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LETTERS TO THE EDITOR

DYNAMIC STIFFNESS OF AN AXIALLY MOVING STRING

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1. INTRODUCTION

The aim of this article is to provide the explicit dynamic stiffness matrix for the transverse oscillation of an axially moving string under a constant tension. This model has important practical applications in the transport of magnetic tapes, paper tapes, textile fibres, aerial haulage cables, power transmission chains, and bandsaw blades.

The dynamic stiffness method is an exact method for the dynamic analysis of various structures. It allows one to model an infinite number of natural modes with a small number of unknowns. It uses the finite element assembly procedure to formulate the system equations of complicated structures, but it does not introduce the discretisation errors associated with finite elements. The dynamic stiffness matrix of a structure is formulated using the exact solutions and hence requires intensive algebraic manipulations. Explicit dynamic stiffness matrices for a range of structural elements have been published, for instance, uniform thin beams [1], Timoshenko beams [2], non-uniform beams [3], etc.

2. FORMULATION

The dynamic stiffness method provides a powerful method for solving forced vibration problems and the explicit dynamic stiffness matrix of the moving string allows it to be assembled into a system which may consist of other structural components such as springs, masses, dampers, beams, and plates. Although the solution for the transverse oscillation of an axially moving string under a constant tension has existed for decades [4], and more recently closed-form solutions have been presented [5, 6], the explicit dynamic stiffness matrix of this system has not been published.

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The equation of motion for an axially moving string under a constant tension is

$$\frac{\partial^2 v}{\partial t^2} + 2c\,\frac{\partial}{\partial t}\frac{\partial v}{\partial x} - (a^2 - c^2)\,\frac{\partial^2 v}{\partial x^2} = 0, \tag{1}$$

where c is the axial speed of the string, a is the wave speed of the stationary string, $a^2 = R/(\rho A)$, R is the tension, ρ is the density, A is cross-sectional area, v is transverse displacement, x is the longitudinal co-ordinate, and t is time.

The boundary conditions on $x = 0$ and $x = L$ are

$$v = 0 \qquad \text{or} \qquad F = \rho A\left(-c\,\frac{\partial v}{\partial t} + (a^2 - c^2)\,\frac{\partial v}{\partial x}\right) = 0, \tag{2}$$

where L is the length of the string, and F is the restoring force.

A separable solution, consisting of a spatial form which varies harmonically with time, may be assumed, that is

$$v(x, t) = V(x)\,\mathrm{e}^{i\omega t}. \tag{3}$$

Substituting v into the equation of motion, gives

$$-\,\omega^2 V(x) + 2i\omega c\,\frac{\mathrm{d}V(x)}{\mathrm{d}x} - (a^2 - c^2)\,\frac{\mathrm{d}^2 V(x)}{\mathrm{d}x^2} = 0. \tag{4}$$

For the case when the transport velocity, c, is less than wave velocity, a, equation (4) is satisfied by solutions of the form [6]

$$V(x) = B_1\,\mathrm{e}^{i\omega x/(a-c)} + B_2\,\mathrm{e}^{-i\omega x/(a+c)}. \tag{5}$$

It should be noted that this solution is not valid when the transport velocity, c, is equal or greater than the wave velocity, a.

Equation (5), which was given in reference [6], will now be used in deriving the dynamic stiffness matrix. Taking the applied end forces as positive in the positive v direction, the end conditions are given by

$$P_1\,\mathrm{e}^{i\omega t} = \rho A\left(i\omega c V - (a^2 - c^2)\,\frac{\mathrm{d}V}{\mathrm{d}x}\right)\mathrm{e}^{i\omega t},$$

$$P_2\,\mathrm{e}^{i\omega t} = \rho A\left(-i\omega c V + (a^2 - c^2)\,\frac{\mathrm{d}V}{\mathrm{d}x}\right)\mathrm{e}^{i\omega t}. \tag{6}$$

