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BOUNDARY CONDITIONING TECHNIQUE FOR STRUCTURAL TUNING

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Boundary conditions of a structure will strongly influence their natural frequencies. By controlling the conditions at the boundaries in a non-homogeneous and non-uniform fashion, it is shown that the structural natural frequencies can be manipulated in a favorable fashion. The technique is named boundary conditioning. An analytical model has been developed for a plate type structure to illustrate the boundary conditioning technique. The results are presented and discussed, along with references to steel pan tuning.

1. INTRODUCTION

The natural frequencies of structures are strongly influenced by the boundary conditions. Changing the boundary conditions from simply supported to clamped conditions in beams and plates can more than double their respective natural frequencies [1, 2]. The boundary conditions of real structures are not always homogeneous, such as clamped or simply supported. The boundary conditions may be flexible either in translation or rotation [3–5]. Furthermore, even along a single edge of a two-dimensional structure such as a plate, the boundary conditions may not be uniform and may change along the edge [6]. This suggests that it is possible to control the natural frequencies by properly controlling the boundary conditions along the edges.

In structures which are supported at several points or lines within the domain of the structures, such as in periodically supported beams or plates, such supports may be considered as providing boundary conditions for the structure. Hence, the designer has a lot of flexibility in positioning the supports in a structure in order to obtain specific natural frequencies for the structure. Such control of the frequency characteristics of a structure has many applications in the control of the noise transmission properties of structures such as in aircraft fuselages. In the present study, the concept of boundary conditioning is investigated on a plate type structure, supported on four translational and rotational type springs distributed along the four edges.

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2. ANALYSIS

Consider a rectangular plate, as shown in Figure 1. The plate is assumed to be supported at all the four edges with both translational and rotational springs [7] with stiffness per unit length of K_T and K_R, respectively.

The plate eigenvalues are estimated using boundary characteristic orthogonal polynomials in the Rayleigh–Ritz method [8]. These orthogonal polynomials are generated so as to satisfy the free edge conditions on all boundaries.

The flexural deflection of the plate is assumed as

$$W(x, y) = \sum_m \sum_n A_{mn}\phi_m(x)\varphi_n(y), \tag{1}$$

where $x = \xi/a$, $y = \eta/b$; ξ and η are co-ordinates of the plate, a and b are dimensions of the plate with $\alpha = a/b$, A_{mn} is the deflection coefficient of each term describing the plate deflection in equation (1), and $\phi_m(x)$ and $\varphi_n(y)$ are boundary characteristic orthogonal polynomials.

The maximum total strain energy U_{max} and the maximum kinetic energy T_{max} of the structure are

$$U_{max} = U_{max,p} + \sum_{s=1}^{4} U_{T,s} + \sum_{s=1}^{4} U_{R,s}, \qquad T_{max} = \frac{1}{2}\rho hab\omega^2 \int_0^1 \int_0^1 W^2(x, y)\,\mathrm{d}x\,\mathrm{d}y, \tag{2, 3}$$

where

$$U_{max,p} = \frac{Dab}{2a^4}\int_0^1 \int_0^1 [W''^2 + \alpha^4 \ddot{W}^2 + 2\upsilon \cdot \alpha^2 \cdot W''\ddot{W} + 2(1-\upsilon)\alpha^2 \dot{W}'^2]\,\mathrm{d}x\,\mathrm{d}y,$$

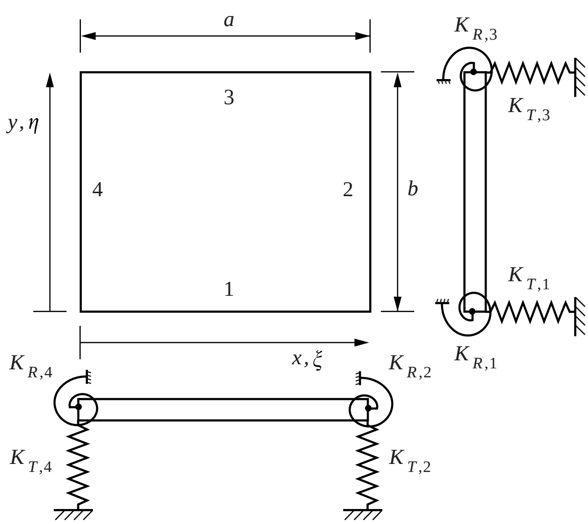

Figure 1. Structural scheme for boundary conditioning.

$$U_{T,s} = \frac{1}{2} K_{T,s} \int_0^{l_s} W_s^2 \, \mathrm{d}l_s, \qquad U_{R,s} = \frac{1}{2} K_{R,s} \int_0^{l_s} W_s'^2 \, \mathrm{d}l_s.$$

