminated plate is given by,

$$U = \frac{1}{2}\sum_{k=1}^{N}\iint_A\int_{h_{k-1}}^{h_k}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_{xz}\varepsilon_{xz} + \sigma_{yz}\varepsilon_{yz} + \sigma_{xy}\varepsilon_{xy})_k\,\mathrm{d}z\,\mathrm{d}A. \tag{4}$$

Similarly, the expression for total kinetic energy T due to the free vibration of laminated plate is,

$$T = \frac{1}{2}\sum_{k=1}^{N}\iint_A\int_{h_{k-1}}^{h_k}\rho_k\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}z\,\mathrm{d}A, \tag{5}$$

in which ρ_k is the mass density of the kth lamina.

As $[\bar{Q}]_k$ changes from layer to layer, it is possible to obtain the equivalent moduli for the entire plate,

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N}\int_{h_k}^{h_{k+1}}(\bar{Q}_{ij})_k(1, z, z^2, z^3, z^4, z^6)\,\mathrm{d}z. \tag{6}$$

Here, all B_{ij} and E_{ij} vanish if the laminae are stacked symmetrically about the mid-plane. For the free vibration problem, the deflection and rotation functions of the laminate mid-plane are periodic in time. Therefore, for small amplitude vibration, one can assume the displacement and rotation components to be expressed in the following forms:

$$u_0(x, y, t) = U(x, y) \sin \omega t,$$

$$v_0(x, y, t) = V(x, y) \sin \omega t,$$

$$w_0(x, y, t) = W(x, y) \sin \omega t,$$

$$\phi_x(x, y, t) = \Theta_u(x, y) \sin \omega t,$$

$$\phi_y(x, y, t) = \Theta_v(x, y) \sin \omega t. \quad (7)$$

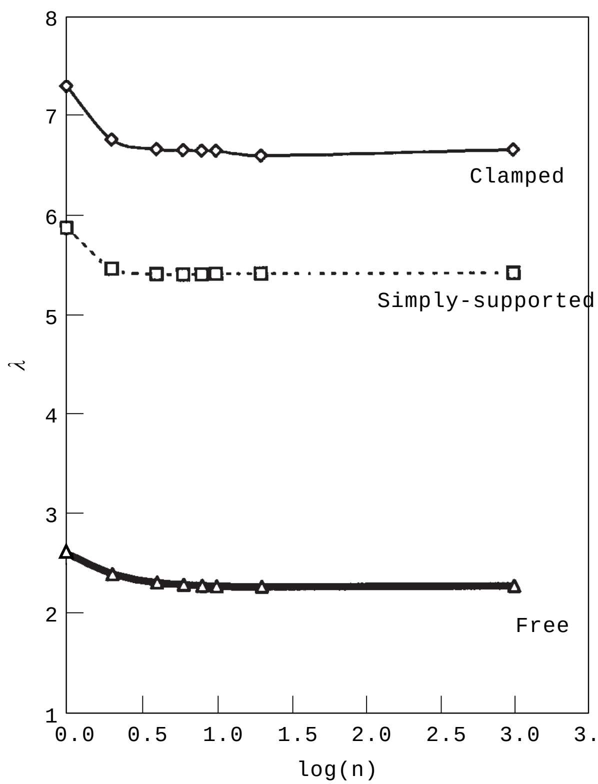


Figure 2. Effect of super elliptical power n on the frequency parameter λ of super elliptical laminate with $a/h = 5$, $a/b = 2$, and stacking sequence $[45/-45]_S$.

TABLE 5

Lowest eight frequency parameters λ for the super elliptical plates with a/h = 5, a/b = 2, and stacking sequence $[45/-45]_S$

n	Boundary condition	Mode sequence number 1	2	3	4	5	6	7	8
1	Free	2·5983	5·5466	5·6871	6·2363	6·7571	8·8092	8·9605	10·3471
	Simply-sup.	5·8709	6·2363	6·7571	8·2904	10·3471	10·9209	12·2416	12·4805
	Clamped	7·2989	9·6234	12·2365	14·1092	15·0761	16·7466	18·0747	19·4689
2	Free	2·3905	4·4875	5·2666	5·2702	6·1986	7·5522	8·1619	9·5922
	Simply-sup.	5·4615	6·1986	6·1986	9·5922	9·8779	11·2300	11·5093	12·0360
	Clamped	6·7518	8·6016	10·9800	13·0685	13·7594	14·8506	16·6438	17·0408
4	Free	2·3058	4·1015	4·8653	5·1628	6·0206	7·0087	7·8226	8·8646
	Simply-sup.	5·4046	6·0206	7·2586	9·4409	9·6891	10·5042	11·3674	11·4598
	Clamped	6·6521	8·3929	10·7465	12·8891	13·5156	14·3640	16·4075	16·4124
6	Free	2·2827	4·0093	4·7657	5·1422	5·9833	6·8665	7·7474	8·6656
	Simply-sup.	5·4021	5·9833	7·2468	9·4236	9·6707	10·2982	11·2774	11·3542
	Clamped	6·6404	8·3662	10·7158	12·8724	13·4803	14·2955	16·3062	16·3867
8	Free	2·2730	3·9732	4·7262	5·1341	5·9696	6·8088	7·7201	8·5827
	Simply-sup.	5·4026	5·9696	7·2459	9·4197	9·6681	10·2128	11·1988	11·3522
	Clamped	6·6374	8·3592	10·7077	12·8688	13·4708	14·2773	16·2767	16·3823
10	Free	2·2680	3·9554	4·7067	5·1300	5·9631	6·7798	7·7073	8·5401
	Simply-sup.	5·4032	5·9631	7·2465	9·4184	9·6685	10·1696	11·1583	11·3520
	Clamped	6·6364	8·3568	10·7053	12·8678	13·4678	14·2712	16·2662	16·3823
20	Free	2·2606	3·9301	4·6786	5·1235	5·9543	6·7376	7·6903	8·4764
	Simply-sup.	5·4051	5·9543	7·2499	9·4175	9·6717	10·1066	11·0989	11·3539
	Clamped	6·5836	8·3219	10·6675	12·7623	13·4078	14·2257	16·2162	16·3286
∞	Free	2·2579	3·9213	4·6688	5·1210	5·9515	6·7226	7·6849	8·4527
	Simply-sup.	5·4064	5·9515	7·2533	9·4174	9·6749	10·0843	11·0780	11·3554
	Clamped	6·6352	8·3544	10·7017	12·8668	13·4632	14·2651	16·2554	16·3739

Substituting equation (7) into equations (4) and (5) yields the maximum strain energy U_{max} and the maximum kinetic energy T_{max} during a vibratory cycle. In the Rayleigh–Ritz method, the governing equation for the free vibration of laminated plate can be established by minimizing the following governing total energy functional,

$$\Pi = U_{max} - T_{max}. \tag{8}$$

The Rayleigh–Ritz method requires the solution to be in the form of a series containing unknown parameters. As a result, the non-dimensional displacement and rotation components can be approximated by assuming a finite set of unknown parameters in the functionals.

$$U(\xi, \eta) = \sum_{i=1}^{m} c_i^u \varphi_i^u(\xi, \eta),$$

$$V(\xi, \eta) = \sum_{i=1}^{m} c_i^v \varphi_i^v(\xi, \eta),$$

$$W(\xi, \eta) = \sum_{i=1}^{m} c_i^w \varphi_i^w(\xi, \eta),$$

$$\Theta_u(\xi, \eta) = \sum_{i=1}^{m} c_i^{\theta_u} \varphi_i^{\theta_u}(\xi, \eta),$$

$$\Theta_v(\xi, \eta) = \sum_{i=1}^{m} c_i^{\theta_v} \varphi_i^{\theta_v}(\xi, \eta), \tag{9}$$