Letting $x = 0$ at end 1 and $x = L$ at end 2, substituting equation (5) into equation (6) and rearranging into matrix form gives

$$\left\{ \begin{matrix} P_1 \\ \\ P_2 \end{matrix} \right\} = \rho A \left[(a^2 - c^2) \begin{bmatrix} \dfrac{-i\omega}{a-c} & \dfrac{i\omega}{a+c} \\ \\ \dfrac{i\omega}{a-c}\, \mathrm{e}^{i\omega L/(a-c)} & \dfrac{-i\omega}{a+c}\, \mathrm{e}^{-i\omega L/(a+c)} \end{bmatrix} + i\omega c \begin{bmatrix} 1 & 1 \\ \\ -\mathrm{e}^{i\omega L/(a-c)} & -\mathrm{e}^{-i\omega L/(a+c)} \end{bmatrix} \right] \left\{ \begin{matrix} B_1 \\ \\ B_2 \end{matrix} \right\},$$

$$\{\mathbf{P}\} = [\mathbf{D}]\{\mathbf{B}\}. \tag{7}$$

The unknown coefficients B_i can be expressed in terms of the maximum end deflections, thus

$$\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ \mathrm{e}^{i\omega L/(a-c)} & \mathrm{e}^{-i\omega L/(a+c)} \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix},$$

$$\{\mathbf{V}\} = [\mathbf{C}]\{\mathbf{B}\}. \tag{8}$$

Equations (7) and (8) can be combined to give the dynamic force–deflection relationship

$$\{\mathbf{P}\} = [\mathbf{D}][\mathbf{C}]^{-1}\{\mathbf{V}\} = [\mathbf{K}]\{\mathbf{V}\}, \tag{9}$$

where $[\mathbf{K}]$ is the dynamic stiffness matrix. After performing the algebraic manipulation, the dynamic stiffness matrix for the moving string is given explicitly as

$$[\mathbf{K}] = \frac{i\omega\rho A a}{\delta} \begin{bmatrix} -\mathrm{e}^{i\omega L/(a-c)} - \mathrm{e}^{-i\omega L/(a+c)} & 2 \\ 2\mathrm{e}^{2i\omega cL/(a^2-c^2)} & -e^{i\omega L/(a-c)} - \mathrm{e}^{-i\omega L/(a+c)} \end{bmatrix}, \tag{10}$$

where

$$\delta = -\mathrm{e}^{i\omega L/(a-c)} + \mathrm{e}^{-i\omega L/(a+c)}\,. \tag{11}$$

The exponential form of the dynamic stiffness can also be conveniently expressed in term of trigonometric functions, hence

$$[\mathbf{K}] = \omega\rho A a \begin{bmatrix} \cot\dfrac{\omega L a}{a^2-c^2} & -\csc\dfrac{\omega L a}{a^2-c^2}\,\mathrm{e}^{-i\omega Lc/(a^2-c^2)} \\ -\csc\dfrac{\omega L a}{a^2-c^2}\,\mathrm{e}^{i\omega Lc/(a^2-c^2)} & \cot\dfrac{\omega L a}{a^2-c^2} \end{bmatrix}. \tag{12}$$

When $c = 0$, the dynamic stiffness matrix reduces to that of a stationary tensioned string.

For a general system, comprised for example of moving strings, masses, springs and dampers, the component dynamic stiffness matrices may be assembled to give a global dynamic stiffness matrix, $[\mathbf{K}_g]$. The natural frequencies of the global system may then be found by solving for the values of ω which satisfy the equation.

$$|[\mathbf{K}_g]| = 0. \tag{13}$$

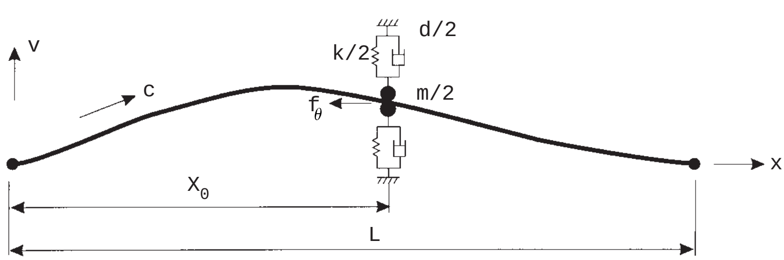

Figure 1. A moving string in contact with a stationary load system.