Substitution of deflection equation (1) into strain energy and kinetic energy expressions (2) and (3), and optimization of the Rayleigh quotient with respect to A_{ij} result in the eigenvalue equation

$$\sum_m \sum_n [C_{mnij} + C^I_{mnij} - \lambda \cdot E^{00}_{mi} F^{00}_{nj}] A_{mn} = 0, \tag{4}$$

where

$$C_{mnij} = E^{22}_{mi} F^{00}_{nj} + \alpha^4 E^{00}_{mi} F^{22}_{nj} + \upsilon \cdot \alpha^2 (E^{02}_{mi} F^{20}_{nj} + E^{20}_{mi} F^{02}_{nj}) + 2 \cdot (1 - \upsilon) \alpha^2 E^{11}_{mi} F^{11}_{nj},$$

$$\begin{aligned} C^I_{mnij} = {} & K^*_{T,1} \alpha E^{00}_{mi} \varphi_n(0) \varphi_j(0) + K^*_{T,2} \phi_m(1) \phi_i(1) F^{00}_{nj} + K^*_{T,3} \alpha E^{00}_{mi} \varphi_n(1) \varphi_j(1) \\ & + K^*_{T,4} \phi_m(0) \phi_i(0) F^{00}_{nj} + K^*_{R,1} \alpha^3 E^{00}_{mi} \varphi^1_n(0) \varphi^1_j(0) + K^*_{R,2} \phi^1_m(1) \phi^1_i(1) F^{00}_{nj} \\ & + K^*_{R,3} \alpha^3 E^{00}_{mi} \varphi^1_n(1) \varphi^1_j(1) + K^*_{R,4} \phi^1_m(0) \phi^1_i(0) F^{00}_{nj}, \end{aligned}$$

$$E^{rs}_{mi} = \int_0^1 \left(\frac{\mathrm{d}^r \phi_m}{\mathrm{d}x^r} \right) \left(\frac{\mathrm{d}^s \phi_i}{\mathrm{d}x^s} \right) \cdot \mathrm{d}x, \qquad F^{rs}_{nj} = \int_0^1 \left(\frac{\mathrm{d}^r \varphi_n}{\mathrm{d}y^r} \right) \left(\frac{\mathrm{d}^s \varphi_j}{\mathrm{d}y^s} \right) \cdot \mathrm{d}y,$$

$$\phi^1_i = \frac{\mathrm{d}\phi_i}{\mathrm{d}x}, \qquad \varphi^1_j = \frac{\mathrm{d}\varphi_j}{\mathrm{d}y}.$$

The solution of equation (4) yields both eigenvalues and mode shapes of the system. The different boundary conditions can be generated by different values of K_T and K_R. The eigenvalues of the system depend upon both translational and rotational stiffness for a given plate. The conditioning of the boundary by altering both the rotational stiffness and translational stiffness affects the flow of distribution of vibrational energy in the structure and hence the eigenvalues and natural frequencies.

The conditions of $K^*_{T,i} = 0$ and $K^*_{R,i} = 0$ correspond to free edge condition, $K^*_{R,i} = 0$ and very high values of $K^*_{T,i}$ correspond to a simply supported condition, while very high values of both $K^*_{T,i}$ and $K^*_{R,i}$ correspond to a clamped condition. The tuning of a structure by boundary conditioning thus means ensuring proper distribution of stiffness on the boundary such that the structure exhibits prescribed patterns of natural frequencies. In engineering applications, the natural frequencies may need to be prescribed in order to be away from exciting frequencies. In order to illustrate the concept of boundary conditioning to manipulate the natural frequencies, the prescribed natural frequencies pattern is chosen as a harmonic combination of natural frequencies in this study. The possible harmonic combinations for a square plate through boundary conditioning are found out by solving equation (4) for eigenfrequencies at different discrete values of four rotational stiffnesses ($K^*_{R,i}$, where $i = 1, 2, 3, 4$) and four translational stiffnesses ($K^*_{T,i}$, where $i = 1, 2, 3, 4$) placed at the edges. It is found that the ranges of rotational stiffness between 0 and 10^3 and translational stiffness between 0 and 10^5 only have a significant effect on the vibration behavior of the plate and hence the boundary conditioning is demonstrated in these ranges of stiffnesses.

The harmonic combination is represented by $h = 1ijk$ where i, j and k are integers and are the ratios of the first three dominating eigenfrequencies with respect to the fundamental eigenfrequency. It was felt that requiring i, j and k to be 2, 3 and 4, respectively, would be too stringent a requirement for compliance. Hence, in the present study, i, j and k were considered to be an acceptable combination if they fall close to any integers. In an actual engineering structure these ratios could be prescribed real numbers indicating the desired natural frequency patterns.