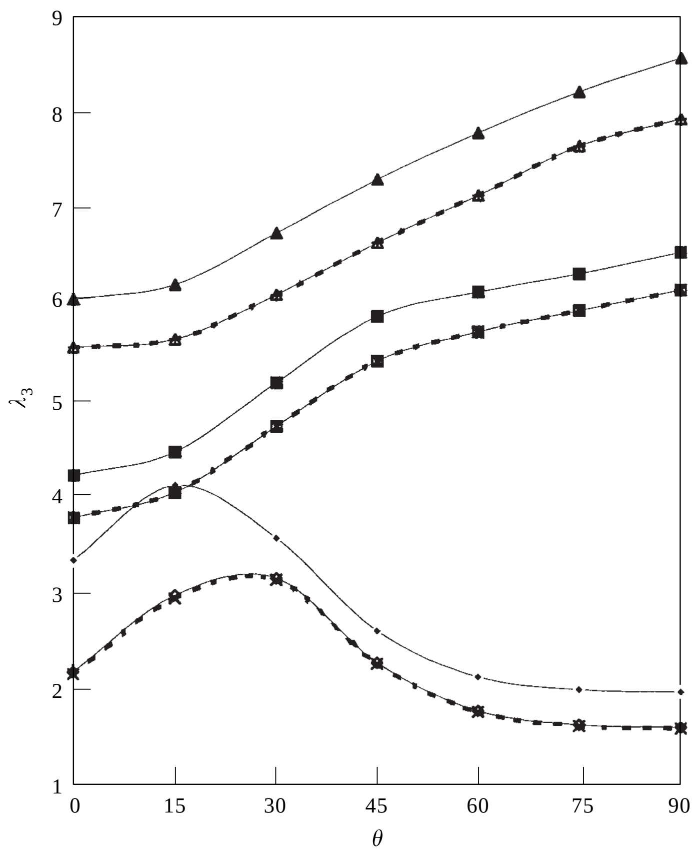
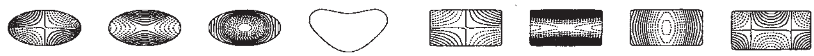
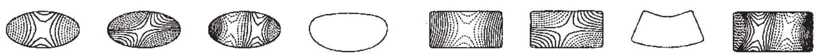
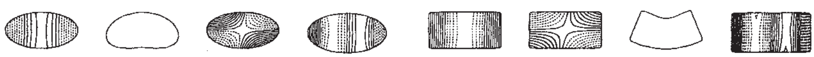
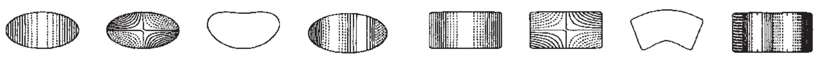


Figure 3. Effect of stacking angle θ on the frequency parameter λ_3 of super elliptical laminate with $a/h = 5$, $a/b = 2$, and stacking sequence. $[\theta/-\theta]_S$. —◆—, $n = 1$, free; —■—, $n = 1$, ss; —▲—, $n = 1$, clamp; —◇—, $n = 10$, free; —□—, $n = 10$, ss; —△—, $n = 10$, clamp; —×—, $n = \infty$, free; —*—, $n = \infty$, free; —+—, $n = \infty$, clamp.

TABLE 6

Displacement contours for lowest four frequencies of free, super elliptical laminate with $n = 1$ and 10, $a/h = 5$, and stacking sequence $[\theta/-\theta]_S$

θ	Mode sequence number, $n=1$: 1	2	3	4	$n=10$: 1	2	3	4
								
0	3.3359	5.3822	6.7392	7.1214	2.1790	5.1207	5.5475	5.9429
								
15	4.1175	5.2346	7.3218	7.4677	2.9709	4.8465	5.6788	5.9233
								
30	3.5623	5.4657	7.0106	8.3758	3.1406	4.1083	6.1491	6.5456
								
45	2.5983	5.5466	5.6871	6.2363	2.2680	3.9553	4.7067	5.1300
								
60	2.1202	4.6003	4.6860	5.2355	1.7616	3.4721	3.6795	4.1825
								
75	1.9846	4.1299	4.3227	4.3574	1.6147	2.9675	3.1344	3.8385
								
90	1.9576	3.1600	3.7627	4.2841	1.5896	2.1169	2.9215	3.7823

where φ_i^u, φ_i^v, φ_i^w, $\varphi_i^{\theta_u}$, and $\varphi_i^{\theta_v}$ are the shape functions and c_i^u, c_i^v, c_i^w, $c_i^{\theta_u}$, and $c_i^{\theta_v}$ are the associated unknown coefficients. ξ and η denote the non-dimensional co-ordinates given by

$$\xi = \frac{x}{a}, \qquad \eta = \frac{y}{b}. \tag{10}$$

The problem now lies in finding suitable shape functions that are general for any boundary conditions and plate geometries.

2.2. *p*-RITZ PROCEDURES

In the *p*-Ritz procedures, the shape functions, φ_i^u, φ_i^y, φ_i^w, $\varphi_i^{\theta_u}$, and $\varphi_i^{\theta_v}$ are assumed to be the product of two-dimensional polynomials and basic functions as follows:

$$\varphi_i^\kappa(\xi,\eta) = f_i(\xi,\eta)\varphi_b^\kappa(\xi,\eta), \tag{11}$$

in which $\kappa = u, v, w, \theta_u$ and θ_v. The functional $f_i(\xi,\eta)$ can be constructed by a two-dimensional polynomial series

$$\sum_{i=1}^{m} f_i(\xi,\eta) = \sum_{q=0}^{p}\sum_{i=0}^{q} \xi^{q-i}\eta^i. \tag{12}$$

Therefore, the number of terms m in equation (9) becomes

$$m = \frac{(p+1)(p+2)}{2}, \tag{13}$$

where p is the highest degree of the set of two-dimensional polynomials.

To satisfy the geometry of a laminated plate, the basic function $\varphi_b^\kappa(\xi, \eta)$ in equation (9) is assumed to be the product of boundary expressions of all supporting edges. The basic function, for the super elliptical laminated plate with super elliptic power n, can be assumed as

$$\varphi_b^\kappa(\xi, \eta) = [(2\xi)^{2n} + (2\eta)^{2n} - 1]^{\Omega^\kappa}, \tag{14}$$

in which Ω^κ represents the associated basic power of boundary expression to ensure automatic satisfaction of the boundary condition of the supporting edge. They are assumed to be 0, 1, and 2 depending on whether the boundary condition of the supporting edge is free, simply supported, or clamped. Note that the simply-supported edges are subject to constraint in the z direction only, i.e., the soft simply-supported condition.