3. TEST EXAMPLES

Chen [7] provided, in graphical form, the approximate natural frequencies of an axially moving string in contact with various stationary load systems, using six-term eigenfunction expansion for the numerical solutions. He studied a uniform string travelling between two fixed supports separated by a distance L. The string was in contact with a stationary load system, consisting of a mass m, a spring k, a damper d and a longitudinal friction force f_θ, at a fixed distance X_0 from the left support (see Figure 1). The string was subjected to a tension R to the right of $x = X_0$, and $R - f_\theta$ to the left. He presented his numerical results in the following non-dimensional parameters [7]:

$$x^* = \frac{x}{L}, \quad v^* = \frac{v}{L}, \quad t^* = t\,\frac{a}{L}, \quad \omega^* = \omega\,\frac{L}{a}, \quad \lambda = i\omega^*, \quad \nu = \frac{c}{a}, \quad \xi = \frac{X_0}{L},$$

$$k^* = k\,\frac{L}{\rho A a^2}, \qquad m^* = \frac{m}{\rho A L}, \qquad d^* = \frac{d}{\rho A a}, \qquad f_\theta^* = \frac{f_\theta}{R},$$

where $a = \sqrt{R/(\rho A)}$ is the wave speed of the string, and λ is the non-dimensional eigenvalue.

The dynamic stiffness method presented in this article will now be used to produce exact solutions for the following six cases of Chen's non-dimensional parameters: (a) no intermediate support; (b) $k^* = 20$ at $\xi = 0{\cdot}3$; (c) $k^* = 1000$ at $\xi = 0{\cdot}3$; (d) $k^* = 3$ and $m^* = 0{\cdot}2$ at $\xi = 0{\cdot}3$; (e) $d^* = 1$ at $\xi = 0{\cdot}3$; and (f) $k^* = 20$ and $f_\theta^* = 0{\cdot}75$ at $\xi = 0{\cdot}5$.

TABLE 1

First three eigenvalues of a moving string, in contact with various load systems, and at various axial speeds, obtained using the dynamic stiffness method

	λ					
ν	Case (a)	Case (b)	Case (c)	Case (d)	Case (e)	Case (f)
0·0	3·1416i	4·2097i	4·4816i	3·2971i	−0·6998 + 3·2991i	3·0655i
	6·2832i	8·0633i	8·9629i	5·7712i	−1·0218 + 6·1452i	5·6221i
	9·4248i	9·6569i	10·4374i	9·3042i	−0·0959 + 9·4131i	6·2832i
0·2	3·0159i	4·0505i	4·3026i	3·1932i	−0·6718 + 3·1671i	2·5862i
	6·0319i	7·7733i	8·6049i	5·5783i	−0·9809 + 5·8993i	5·0803i
	9·0478i	9·2802i	10·0212i	8·9373i	−0·0921 + 9·0365i	5·6279i
0·4	2·6390i	3·5687i	3·7654i	2·8718i	−0·5878 + 2·7712i	1·1216i
	5·2779i	6·8913i	7·5306i	4·9940i	−0·8583 + 5·1619i	2·2423i
	7·9168i	8·1494i	8·7720i	7·8351i	−0·0806 + 7·9070i	3·3602i
0·6	2·0106i	2·7520i	2·8697i	2·2996i	−0·4478 + 2·1114i	
	4·0212i	5·3738i	5·7393i	3·9938i	−0·6539 + 3·9329i	
	6·0319i	6·2613i	6·6879i	5·9928i	−0·0614 + 6·0244i	
0·8	1·1310i	1·5761i	1·6148i	1·4033i	−0·2519 + 1·1877i	
	2·2619i	3·1241i	3·2297i	2·4967i	−0·3678 + 2·2123i	
	3·3929i	3·5933i	3·7654i	3·4000i	−0·0345 + 3·3887i	

In dimensional form for cases (a)–(e), the string may be modelled as two elements, one with length X_0, and one with length $L - X_0$, the spring, mass, and damper being added to the node at $x = X_0$. The global dynamic stiffness matrix, $[\mathbf{K}_g]$ may be found by assembling the elemental matrices giving

$$[\mathbf{K}_g] = \begin{bmatrix} \omega\rho A a_1 \cot \dfrac{\omega L a_1}{a_1^2 - c^2} & -\omega\rho A a_1 \csc \dfrac{\omega L a_1}{a_1^2 - c^2}\, \mathrm{e}^{-i\omega Lc/(a_1^2 - c^2)} & 0 \\ -\omega\rho A a_1 \csc \dfrac{\omega L a_1}{a_1^2 - c^2}\, \mathrm{e}^{i\omega Lc/(a_1^2 - c^2)} & K_{22} & -\omega\rho A a_2 \csc \dfrac{\omega L a_2}{a_2^2 - c^2}\, \mathrm{e}^{-i\omega Lc/(a_2^2 - c^2)} \\ 0 & -\omega\rho A a_2 \csc \dfrac{\omega L a_2}{a_2^2 - c^2}\, \mathrm{e}^{i\omega Lc/(a_2^2 - c^2)} & \omega\rho A a_2 \cot \dfrac{\omega L a_2}{a_2^2 - c^2} \end{bmatrix}, \tag{14}$$