At each boundary condition defined by different values of boundary stiffnesses, the closest available harmonic combination of $h = 1ijk$ is the one which gives minimum error function (Λ_h), given by

$$\Lambda_h = \sum_{n=2}^{N_f} \min\left[\left(\frac{\lambda_n - q\lambda_1}{q\lambda_1}\right)^2 \forall q = i, j, k\right],$$

where N_f is the number of eigenfrequencies considered in the combination = 4, and λ_n is the nth eigenvalue. In this study, the integers i, j and k that could give a minimum value for Λ_h were checked up to a value of 25.

The value of Λ_h indicates the relative closeness of the arrangement of eigenfrequencies to a harmonic combination of $h = 1ijk$. A value of Λ_h closer to zero shows the availability of a perfect harmonic combination of h. In the present investigation, a value of 0·02 for Λ_h is considered accurate enough to consider the natural frequencies as a harmonic combination. Thus, it becomes possible to manipulate the natural frequencies into a required harmonic combination by minimizing Λ_h through boundary conditioning, as explained in the next section.

3. RESULTS AND DISCUSSION

In order to illustrate the boundary conditioning technique, the boundary stiffness distribution of a square plate is arranged in such a way as to have harmonic relations among the natural frequencies of the plate. Denoting the harmonic order of the natural frequencies with respect to the fundamental natural frequency by their corresponding numbers, the different sets of harmonic combinations obtained for a square plate by boundary conditioning of the four edges are 1234, 1235, 1236, 1346, 1356, 1357, 1358 and 1368 for $\Lambda_h \leqslant 0{\cdot}02$. A value of $\Lambda_h \leqslant 0{\cdot}02$, which is considered to indicate the presence of a harmonic combination, is possible to obtain over a range of boundary conditions. But the optimum or reference boundary condition for a set of harmonic combinations results in the least value of Λ_h ideally close to zero. The increase of Λ_h from 0·02 as boundary conditions (stiffness) are changed, indicates either the destruction of the present harmonic combination or the presence of other harmonic combinations.

The following results present the effect of boundary conditioning on achieving a particular set of harmonic combination. The results are presented for a few

TABLE 1

Reference stiffness values for optimum conditioning

Harmonics	Reference values $K^*_{T,1,ref}$	$K^*_{T,2,ref}$	$K^*_{T,3,ref}$	$K^*_{T,4,ref}$	$K^*_{R,1,ref}$	$K^*_{R,2,ref}$	$K^*_{R,3,ref}$	$K^*_{R,4,ref}$
1234	10^5	10	10^5	10^3	0	10	10	0
1236	10	10	10	10	10	10	10^2	10^2
1357	10	10	10^3	10^3	10	10	0	0

combinations of harmonics taken four at a time. In order to study the effect of boundary conditioning on the objective function of a particular combination of harmonics, a relative stiffness parameter is defined with respect to the boundary stiffness values corresponding to the optimum conditioning, called reference values, as follows:

$$\sigma_h = \sqrt{\frac{\left[\sum_{i=1}^{4} (K^*_{T,i} - K^*_{T,i,ref})^2 + \sum_{i=1}^{4} (K^*_{R,i} - K^*_{R,i,ref})^2\right]}{\sum_{i=1}^{4} K^{*2}_{T,i,ref} + \sum_{i=1}^{4} K^{*2}_{R,i,ref}}},$$

where $K^*_{,,ref}$ is the reference value corresponding to optimum boundary conditioning.

The reference stiffness values for three sets of combinations of harmonics corresponding to the minimum objective function Λ_{1ijk} is given for a combination

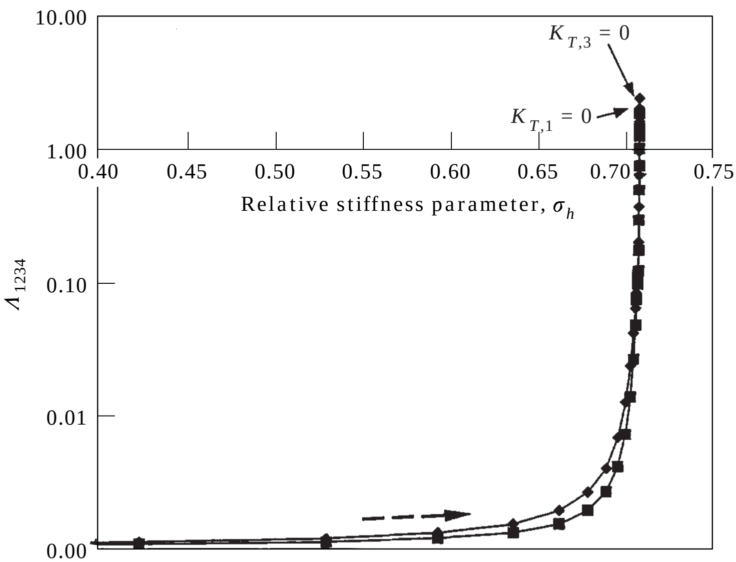

Figure 2. Effect of $K_{T,1}$ and $K_{T,3}$ on Λ_{1234}: –■–, $K_{T,1}$; –◆–, $K_{T,3}$.

