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LETTERS TO THE EDITOR

ANALYTICAL AND NUMERICAL EXPERIMENTS ON VIBRATING CIRCULAR ANNULAR PLATES OF RECTANGULAR ORTHOTROPY

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1. Introduction

Consider an orthotropic plate where the x- and y-axes coincide with the principal directions of elasticity. Such structural elements find common application in present technology in view of the always increasing use of fiber-reinforced materials.

On the other hand, orthotropic characteristics may be induced by certain metallurgical processes [1] and also by certain artificially made differences between flexural rigidities for two orthgonal directions as in the case of corrugated plates or the situation where stiffening ribs are attached to the plate element, etc.

When dealing with such orthotropic plates the static and dynamic treatments are rather straightforward† in the case of rectangular plates whose sides are parallel to the principal directions of elasticity.

Obviously the analysis is considerably more complicated when dealing with other plate geometries and in the case of annular circular plates with an inner free edge one encounters the fact that satisfying the Kirchhoff–Kelvin conditions is exceedingly difficult. Recent results have been obtained on this problem [3, 4]. The present study deals with vibrating, simply supported and clamped annular plates with a free inner edge by means of: (1) analytical approximations using simple polynomial co-ordinate functions to represent the fundamental mode shape; (2) numerical determinations using, two well known and extremely accurate finite element codes [5–6].

† In the sense that one is able to follow and use the isotropic plate solutions [2].

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TABLE 1

Finite element analysis [5] *of annular isotropic and orthotropic plates with free inner edge*

				Simply supported			Clamped		
b/a	n	m	No. of nodes	No. of equations	Ω_1		No. of equations	Ω_1	
					Isotr.	Orthotr.		Isotr.	Orthotr.
0·1	80	80	6561	19440	4·8901	4·4004	19280	10·1353	9·1249
0·2	80	75	6156	18235	4·7325	4·2547	18075	10·3475	9·3146
0·3	90	70	6461	19150	4·6592	4·1871	18970	11·3381	10·2121
0·4	100	60	6161	18260	4·7435	4·2627	18060	13·5001	12·1591
0·5	120	50	6171	18290	5·0427	4·5305	18050	17·5961	15·8113
0·6	140	45	6486	19225	5·6624	5·0822	18945	25·5362	22·7508
0·7	160	40	6601	19560	6·8634	6·1477	19240	42·9705	37·3192
0·8	200	30	6231	18430	9·4524	8·4405	18030	92·7737	76·3391
0·9	300	20	6321	18620	17·4997	15·5653	18020	359·5490	275·7468

2. ANALYTICAL SOLUTION

Using Lekhnitskii's well known and accepted notation [7] one expresses the governing functional in the form

$$J[W] = \frac{1}{2}\iint\left[D_1\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + 2D_1\nu_2\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + D_2\left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 4D_k\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y - \frac{\rho\omega^2}{2}\iint hW^2\,\mathrm{d}x\,\mathrm{d}y, \qquad (1)$$

where, when substituting $W(x, y)$ by an approximation, $W_a(x, y)$, it will suffice if $W_a(x, y)$ satisfies, at least, the essential boundary conditions of the structural system.

In the case of an annular plate with a clamped outer boundary the following approximations are used (isotropic plate)

$$\alpha r^3 + \beta r^2 + 1, \qquad \alpha r^4 + \beta r^2 + 1, \qquad \alpha r^4 + \beta r^3 + 1, \qquad (2a–c)$$

$$\alpha r^\gamma + \beta r^2 + 1, \qquad \alpha r^3 + \beta r^\gamma + 1, \qquad (\alpha r^3 + \beta r^\gamma + 1)(1 + \eta_1 \sin^2\Theta + \eta_2\cos^2\Theta), \qquad (2d–f)$$

where the α's and β's are determined by substituting each co-ordinate function in the prescribed boundary conditions at the outer edge:

$$W_a = \frac{\mathrm{d}W_a}{\mathrm{d}r} = 0 \qquad \text{for } r = a. \qquad (3)$$

The exponential parameter, γ, allows for minimization of the upper bound [8].