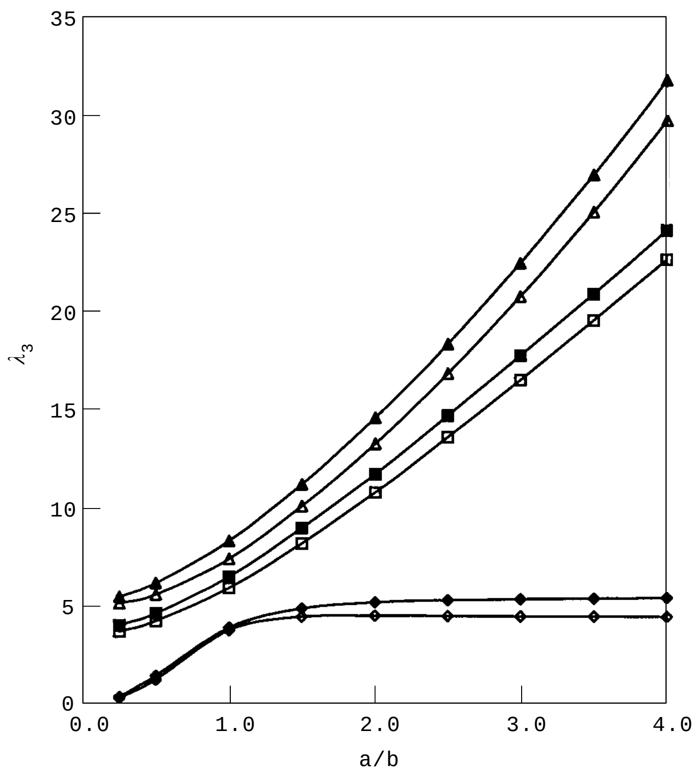


Figure 4. Effect of aspect ratio a/b on the frequency parameter λ_3 of super elliptical laminate with $a/h = 5$, and stacking sequence $[45/-45]_S$. —◆—, $n = 1$, free; —■—, $n = 1$, ss; —▲—, $n = 1$, clamp; —◇—, $n = 10$, free; —□—, $n = 10$, ss; —△—, $n = 10$, clamp.

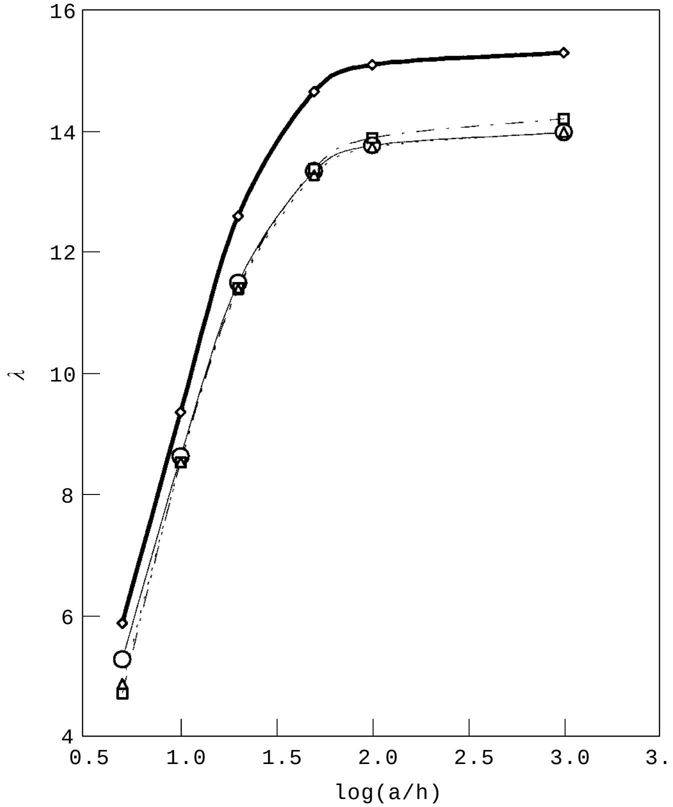


Figure 5. Effect of length-to-thickness ratio a/h on the frequency parameter λ of simply-supported, super elliptical laminate with $a/b = 2$ and stacking sequence $[45/-45]_S$. —◇—, $n = 1$; —○—, $n = 2$; —△—, $n = 4$; —□—, $n = 10$.

Substituting equation (9) into equation (7) and minimizing the total energy functional Π with respect to the unknown coefficients yields the governing eigenvalue equation

$$\{[\mathbf{K}] - \lambda^2[\mathbf{M}]\}\{c\} = \{0\}, \tag{15}$$

where $\{c\} = \{c^u\ c^v\ c^w\ ac^{\theta_u}\ bc^{\theta_v}\}^T$.

If all laminae are made of the same material, the non-dimensional frequency parameter λ can be expressed in terms of frequency, plate dimensions, D_0, and mass density per unit volume ρ as

$$\lambda = \omega ab\sqrt{\frac{\rho h}{D_0}}, \tag{16}$$

where

$$D_0 = \frac{E_{11}h^3}{12(1 - \nu_{12}\nu_{21})}. \tag{17}$$

The vibration frequencies and mode shapes of super elliptical laminates are then obtained by solving λ. In addition, the stiffness matrix K and the mass matrix M in equation (15) are given by

$$[\mathbf{K}] = \frac{1}{D_0}\begin{bmatrix} [\mathbf{K}^{uu}] & [\mathbf{K}^{uv}] & 0 & 0 & 0 \\ & [\mathbf{K}^{vv}] & 0 & 0 & 0 \\ & & [\mathbf{K}^{ww}] & [\mathbf{K}^{w\theta_u}] & [\mathbf{K}^{w\theta_v}] \\ & \text{sym.} & & [\mathbf{K}^{\theta_u\theta_u}] & [\mathbf{K}^{\theta_u\theta_v}] \\ & & & & [\mathbf{K}^{\theta_v\theta_v}] \end{bmatrix} \tag{18}$$

TABLE 7

Displacement contours and mode shapes for lowest four frequencies of elliptical laminate with $a/b = 2$, $a/h = 5$, and stacking sequence $[(45/-45)_2]_S$

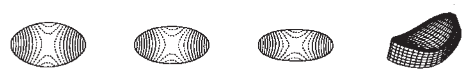
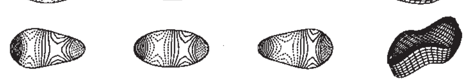
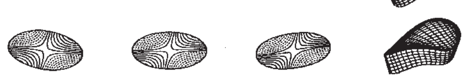
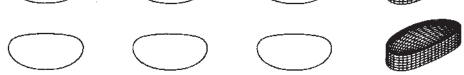
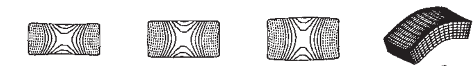
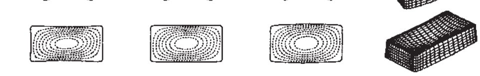
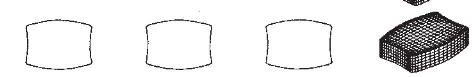
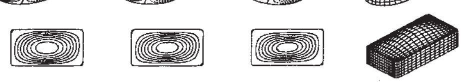
Boundary condition	Mode number	Displacement contour: Top surface	Displacement contour: Middle surface	Displacement contour: Bottom surface	3-D mode shape	Frequency parameter
Free	1					2.6506
	2					5.9443
	3					6.2079
	4					6.2363
Simply-supported	1					6.2363
	2					6.3032
	3					6.7571
	4					8.8983
Clamped	1					7.7140
	2					10.2645
	3					13.1000
	4					14.9326

TABLE 8

Displacement contours and mode shapes for lowest four frequencies of super elliptical laminate with $n = 10$, $a/b = 2$, $a/h = 5$, and stacking sequence $[(45/-45)_2]_S$