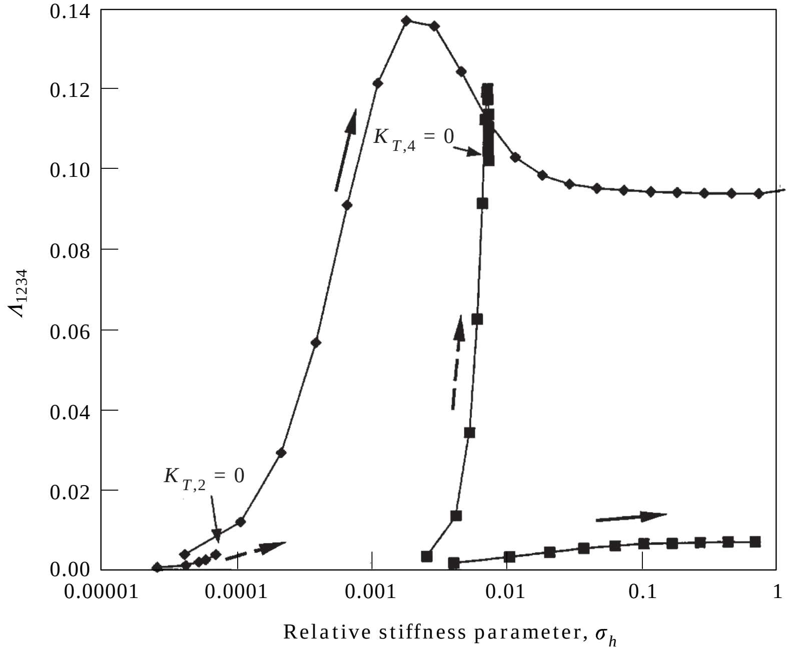

Figure 3. Effect of $K_{T,2}$ and $K_{T,4}$ on Λ_{1234}: –◆–, $K_{T,2}$; –■–, $K_{T,4}$.

of harmonics $h = 1ijk$ in order to demonstrate the effect of boundary conditioning; see Table 1. A value of 10^5 in the table indicates an infinitely stiff condition.

At the optimum boundary stiffness distribution, it can be seen that the relative stiffness parameter σ_h is equal to 0. The effect of changing the individual boundary stiffness values on the natural frequencies are presented in Figures 2–10. In these figures, the parameter Λ_{1ijk} showing the deviation from the desired harmonic combination is plotted against the relative stiffness parameter σ_h, which is affected by changing individual boundary stiffness values. In all these figures, while the boundary stiffness value under study is varied from 0 to the optimum value and beyond, the other stiffnesses are held at their optimum values. In all the figures, the solid arrows indicate the direction of relative stiffening of the boundary with respect to the optimum, while the broken arrows indicate a relative weakening of the boundary.

3.1. HARMONICS 1234

The variation of error function corresponding to 1234 harmonics (Λ_{1234}) in relation to relative boundary stiffness parameter σ_h are given in Figures 2–5.

3.1.1. *Effect of translational stiffness*

The influence of translational stiffnesses on sides 1 and 3 are shown in Figure 2. When $K_{T,1}$ is varied from 0 to 10^5, the deviation parameter Λ_{1234} reduces from 1·86 towards zero. The behavior is identical when $K_{T,3}$ is varied from 0 to 10^5 while keeping the other stiffness values at the optimum. The influence of translational stiffnesses on sides 2 and 4 are shown in Figure 3. The relative weakening of $K_{T,2}$