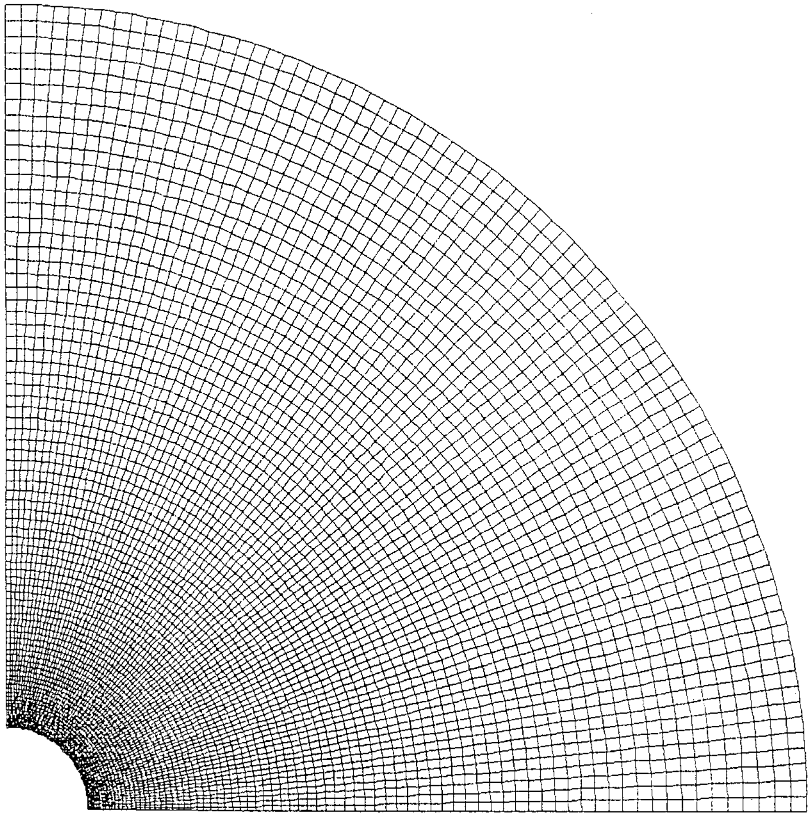

Figure 1. Finite element mesh of one-quarter of the annular plate ($b/a = 0{\cdot}1$).

In view of the trend of the eigenvalues obtained, in the case of the orthotropic plate clamped at $r = a$, the co-ordinate functions (2a) and (2e) were employed.

When dealing with the simply supported isotropic and orthotropic annular plate with a free edge the same approximations were used and where α and β are obtained substituting the co-ordinate functions in

$$W(a) = 0, \qquad \left.\frac{d^2 W}{dr^2} + \frac{\nu_2}{r}\frac{dW}{dr}\right|_{r=a} = 0. \qquad (4a, b)$$

Equation (4b) is an approximate condition for the orthotropic case [4].

Admittedly, when dealing with orthotropic structural elements, the azimuthal variable also comes into play in view of the constitutive properties of the plate material. Hence, the above approximations are first order representations of the fundamental mode shapes when dealing with isotropic systems which possess less degree of accuracy for orthotropic plates.

3. NUMERICAL RESULTS

For the isotropic situations Poisson's ratio was taken equal to 1/3. On the other hand, the calculations corresponding to the orthotropic problems were performed assuming $D_2/D_1 = 1/2$; $D_k/D_1 = 1/3$ and $\nu_2 = 1/3$.

When using the finite element method [5] one-quarter of the plate was subdivided taking "n" divisions in the circumferential direction and "m" divisions, radially. The corresponding values of "n" and "m" are indicated in Table 1. Figures 1 and 2 depict the corresponding finite element meshes for $b/a = 0{\cdot}1$ and $0{\cdot}5$, respectively.