Boundary condition	Mode number	Displacement contour: Top surface, Middle surface, Bottom surface; 3-D mode shape	Frequency parameter
Free	1		2.3509
	2		4.4115
	3		4.7067
	4		5.3671
Simply-supported	1		4.7067
	2		5.8077
	3		5.9631
	4		7.7897
Clamped	1		7.0291
	2		8.9015
	3		11.4386
	4		13.6623

and

$$[\mathbf{M}] = \begin{bmatrix} [\mathbf{M}^{uu}] & 0 & 0 & 0 & 0 \\ & [\mathbf{M}^{vv}] & 0 & 0 & 0 \\ & & [\mathbf{M}^{ww}] & [\mathbf{M}^{w\theta_u}] & [\mathbf{M}^{w\theta_v}] \\ & \text{sym.} & & [\mathbf{M}^{\theta_u\theta_u}] & 0 \\ & & & & [\mathbf{M}^{\theta_v\theta_v}] \end{bmatrix}. \tag{19}$$

TABLE 9

Lowest eight frequency parameters λ for free elliptical laminate with a/h = 5 and stacking sequence $[(\theta/-\theta)_2]_S$

			Mode sequence number							
n	a/b	θ	1	2	3	4	5	6	7	8
1	1	0	3·8355	3·8529	7·9348	8·2319	9·9345	9·9391	10·0149	11·8522
		15	3·8740	6·5375	8·4631	10·2392	10·8724	11·1064	12·2286	13·5895
		30	3·9148	8·3484	9·8057	10·2638	12·0214	12·1627	12·4177	15·7178
		45	3·9542	8·8445	9·9859	11·5871	11·9448	12·4695	14·5405	14·5416
	2	0	3·3359	5·3822	6·7392	7·1214	7·2982	7·5228	8·0060	9·4220
		15	4·7554	5·3504	7·4126	8·2400	8·7515	9·4957	10·6249	11·7389
		30	3·7594	6·0876	7·5448	8·3758	8·9408	9·5433	10·8490	11·2601
		45	2·6506	5·9443	6·2079	6·2363	6·7571	9·5001	9·6956	10·3471
		60	2·1292	4·6003	4·7504	5·5990	5·9170	7·7753	9·3017	9·3387
		75	1·9867	4·1299	4·3715	4·9561	5·2199	7·0919	8·1632	8·2162
		90	1·9576	3·1600	3·7627	4·2841	5·1030	5·7620	6·5888	6·9207
	4	0	2·0750	3·1010	3·3501	3·7560	4·6385	4·8308	5·4232	6·6055
		15	2·5733	3·0238	4·3726	5·3614	5·4861	6·3102	7·5503	7·7521
		30	2·1160	2·9655	3·4556	3·8548	5·4207	5·7005	6·1616	7·3944
		45	1·3891	1·8632	2·9139	2·9307	3·7558	4·1840	4·5754	4·9846
		60	1·0817	1·3745	2·3673	2·7859	2·8569	3·0535	3·8250	4·6572
		75	0·9993	1·2444	2·2028	2·5095	2·6218	2·7276	3·5784	4·2236
		90	0·9835	1·1788	1·9636	2·1667	2·4477	2·5733	3·4263	3·5188
10	1	0	2·4422	3·1796	5·3338	7·4437	7·5641	8·3404	8·6276	9·2342
		15	3·2413	4·3501	7·6215	7·7276	7·9058	8·7520	10·4285	11·0553
		30	3·5714	5·7322	8·6906	8·7032	8·9507	9·1548	9·3485	11·8926
		45	3·8418	6·0597	8·7538	9·1791	9·3469	9·4178	11·3303	11·3304
	2	0	2·1790	5·1207	5·5475	5·9429	5·9442	6·2058	6·5524	8·1487
		15	3·5088	4·8728	5·7269	6·4878	6·8453	8·5682	8·6601	8·6960
		30	3·5430	4·3474	6·5351	7·0646	7·2709	7·9042	8·4286	8·7014
		45	2·3509	4·4115	4·7067	5·3671	5·9631	7·5711	8·0423	9·4184
		60	1·7774	3·4721	4·1747	4·2493	4·7518	7·1492	7·3594	7·5729
		75	1·6166	3·1344	3·4195	3·8451	4·3856	6·3008	6·4937	6·6911
		90	1·5896	2·1169	2·9215	3·7823	4·2043	4·3141	5·3901	6·3094
	4	0	1·5199	2·7692	2·8515	3·1639	4·2732	4·4540	4·9258	6·2586
		15	2·0254	2·6340	3·9082	4·3912	4·6516	5·8604	6·7877	7·1964
		30	1·8171	2·3245	2·7677	3·5037	4·7583	5·2835	5·4233	6·6695
		45	1·1521	1·4970	2·2618	2·5946	3·1816	3·7140	4·2036	4·5360
		60	0·8798	1·1035	2·0738	2·1380	2·3964	2·7230	3·4775	4·2530
		75	0·8078	1·0008	1·9064	1·9198	2·1940	2·4385	3·2412	3·7956
		90	0·7948	0·9528	1·4501	1·8912	2·1571	2·1937	2·8976	3·1963

TABLE 9 (*Continued*)

∞	1	0	2·4132	3·1608	5·2635	7·3768	7·5286	8·3086	8·5967	9·1287
		15	3·2234	4·3045	7·5359	7·7023	7·8326	8·7239	10·3675	10·9556
		30	3·5605	5·6770	8·6267	8·6858	8·8597	9·1394	9·2790	11·8042
		45	3·8421	6·0011	8·6606	9·1040	9·2978	9·4174	11·2521	11·2521
	2	0	2·1548	5·0995	5·5184	5·8798	5·8846	6·1769	6·5340	8·0607
		15	3·4757	4·8707	5·6738	6·4239	6·7952	8·4882	8·6451	8·6480
		30	3·5344	4·3122	6·5210	7·0101	7·2053	7·9031	8·3637	8·6073
		45	2·3413	4·3730	4·6688	5·3617	5·9515	7·5085	8·0174	9·3327
		60	1·7678	3·4439	4·1372	4·2352	4·7370	7·1393	7·2977	7·4992
		75	1·6073	3·1092	3·3874	3·8277	4·3701	6·2415	6·4710	6·6247
		90	1·5804	2·0956	2·8995	3·7643	4·1576	4·2984	5·3481	6·2369
	4	0	1·5077	2·7575	2·8384	3·1395	4·2568	4·4397	4·8897	6·2324
		15	2·0110	2·6218	3·8898	4·3634	4·6236	5·8359	6·7577	7·1556
		30	1·8080	2·3074	2·7484	3·4931	4·7253	5·2746	5·4116	6·6181
		45	1·1458	1·4864	2·2446	2·5846	3·1714	3·6846	4·1936	4·5045
		60	0·8748	1·0958	2·0643	2·1218	2·3879	2·7016	3·4651	4·2233
		75	0·8031	0·9938	1·8918	1·9107	2·1860	2·4196	3·2281	3·7683
		90	0·7902	0·9464	1·4388	1·8821	2·1492	2·1784	2·8754	3·1832