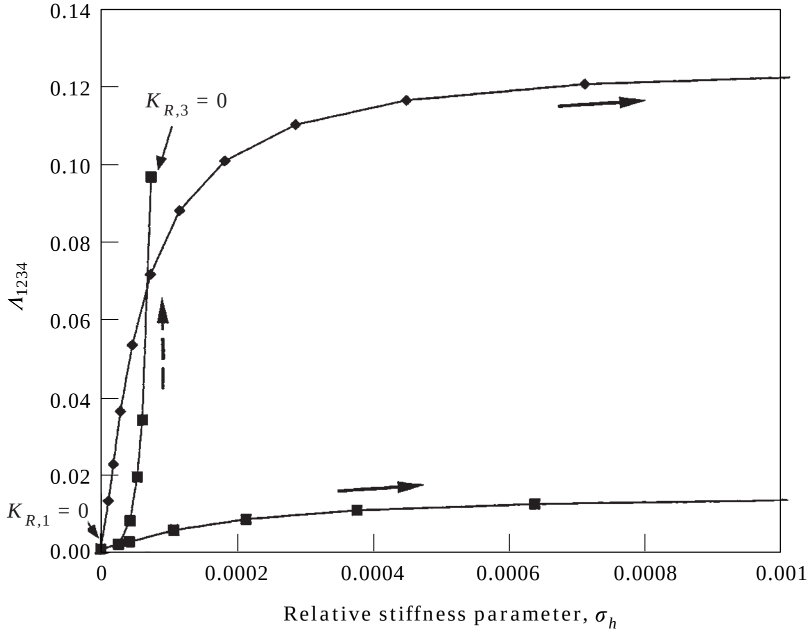

Figure 4. Effect of $K_{R,1}$ and $K_{R,3}$ on Λ_{1234}: –◆–, $K_{R,1}$; –■–, $K_{R,3}$.

from the optimum stiffness value of 10 has insignificant effect on the quality of harmonic combination 1234, while the relative weakening of $K_{T,4}$ from the optimum value of 10^3 destroys the harmonic combination 1234 as shown in Figure 3.

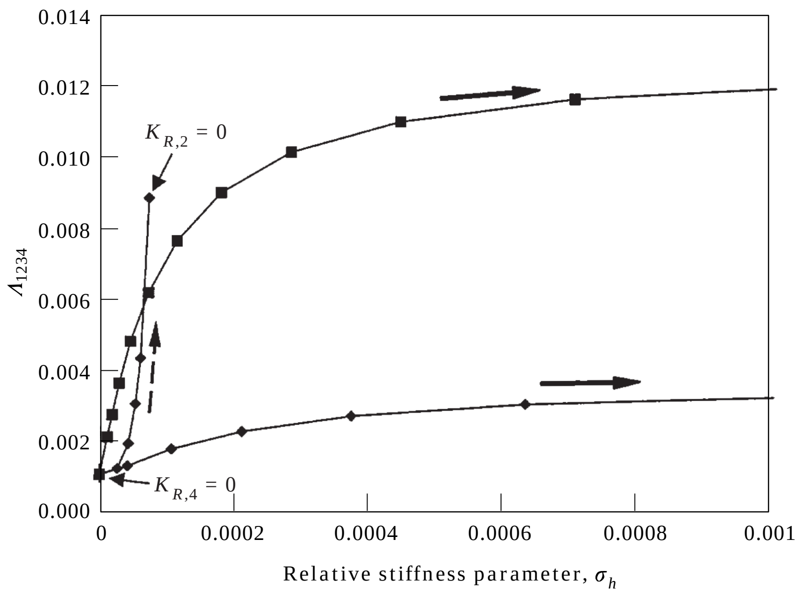

Figure 5. Effect of $K_{R,2}$ and $K_{R,4}$ on Λ_{1234}: –◆–, $K_{R,2}$; –■–, $K_{R,4}$.

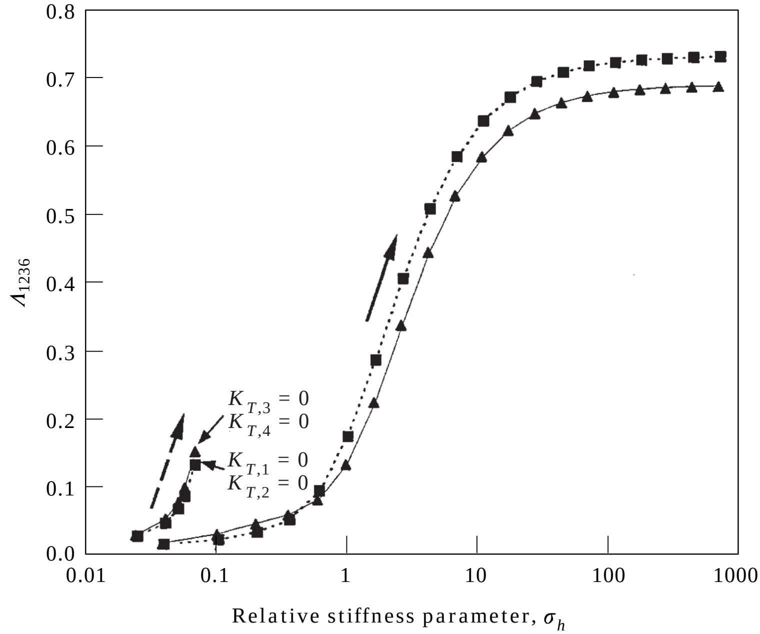

Figure 6. Effect of $K_{T,1}$, $K_{T,2}$, $K_{T,3}$ and $K_{T,4}$, on Λ_{1236}: ■, $K_{T,1}$; - - - - , $K_{T,2}$; ▲, $K_{T,3}$; —, $K_{T,4}$.