where

$$K_{22} = \omega\rho A a_1 \cot\frac{\omega L a_1}{a_1^2 - c^2} + \omega\rho A a_2 \cot\frac{\omega L a_2}{a_2^2 - c^2} + k + i\omega d - \omega^2 m,$$

and $a_1 = \sqrt{(R - f_\theta)/(\rho A)}$ is the wave speed of element 1, and $a_2 = a = \sqrt{R/(\rho A)}$ is the wave speed of element 2.

Taking into account the fixed end conditions, that is zero displacement at $x = 0$ and $X = L$, the system force–deflection relationship reduces equation (14) to

$$[\mathbf{K}_g] = \omega\rho A\left(a_1 \cot\frac{\omega X_0 a_1}{a_1^2 - c^2} + a_2 \cot\frac{\omega(L - X_0)a_2}{a_2^2 - c^2}\right) + k + i\omega d - \omega^2 m, \quad (15)$$

which may be expressed in the above non-dimensional parameters as

$$[\mathbf{K}_g^*] = \omega^*\left(\sqrt{1 - f_\theta^*}\cot\frac{\omega^*\xi\sqrt{1 - f_\theta^*}}{(1 - f_\theta^*) - v^2} + \cot\frac{\omega^*(1 - \xi)}{1 - v^2}\right)$$
$$+ k^* + i\omega^* d^* - \omega^{*2} m^*, \quad (16)$$

For case (f), it is necessary to use four elements to represent the string because there is a node of the second mode of vibration at $X_0 = L/2$. When assembled and reduced in the manner described above, this leads to a 3×3 global dynamic stiffness matrix which was used to solve for natural frequencies. It should be noted

TABLE 2

First three eigenvalues of a moving string, in contact with various load systems, and at various axial speeds, digitized from reference [7]

	λ					
v	Case (a)	Case (b)	Case (c)	Case (d)	Case (e)	Case (f)
0·0	3·1i	4·2i	4·5i	3·3i	−0·70 + 3·3i	3·1i
	6·3i			5·8i	−1·00 + 6·1i	5·7i
						6·3i
0·2	3·0i	4·1i	4·3i	3·2i	−0·70 + 3·2i	2·6i
	6·0i			5·6i	−0·95 + 5·9i	5·2i
						5·6i
0·4	2·6i	3·6i	3·8i	2·9i	−0·60 + 2·8i	1·1i
	5·3i	7·0i		5·0i	−0·85 + 5·1i	2·2i
						3·4i
0·6	2·0i	2·8i	2·9i	2·3i	−0·45 + 2·1i	
	4·0i	5·5i	5·8i	4·0i	−0·65 + 3·9i	
	6·0i	6·3i	6·8i	6·0i	−0·05 + 6·0i	
0·8	1·1i	1·6i	1·6i	1·4i	−0·25 + 1·2i	
	2·3i	3·1i	3·2i	2·5i	−0·35 + 2·2i	
	3·4i	3·6i	3·8i	3·4i	−0·05 + 3·4i	

that for this case, due to the friction reducing the tension in the first span to one-quarter of that in the second span, the transport velocity reaches a critical value in the first span when $c = a_1$ corresponding to $c = a_2/2$ (or $v = 0{\cdot}5$). Natural frequencies beyond the critical speed of $v = 0{\cdot}5$ were not calculated because the solutions given in equation (5) for the sub-critical cases are not valid.

Table 1 shows the first three eigenvalues of the six cases for various axial velocities, v. These results may be used to check the accuracy of the graphical results presented by Chen [7]. Table 2 presents a selection of eigenvalues digitized by the present authors from the graphs in reference [7]. It may be seen that his results are in agreement with the exact results.

Chen in his Figure 7 also presented a set of results for conditions above the critical speed. But in view of the fact that his model is also a linear model, the present authors believe that those results are invalid.

4. CONCLUSION

The explicit dynamic stiffness of an axially moving string under constant tension has been given. The exact eigenvalues of an axially moving string in contact with a stationary load system have been calculated and compared with the published approximate results in reference [7].

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