More explicitly, the elements of **K** can be expressed as

$$\mathbf{K}_{ij}^{uu} = A_{66}\left(\frac{a^2}{h^3}\right)\mathbf{R}^{0101}_{\phi_i^u\varphi_j^u} + A_{16}\left(\frac{ab}{h^3}\right)[\mathbf{R}^{0110}_{\varphi_i^u\varphi_j^u} + \mathbf{R}^{1001}_{\varphi_i^u\varphi_j^u}] + A_{11}\left(\frac{b^2}{h^3}\right)\mathbf{R}^{1010}_{\varphi_i^u\varphi_j^u},$$

$$\mathbf{K}_{ij}^{uv} = A_{26}\left(\frac{a^2}{h^3}\right)\mathbf{R}^{0101}_{\phi_i^u\varphi_j^v} + \left(\frac{ab}{h^3}\right)[A_{66}\mathbf{R}^{0110}_{\varphi_i^u\varphi_j^v} + A_{12}\mathbf{R}^{1001}_{\varphi_i^u\varphi_j^v}] + A_{16}\left(\frac{b^2}{h^3}\right)\mathbf{R}^{1010}_{\varphi_i^u\varphi_j^v},$$

$$\mathbf{K}_{ij}^{vv} = A_{22}\left(\frac{a^2}{h^3}\right)\mathbf{R}^{0101}_{\phi_i^v\varphi_j^v} + A_{26}\left(\frac{ab}{h^3}\right)[\mathbf{R}^{0110}_{\varphi_i^v\varphi_j^v} + \mathbf{R}^{1001}_{\varphi_i^v\varphi_j^v}] + A_{66}\left(\frac{b^2}{h^3}\right)\mathbf{R}^{1010}_{\varphi_i^v\varphi_j^v},$$

$$\begin{aligned}\mathbf{K}_{ij}^{ww} = {} & \left[A_{44}\left(\frac{a^2}{h^3}\right) - D_{44}\left(\frac{8a^2}{h^5}\right) + F_{44}\left(\frac{16a^2}{h^7}\right)\right]\mathbf{R}^{0101}_{\varphi_i^w\varphi_j^w} + \left[A_{45}\left(\frac{ab}{h^3}\right)\right. \\ & \left. - D_{45}\left(\frac{8ab}{h^5}\right) + F_{45}\left(16\frac{ab}{h^7}\right)\right][\mathbf{R}^{0110}_{\varphi_i^w\varphi_j^w} + \mathbf{R}^{1001}_{\varphi_i^w\varphi_j^w}] + \left[A_{55}\left(\frac{b^2}{h^3}\right) - D_{55}\left(\frac{8b^2}{h^5}\right)\right. \\ & \left. + F_{55}\left(\frac{16b^2}{h^7}\right)\right]\mathbf{R}^{1010}_{\varphi_i^w\varphi_j^w} + H_{22}\left(\frac{16a^2}{9b^2h^7}\right)\mathbf{R}^{0202}_{\varphi_i^w\varphi_j^w} + H_{26}\left(\frac{32a}{9bh^7}\right)[\mathbf{R}^{0211}_{\varphi_i^w\varphi_j^w} \\ & + \mathbf{R}^{1102}_{\varphi_i^w\varphi_j^w}] + H_{66}\left(\frac{64}{9h^7}\right)\mathbf{R}^{1111}_{\varphi_i^w\varphi_j^w} + H_{16}\left(\frac{32b}{9ah^7}\right)[\mathbf{R}^{1120}_{\varphi_i^w\varphi_j^w} + \mathbf{R}^{2011}_{\varphi_i^w\varphi_j^w}] \\ & + H_{12}\left(\frac{16}{9h^7}\right)[\mathbf{R}^{0220}_{\varphi_i^w\varphi_j^w} + \mathbf{R}^{2002}_{\varphi_i^w\varphi_j^w}] + H_{11}\left(\frac{16b^2}{9a^2h^7}\right)\mathbf{R}^{2020}_{\varphi_i^w\varphi_j^w},\end{aligned}$$

TABLE 10

Lowest eight frequency parameters λ for simply-supported, super elliptical laminate with $n = 10$, $a/h = 5$, and stacking sequence $[(\theta/-\theta)_2]_S$

			Mode sequence number							
n	a/b	θ	1	2	3	4	5	6	7	8
1	1	0	6·0218	8·9537	9·9345	9·9391	10·0149	13·1332	13·2958	13·9147
		15	6·2257	9·6896	10·2392	11·1064	13·2903	14·1577	16·5516	18·5338
		30	6·7141	10·2638	11·2900	12·1627	13·4590	16·3961	16·9869	17·6483
		45	7·0343	9·9859	12·3593	13·5601	14·5405	14·5416	17·8294	18·2566
	2	0	4·2239	7·1214	7·5015	7·5228	9·2323	9·4220	10·8543	10·8946
		15	4·5858	7·7968	9·4957	9·7773	11·3612	11·8185	12·9794	14·0286
		30	5·5089	8·3758	8·4169	10·8490	11·2601	11·4316	11·6121	14·2967
		45	6·2363	6·3032	6·7571	8·8983	10·3471	11·7060	12·4805	12·9732
		60	4·6003	5·5990	6·4745	8·5047	9·3387	10·0348	10·8686	13·4758
		75	4·1299	5·2199	6·4787	7·9765	8·2162	9·5996	9·9458	12·2583
		90	3·7627	5·1030	6·5329	6·5888	7·8253	8·7004	9·2121	9·5456
	4	0	3·3501	4·5742	4·6385	5·9758	6·6055	7·5102	7·6035	7·6410
		15	4·8222	5·3614	6·2609	7·6509	7·6914	7·9651	8·2857	8·5828
		30	3·4556	5·6509	6·1616	6·9808	7·5640	7·7748	7·8111	8·1697
		45	1·8682	3·7558	4·1840	6·4103	6·5880	7·0229	7·7208	7·8839
		60	1·3745	2·8569	3·0535	5·1009	5·2461	6·7557	7·3930	7·5459
		75	1·2444	2·6218	2·7276	4·5052	4·8396	6·4755	6·9217	7·0295
		90	1·1788	2·4477	2·5733	3·8652	4·7321	5·3569	6·8588	6·8892
10	1	0	5·5139	7·4437	7·5911	8·3404	8·6276	11·6807	12·3482	12·8785
		15	5·7915	7·9058	8·4019	8·7520	12·3620	12·6074	14·0926	14·9557
		30	6·2049	8·7032	9·1548	10·0722	12·2543	14·7257	15·5634	15·8216
		45	6·4412	9·4178	11·3303	11·3304	11·3681	11·9767	12·7975	15·5033
	2	0	3·7805	6·2058	6·5524	6·6801	8·2788	8·3813	8·6247	9·8631
		15	4·1696	6·9830	8·5682	8·6601	8·8967	10·2420	10·8516	11·4450
		30	5·0301	7·4528	7·9042	8·4286	9·1073	10·0283	10·4381	10·5896
		45	4·7067	5·8077	5·9631	7·7897	9·4184	10·1696	10·3722	11·1583
		60	3·4721	4·7518	6·0334	7·3778	7·5729	9·1544	9·4281	11·9138
		75	3·1344	4·3856	6·0940	6·6911	6·8442	8·3920	8·7507	10·5338
		90	2·9215	4·3141	5·3901	6·1470	6·6350	7·9821	8·0470	8·0996
	4	0	2·8515	4·1783	4·2732	5·0114	6·2586	6·3583	6·5740	7·6035
		15	4·3912	4·4356	5·3006	6·6478	7·6509	7·6908	8·1431	8·1776
		30	2·7677	5·2864	5·4233	6·0165	6·6695	7·1884	7·7748	7·8104
		45	1·4970	3·1816	3·7140	5·9631	6·0223	6·5249	6·6994	7·7252
		60	1·1035	2·3964	2·7230	4·7513	4·7754	6·4272	6·7814	7·0201
		75	1·0008	2·1940	2·4385	4·2302	4·3850	6·2181	6·5704	6·7032
		90	0·9528	2·1571	2·1937	3·6288	4·3135	5·0804	6·4663	6·4686