3.1.2. *Effect of rotational stiffness*

The effect of varying the rotational stiffnesses on sides 1 and 3 are shown in Figure 4. The relative stiffening of $K_{R,1}$ from the reference value of zero aggravates the harmonics while the relative stiffening of $K_{R,3}$ is ineffective. The relative weakening of $K_{R,3}$ from the reference value affects the harmonics drastically. But Figure 5 indicates that this harmonic combination is insensitive to relative changes in stiffnesses $K_{R,2}$ and $K_{R,4}$.

3.2. HARMONICS 1236

The effect of boundary conditioning by changing the translational stiffnesses at the edges is shown in Figure 6. This figure shows that the harmonic combination 1236 is quite sensitive to translational stiffness on any edge. Translational strengthening and weakening on any side in relation to reference values destroy the harmonics 1236 rapidly as shown in Figure 6. The effect of boundary conditioning by changing the rotational stiffnesses is shown in Figure 7. It is seen that the harmonic combination 1236 is less sensitive to changes in rotational stiffnesses than translational stiffnesses. Further, rotational weakening affects the harmonics more rapidly than the rotational strengthening.

3.3. HARMONICS 1357

The effect of boundary conditioning on harmonic combination 1357 is shown in Figures 8–10. The effect of change in rotational stiffnesses is less significant than that due to change in translational stiffnesses. Rotational stiffnesses on sides 3 and 4 have similar effect on harmonics and rotational strengthening on these sides deteriorates the harmonics 1357, while the variation of rotational stiffnesses on sides 1 and 2 has negligible effect on the desired harmonic combination. The relative change in translational stiffnesses from their reference values has a significant effect on harmonic combination 1357 as seen from Figures 9 and 10.

The above results confirm that the method of obtaining a combination of harmonics for a given structure through boundary conditioning is quite feasible. These results also show the complexity and uniqueness associated with each harmonic combination in terms of distribution of boundary stiffness.

4. ROLE OF BOUNDARY CONDITIONING IN TUNING OF STEEL PANS

The boundary conditioning technique discussed above can be seen as a general technique to manipulate the natural frequencies of a structure by changing the boundary conditions of the existing or newly created boundaries. Since the vibration behavior of the structure is dependent on the distribution of material properties, structural geometry and boundary conditions, the required pattern of natural frequencies can be achieved by conditioning any of the mentioned parameters in isolation or in combination.

Part of the tuning process of a steel pan instrument employs this boundary conditioning technique. A steel pan is essentially a hemispherical shell type structure, made by hammering the flat end of a tar drum. Circular, elliptical or trapezoidal areas called notes are marked on the hemispherical surface by indenting the periphery of these regions using dot punches resulting in the creation

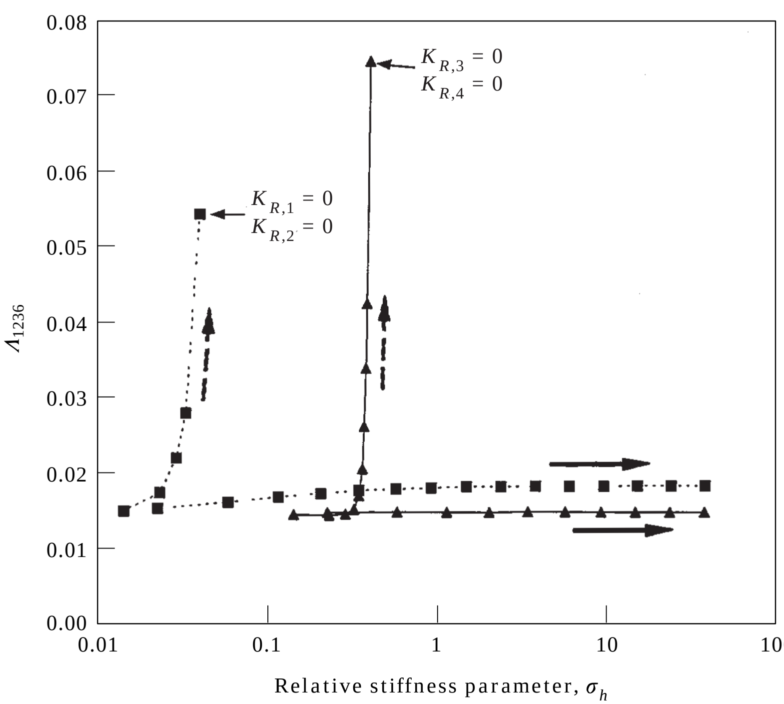

Figure 7. Effect of $K_{R,1}$, $K_{R,2}$, $K_{R,3}$ and $K_{R,4}$, on Λ_{1236}: ■, $K_{R,1}$; - - - -, $K_{R,2}$; ▲, $K_{R,3}$; —, $K_{R,4}$.

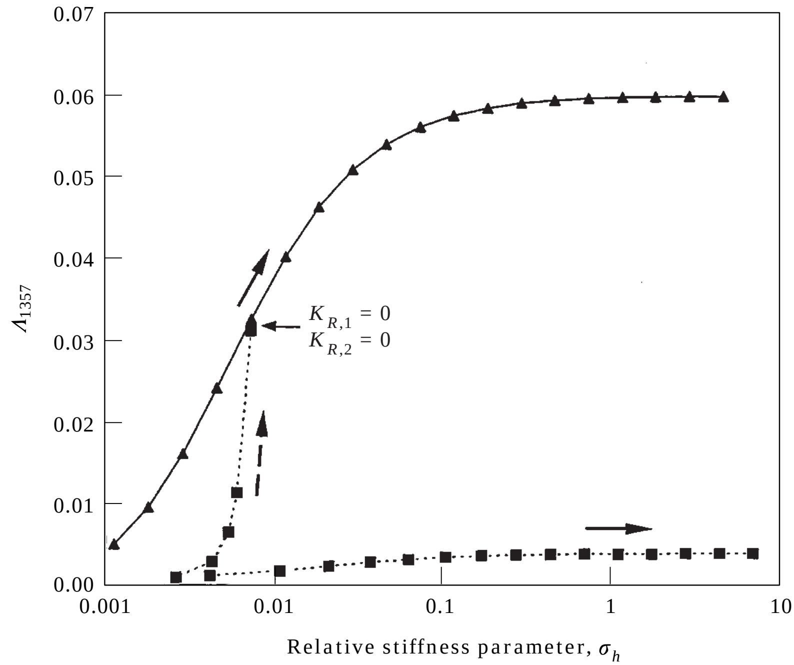

Figure 8. Effect of $K_{R,1}$, $K_{R,2}$, $K_{R,3}$ and $K_{R,4}$, on Λ_{1357}: ■, $K_{R,1}$; - - - -, $K_{R,2}$; ▲, $K_{R,3}$; —, $K_{R,4}$.

of new rotationally weak boundaries of the notes. During the process of steel pan making, the areas in between these notes are hardened by heat treatment representing translationally stiff boundaries. The localization of note vibration and

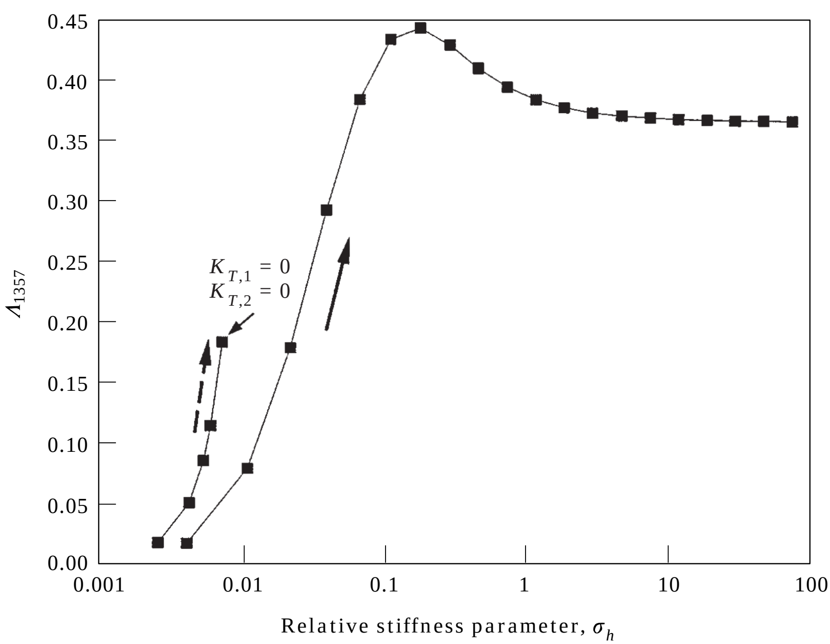

Figure 9. Effect of $K_{T,1}$ and $K_{T,2}$ on Λ_{1357}: ■, $K_{T,1}$; —, $K_{T,2}$.