TABLE 10 (*Continued*)

∞	1	0	5·5153	7·3768	7·5927	8·3086	8·5967	11·6801	12·3501	12·8138
		15	5·7954	7·8326	8·4060	8·7239	12·3664	12·6082	14·1003	14·8094
		30	6·2099	8·6267	9·1394	10·0758	12·2596	14·7291	15·5713	15·8170
		45	6·4464	9·4174	11·2521	11·2521	11·3694	11·9846	12·6109	15·5087
	2	0	3·7817	6·1769	6·5340	6·6822	8·1903	8·3813	8·5963	9·8137
		15	4·1722	6·9883	8·4882	8·6451	8·8976	10·2454	10·8561	11·3357
		30	5·0329	7·4594	7·9031	8·3637	9·0549	9·8884	10·4439	10·5931
		45	4·6688	5·8107	5·9515	7·7968	9·4174	10·0843	10·3795	11·0780
		60	3·4439	4·7370	6·0361	7·3836	7·4992	9·1397	9·4337	11·9176
		75	3·1092	4·3701	6·0960	6·6247	6·8481	8·3936	8·7239	10·5329
		90	2·8995	4·2984	5·3481	6·1479	6·6361	7·9802	7·9812	8·8012
	4	0	2·8384	4·1784	4·2568	5·0114	6·2324	6·3576	6·5554	7·6035
		15	4·3634	4·4362	5·3027	6·6508	7·6509	7·6908	8·0926	8·1756
		30	2·7484	5·2883	5·4116	6·0229	6·6181	7·1982	7·7748	7·8104
		45	1·4864	3·1714	3·6846	5·9518	6·0247	6·4686	6·7075	7·7370
		60	1·0958	2·3879	2·7016	4·7334	4·7370	6·4289	6·7857	7·0040
		75	0·9938	2·1860	2·4196	4·1937	4·3701	6·1603	6·5501	6·7043
		90	0·9464	2·1492	2·1784	3·6016	4·2984	5·0392	6·4120	6·4476

$$\mathbf{K}_{ij}^{w\theta_u} = \left[A_{45}\left(\frac{ab}{h^3}\right) - D_{45}\left(\frac{8ab}{h^5}\right) + F_{45}\left(\frac{16ab}{h^7}\right)\right]\mathbf{R}^{0100}_{\varphi_i^w\varphi_j^u} + \left[H_{26}\left(\frac{16a}{9bh^7}\right)\right.$$

$$\left. - F_{26}\left(\frac{4a}{3bh^5}\right)\right]\mathbf{R}^{0201}_{\varphi_i^w\varphi_j^u} + \left[H_{12}\left(\frac{16}{9h^7}\right) - F_{12}\left(\frac{4}{3h^5}\right)\right]\mathbf{R}^{0210}_{\varphi_i^w\varphi_j^{\theta_u}} + \left[A_{55}\left(\frac{b^2}{h^3}\right)\right.$$

$$\left. - D_{55}\left(\frac{8b^2}{h^5}\right) + F_{55}\left(\frac{16b^2}{h^7}\right)\right]\mathbf{R}^{1000}_{\varphi_i^w\varphi_j^{\theta_u}} + \left[H_{66}\left(\frac{32}{9h^7}\right) - F_{66}\left(\frac{8}{3h^5}\right)\right]\mathbf{R}^{1101}_{\varphi_i^w\varphi_j^{\theta_u}}$$

$$+ \left[H_{16}\left(\frac{16b}{9ah^7}\right) - F_{16}\left(\frac{4b}{3ah^5}\right)\right][2\mathbf{R}^{1110}_{\varphi_i^w\varphi_j^{\theta_u}} + \mathbf{R}^{2001}_{\varphi_i^w\varphi_j^{\theta_u}}] + \left[H_{11}\left(\frac{16b^2}{9a^2h^7}\right)\right.$$

$$\left. - F_{11}\left(\frac{4b^2}{3a^2h^5}\right)\right]\mathbf{R}^{2010}_{\varphi_i^w\varphi_j^{\theta_u}},$$

$$\mathbf{K}_{ij}^{w\theta_v} = \left[A_{44}\left(\frac{a^2}{h^3}\right) - D_{44}\left(\frac{8a^2}{h^5}\right) + F_{44}\left(\frac{16a^2}{h^7}\right)\right]\mathbf{R}^{0100}_{\varphi_i^w\varphi_j^{\theta_v}} + \left[H_{22}\left(\frac{16a^2}{9b^2h^7}\right)\right.$$

$$\left. - F_{22}\left(\frac{4a^2}{3b^2h^5}\right)\right]\mathbf{R}^{0201}_{\varphi_i^w\varphi_j^{\theta_v}} + \left[H_{26}\left(\frac{16a}{9bh^7}\right) - F_{26}\left(\frac{4a}{3bh^5}\right)\right]\mathbf{R}^{0210}_{\varphi_i^w\varphi_j^{\theta_v}}$$

$$+ \left[A_{45}\left(\frac{ab}{h^3}\right)\right.$$

$$\left. - D_{45}\left(\frac{8ab}{h^5}\right) + F_{45}\left(\frac{16ab}{h^7}\right)\right]\mathbf{R}^{1000}_{\varphi_i^w\varphi_j^{\theta_v}}$$

$$+ \left[H_{26}\left(\frac{32a}{9bh^7}\right) - F_{26}\left(\frac{8a}{3bh^5}\right)\right]\mathbf{R}^{1101}_{\varphi_i^w\varphi_j^{\theta_v}}$$

$$+ \left[H_{66}\left(\frac{32}{9h^7}\right) - F_{66}\left(\frac{8}{3h^5}\right)\right][\mathbf{R}^{1110}_{\varphi_i^w\varphi_j^{\theta_v}} + \left[H_{12}\left(\frac{16}{9h^7}\right) - F_{12}\left(\frac{4}{3h^5}\right)\right]\mathbf{R}^{2001}_{\varphi_i^w\varphi_j^{\theta_v}}]$$

$$+ \left[H_{16}\left(\frac{16b}{9ah^7}\right) - F_{16}\left(\frac{4b}{3ah^5}\right)\right]\mathbf{R}^{2010}_{\varphi_i^w\varphi_j^{\theta_v}},$$

$$\mathbf{K}_{ij}^{\theta_u\theta_u} = \left[A_{55}\left(\frac{b^2}{h^3}\right) - D_{55}\left(\frac{8b^2}{h^5}\right) + F_{55}\left(\frac{16b^2}{h^7}\right)\right]\mathbf{R}^{0000}_{\varphi_i^{\theta_u}\varphi_j^{\theta_u}} + \left[D_{66}\left(\frac{1}{h^3}\right)\right.$$

$$\left. + H_{66}\left(\frac{16}{9h^7}\right) - F_{66}\left(\frac{8}{3h^5}\right)\right]\mathbf{R}^{0101}_{\varphi_i^{\theta_u}\varphi_j^{\theta_u}} + \left[D_{16}\left(\frac{b}{ah^3}\right) + H_{16}\left(\frac{16b}{9ah^7}\right)\right.$$

$$\left. - F_{16}\left(\frac{8b}{3ah^5}\right)\right][\mathbf{R}^{0110}_{\varphi_i^{\theta_u}\varphi_j^{\theta_u}} + \mathbf{R}^{1001}_{\varphi_i^{\theta_u}\varphi_j^{\theta_u}}] + \left[D_{11}\left(\frac{b^2}{a^2h^3}\right) + H_{11}\left(\frac{16b^2}{9a^2h^7}\right)\right.$$

$$\left. - F_{11}\left(\frac{8b^2}{3a^2h^5}\right)\right]\mathbf{R}^{1010}_{\varphi_i^{\theta_u}\varphi_j^{\theta_u}}$$

$$\mathbf{K}_{ij}^{\theta_u\theta_v} = \left[A_{45}\left(\frac{ab}{h^3}\right) - D_{45}\left(\frac{8ab}{h^5}\right) + F_{45}\left(\frac{16ab}{h^7}\right)\right]\mathbf{R}^{0000}_{\varphi_i^{\theta_u}\varphi_j^{\theta_v}} + \left[D_{26}\left(\frac{a}{bh^3}\right) + H_{26}\left(\frac{16a}{9bh^7}\right) - F_{26}\left(\frac{8a}{3bh^5}\right)\right]\mathbf{R}^{0101}_{\varphi_i^{\theta_u}\varphi_j^{\theta_v}} + \left[D_{66}\left(\frac{1}{h^3}\right) + H_{66}\left(\frac{16}{9h^7}\right) - F_{66}\left(\frac{8}{3h^5}\right)\right]\mathbf{R}^{0110}_{\varphi_i^{\theta_u}\varphi_j^{\theta_v}} + \left[D_{12}\left(\frac{1}{h^3}\right) + H_{12}\left(\frac{16}{9h^7}\right) - F_{12}\left(\frac{8}{3h^5}\right)\right]\mathbf{R}^{1001}_{\varphi_i^{\theta_u}\varphi_j^{\theta_v}} + \left[D_{16}\left(\frac{b}{ah^3}\right) + H_{16}\left(\frac{16b}{9ah^7}\right) - F_{16}\left(\frac{8b}{3ah^5}\right)\right]\mathbf{R}^{1010}_{\varphi_i^{\theta_u}\varphi_j^{\theta_v}},$$

$$\mathbf{K}_{ij}^{\theta_v\theta_v} = \left[A_{44}\left(\frac{a^2}{h^3}\right) - D_{44}\left(\frac{8a^2}{h^5}\right) + F_{44}\left(\frac{16a^2}{h^7}\right)\right]\mathbf{R}^{0000}_{\varphi_i^{\theta_v}\varphi_j^{\theta_v}} + \left[D_{22}\left(\frac{a^2}{b^2h^3}\right) + H_{22}\left(\frac{16a^2}{9b^2h^7}\right) - F_{22}\left(\frac{8a^2}{3b^2h^5}\right)\right]\mathbf{R}^{0101}_{\varphi_i^{\theta_v}\varphi_j^{\theta_v}} + \left[D_{26}\left(\frac{a}{bh^3}\right) + H_{26}\left(\frac{16a}{9bh^7}\right) - F_{26}\left(\frac{8a}{3bh^5}\right)\right][\mathbf{R}^{0110}_{\varphi_i^{\theta_v}\varphi_j^{\theta_v}} + \mathbf{R}^{1001}_{\varphi_i^{\theta_v}\varphi_j^{\theta_v}}] + \left[D_{66}\left(\frac{1}{h^3}\right) + H_{66}\left(\frac{16}{9h^7}\right) - F_{66}\left[\frac{8}{3h^5}\right)\right]\mathbf{R}^{1010}_{\varphi_i^{\theta_v}\varphi_j^{\theta_v}} . \tag{20}$$

Accordingly, the elements in $\mathbf{M}$ can be further expanded as

$$\mathbf{M}_{ij}^{uu} = h\mathbf{R}^{0000}_{\varphi_i^u\varphi_j^u}, \qquad \mathbf{M}_{ij}^{vv} = h\mathbf{R}^{0000}_{\varphi_i^v\varphi_j^v},$$

$$\mathbf{M}_{ij}^{ww} = h\mathbf{R}^{0000}_{\varphi_i^w\varphi_j^w} + \left(\frac{h^3}{252b^2}\right)\mathbf{R}^{0101}_{\varphi_i^w\varphi_j^w} + \left(\frac{h^3}{252a^2}\right)\mathbf{R}^{1010}_{\varphi_i^w\varphi_j^w},$$

$$\mathbf{M}_{ij}^{w\theta_u} = \left(\frac{-4h^3}{315a^2}\right)\mathbf{R}^{1000}_{\varphi_i^w\varphi_j^{\theta_u}}, \qquad \mathbf{M}_{ij}^{w\theta_v} = \left(\frac{-4h^3}{315b^2}\right)\mathbf{R}^{0100}_{\varphi_i^w\varphi_j^{\theta_v}},$$

$$\mathbf{M}_{ij}^{\theta_u\theta_u} = \left(\frac{17h^3}{315a^2}\right)\mathbf{R}^{0000}_{\varphi_i^{\theta_u}\varphi_j^{\theta_u}}, \qquad \mathbf{M}_{ij}^{\theta_v\theta_v} = \left(\frac{17h^3}{315b^2}\right)\mathbf{R}^{0000}_{\varphi_i^{\theta_v}\varphi_j^{\theta_v}}, \tag{21}$$

where

$$\mathbf{R}^{defg}_{\varphi_i^\alpha \theta_j^\beta} = \int\int_A \frac{\partial^{d+e}\varphi_i^\alpha(\xi,\eta)}{\partial\xi^d\,\partial\eta^e}\frac{\partial^{f+g}\theta_j^\beta(\xi,\eta)}{\partial\xi^f\,\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{22}$$

in which φ^α, $\theta^\beta = \varphi^u$, φ^v, φ^w, φ^{θ_u}, φ^{θ_v} and i, $j = 1, 2, \ldots, m$. In this study, the integrand $\mathbf{R}$ in equation (22) was obtained by using the Gaussian quadrature method.

3. NUMERICAL STUDIES AND DISCUSSIONS

Numerical results have been obtained using the proposed method for the symmetrically laminated, super elliptical plates subject to a variety of aspect ratios, length-to-thickness ratios, super elliptical powers, number of plies, stacking angles, and boundary conditions. A convergence study has been carried out to ensure a sufficient number of polynomials is employed in the integration. The results have been compared with published solutions. The material properties assumed for the examples of thick laminated plates are: $E_1/E_2 = 40$, $G_{12}/E_2 = 0{\cdot}6$, $G_{23}/E_2 = 0{\cdot}5$, $G_{13} = G_{12}$, $\nu_{12} = 0{\cdot}25$, and a length-to-thickness ratio of 5.

In the first example, a four-ply laminate with an aspect ratio of 2, stacking sequence $[\theta/-\theta]_s$, and subjected to free, simply-supported, and clamped boundary conditions have been examined. The stacking angle θ varies from 0 to 90° with 15° increment. As shown in Table 1, convergence of the lowest four frequency parameters λ has been studied by increasing the degree of polynomials p. It is seen that the errors between $p = 11$ and $p = 15$ are less than 0·4% in all cases and even smaller for the fundamental modes. Therefore, it is concluded that $p = 15$ is able to ensure convergence of results and has been adopted in all examples.