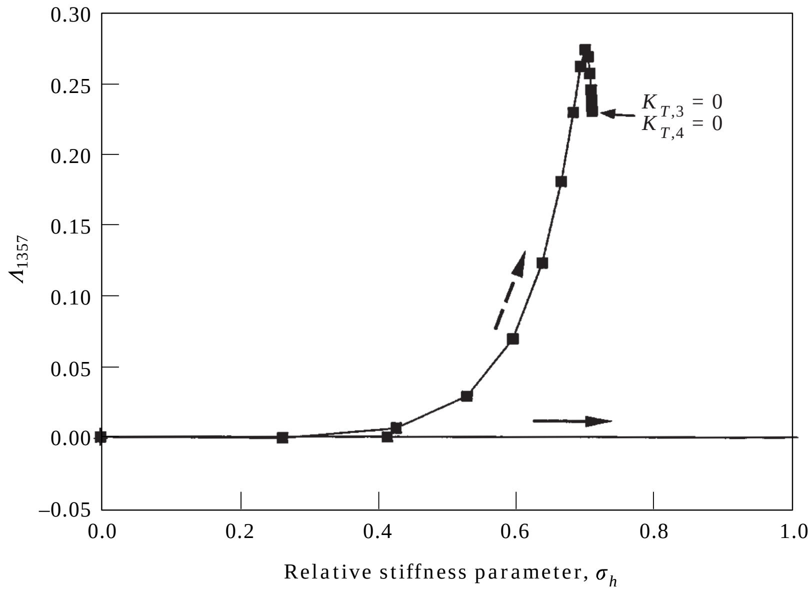

Figure 10. Effect of $K_{T,3}$ and $K_{T,4}$ on Λ_{1357}: ■, $K_{T,3}$; —, $K_{T,4}$.

the coarse tuning are achieved by grooving, punching, hammering, peening, stretching, heat treatment, etc. [9]. Hence, the individual note regions that act as individual vibrator elements are capable of producing distinct tones by tuning each note to generate a distinct fundamental natural frequency and a combination of harmonics in isolation or in pair with adjacent notes. Considering the individual note as a shell-like structure and replacing its surrounding areas with equivalent springs, has been suggested by Achong [10].

The model of a rectangular plate with rotational and translational stiffnesses at the edges may be considered as a very simplistic example of an individual localized vibrator, and varying the stiffness values in order to achieve a certain relation among natural frequencies resembles the boundary conditioning part of the tuning operation in steel pans. The final tuning process of the steel pan includes the manipulation of several other non-linear parameters, particularly in view of the curvature of the shell note area, residual stresses, etc., as suggested by Achong [10, 11]. It can be seen from the above results and discussion that the boundary conditioning becomes one of the techniques, like modifying geometry and material properties, responsible for structural tuning of a particular note in order to obtain a desired harmonic relation among its natural frequencies.

5. CONCLUSIONS

A method has been developed to study the effect of boundary conditioning on vibration behavior and hence the harmonics of a rectangular plate. The boundary conditioning procedure implies the modification of translational and rotational stiffness distribution on the edges in order to achieve the required results. The boundary conditioning technique can be applied to different problems of interest, such as the noise transmission into an aircraft fuselage interior. The results of the present study confirm that the natural frequencies are strongly influenced by boundary conditioning.

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APPENDIX: NOMENCLATURE

A	deflection coefficient
a	dimension of the plate in the x direction
b	dimension of the plate in the y direction
D	flexural rigidity of the plate
h	thickness of the plate
$K^*_{T,i}$	normalized translational stiffness $= K_{T,i}a^3/D$
$K^*_{R,i}$	normalized rotational stiffness $= K_{R,i}a/D$
l_s	length of the sth side
$U_{max,p}$	strain energy of the plate due to bending
$U_{T,s}$	potential energy of the sth side translational spring
$U_{R,s}$	potential energy of the sth side rotational spring
W_s	deflection along the sth side
W'_s	slope along the sth side
x	normalized co-ordinate $= \xi/a$

y	normalized co-ordinate $= \eta/b$
$\alpha = a/b$	
λ	eigenvalue $= \rho h a^4 \omega^2 / D$
η	co-ordinate of the plate
ρ	mass density of the plate
ω	natural frequency
ξ	co-ordinate of the plate
$(\ )'$	differentiation with respect to x
$(\dot{\ })$	differentiation with respect to y