To check the accuracy of results, a comparison of the lowest six frequency parameters, $\lambda_1 = \omega(a/\pi)^2\sqrt{\rho h/D_0}$, has been presented in Table 2 for isotropic, super elliptical, thin plates with $a/b = 2$, $\nu = 0{\cdot}3$, simply-supported, and clamped boundary conditions. The results from this analysis show close agreement with published solutions [2, 3, 7] obtained from classical thin plate theory. In Tables 3 and 4, the present method has been applied to: (i) isotropic, super elliptical, thick plates with $a/b = 1$, $h/a = 0{\cdot}3$, and $\nu = 0{\cdot}3$; (ii) 16-ply, clamped, circular, thin laminates of E-glass/epoxy. As anticipated, published results [10, 15] using the Reddy's higher-order theory and the classical laminate theory are in excellent agreement with the results obtained by the present method.

To further understand the complicated effect of plate geometry, boundary conditions, and stacking angles on the non-dimensional frequency parameter of the first flexural mode, an extensive study has been conducted. First, the effect of super elliptical power n on the frequency parameter λ for thick super elliptical laminates with $a/b = 2$ and stacking sequence $[45/-45]_S$ has been illustrated in Figure 2. More details of the influence of super elliptical powers n and boundary conditions on the lowest eight frequency parameters of the same laminated plates are given in Table 5. From Figure 2, it is clear that the frequency parameters tend to converge when super elliptical power n is greater than 10 because of the similarity between such super ellipse and the rectangle which is the super ellipse with n approaching infinity. Lower frequency parameters are obtained for higher super elliptical power n because the increase in n leads to an increase of mass in the laminated super elliptical plate. Furthermore, stiffer boundary constraints lead to higher frequency parameters in all modes.

Next, the effect of stacking angle on the non-dimensional frequency parameter λ for the four-ply, thick, super elliptical laminate subject to free, simply-supported, and clamped boundary conditions is given in Figure 3. Super ellipse powers n of 1, 10, and infinity have been analyzed, with a laminate aspect ratio of 2, and the layering angle of $[\theta/-\theta]_s$. It is found the frequency parameters for clamped and simply-supported laminates increase with larger stacking angle. However, for free super elliptical laminates, the first flexural mode undergoes a transition from torsional mode to bending mode while the stacking angle increases. Table 6 provides further insight about this transition and it is noted that the displacement contours on the mid-plane of free super elliptical laminates are affected by the stacking angle. One may observe the frequencies increase with stacking angle for torsional modes but decrease for bending modes for free elliptical laminates. A similar trend can also be seen on the curves of $n = 10$ and $n = \infty$.

Figure 4 shows the effect of aspect ratio on the non-dimensional frequency parameters λ_3 of the thick, super elliptical laminate. Here, super elliptical powers n of 1 and 10 have been considered and the laminates' aspect ratios vary from 0·25, 0·5, 1·0, 1·5, 2·0, 2·5, 3·0, 3·5, to 4·0. A stacking sequence of $[45/-45]_S$ was studied in this example. As expected, the frequencies of super elliptical laminate with $n = 1$ are higher than those of $n = 10$. It is also seen that higher frequencies were obtained for clamped, simply-supported super elliptical laminates with higher aspect ratios. The increase in aspect ratio leads to the decrease in mass and consequently increases the frequency. Interestingly, the frequencies of free super elliptical laminates exhibit the tendency to converge when their aspect ratios are over 1·5. A similar trend was seen in Narita's investigation on the free, elliptical, orthotropic thin plates [5]. It should be addressed that the first flexural modes compared in Figures 2–4 may not be the lowest modes. This is because some lowest modes of simply-supported and free laminates are in-plane modes whose transverse displacement magnitude is far smaller in comparison with the in-plane displacements. As shown in Table 6, the in-plane modes are completely without contour lines but with significantly deformed outlines. It is also worth noting that the first in-plane modes of both simply-supported and free laminates with the same plate geometry yield similar frequency results since in-plane displacements between simply-supported and free laminates are similar.

Simply-supported, super elliptical, thick laminates have been studied on the effect of length-to-thickness ratio on the fundamental frequency parameters by varying length-to-thickness ratio a/h from 5, 10, 20, 50, 100, to 1000. As can be seen in Figure 5, the fundamental frequency converges as the laminate becomes thinner ($a/h > 100$). Super elliptical laminates with $n = 4$ and $n = 10$ show very similar curves which indicates that the geometric differences are not significant. Moreover, by comparing the curves of $n = 2$, 4, and 10 in Figure 6, it is found that the increase in n does not guarantee a higher fundamental frequency if a/h is greater than 50.

The following examples consider eight-ply, super elliptical, thick laminates subjected to free, simply-supported, and clamped boundary conditions. Selected displacement contours and mode shapes for the lowest four modes of the eight-ply, super elliptical, thick laminates with $a/b = 2$, $a/h = 5$, and stacking sequence $[(45/-45)_2]_S$ have been plotted. Tables 7 and 8 show the 3-D mode shapes and the displacement contours on top, middle, and bottom surfaces as the contours may change along the thickness direction of the thick laminates. A solid line denotes the displacement contours along the positive z direction while a dashed line has been used for the negative z direction. The displacement contours of modes 1 and 3 in simply-supported cases disappear because the in-plane displacements are dominant in these two modes.

A cursory investigation of the effects of super elliptical power, aspect ratio, and stacking angle on the frequency parameter λ is tabulated in Tables 9 and 10. The super elliptical power n is assumed to be 1 and 10 for Tables 9 and 10 respectively. The laminates are assumed to be subject to free, simply-supported and clamped boundary conditions and stacked in the sequence of $[(\theta/-\theta)_2]_s$. It is observed that the effect of stacking angle varies with the aspect ratio and boundary conditions of the laminates. From the results in Tables 9 and 10, it is found that maximum frequency parameters occur: (i) for free edge condition, with a stacking angle between 30 and 45° and an aspect ratio of 1; (ii) for free edge condition, with a stacking angle between 0 and 45° and an aspect ratio of 2; (iii) for free and simply-supported edge conditions, with a stacking angle between 0 and 30° and an aspect ratio of 4; (iv) for simply-supported and clamped edge conditions, with a stacking angle between 15 and 45° and an aspect ratio of 1; (v) for simply-supported edge condition, with a stacking angle between 15 and 60° and an aspect ratio of 2; (vi) for clamped edge condition, with a stacking angle between 45 and 75° and an aspect ratio of 2 or 4. Lastly, the similarity between the results of eight-ply, thick super elliptical laminates and those of four-ply cases in preceding examples indicates that the number of plies does not have a pronounced influence.

4. CONCLUSIONS

The natural frequencies and mode shapes of a class of plates with rounded corners have been obtained using the p-Ritz method. The plate planform is defined by a super elliptic function which can form a plate shape varying from a square or rectangle to a circle or ellipse. Numerical examples for symmetrically laminated super elliptic plates for free, simply-supported, and clamped boundary conditions have been considered by varying the fiber stacking sequence and degree of super ellipticity.

Although the p-Ritz method has been extensively used in plate vibration problems, in this paper the method has been applied successfully for the first time to the problem of super elliptical planform for symmetrically laminated plates with the inclusion of transverse shear deformation effects. This has been done by employing Reddy's higher-order plate deformation theory. The method could be readily extended to the calculation of natural frequencies and mode shapes of more complex geometry and boundary conditions, such as a super elliptical plate with internal line or ring supports.

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