When using the SAMCEF system [6] the modelling was similar. Table 2 depicts eigenvalues determined using the co-ordinate functions (2) and the finite element (FE) results for the isotropic case. In general the values determined using (2e) are lower, and hence more accurate than the eigenvalues determined using (2a), (2b), (2c) and (2d). For the solid plate an excellent agreement with the exact result is achieved. The analytical predictions agree well with the finite element calculations and also for the annular situations, with the exact results recently determined [9].

Table 3 deals with the orthotropic case. The finite element determinations are in excellent agreement between themselves. The eigenvalues obtained using (2f) are in good agreement with the finite element calculations for $0{\cdot}2 \leqslant b/a \leqslant 0{\cdot}9$. It is

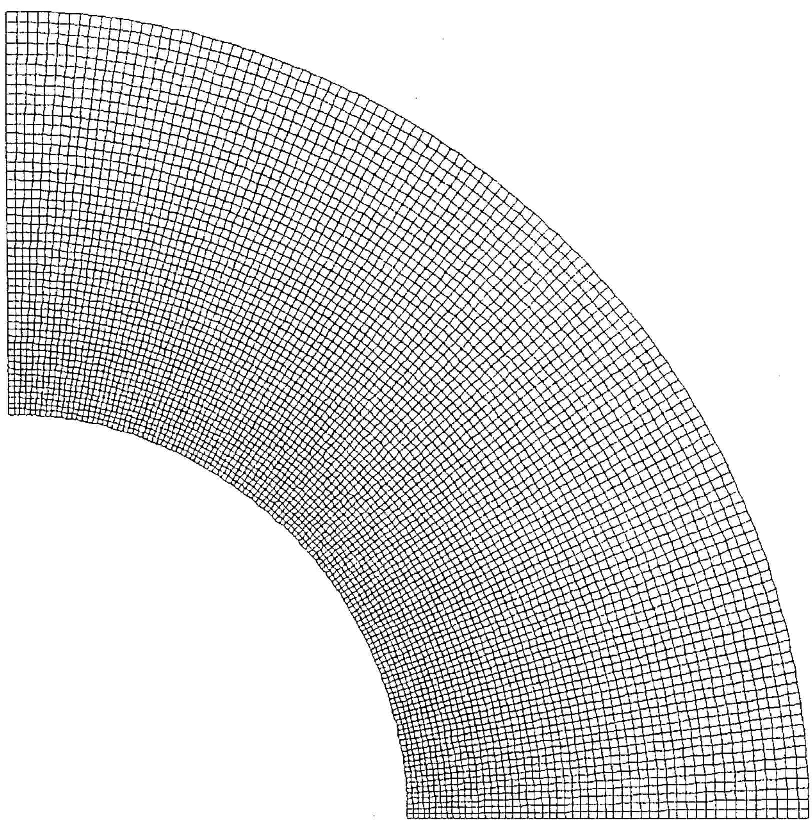

Figure 2. Finite element mesh of one-quarter of the annular plate ($b/a = 0{\cdot}5$).

TABLE 2

Isotropic, annular plate with a free edge, clamped at the outer boundary

b/a	Values of Ω_1 determined using (2a)	(2b)	(2c)	(2d)	(2e)	Ω_1 exact [9]	Ω_1 using FE[a]
0	10·246	10·328	11·224	10·226	10·221	10·2158	–
0·1	10·319	10·486	11·434	10·319	10·310	10·1348	10·135
0·2	10·696	11·007	12·070	10·604	10·523	10·3470	10·347
0·3	11·651	12·055	13·212	11·494	11·442	11·3379	11·338
0·4	13·643	14·015	15·147	13·582	13·564	13·5004	13·500
0·5	17·643	17·777	18·654	17·635	17·638	17·5979	17·596
0·6	25·974	25·575	25·830	25·557	25·567	25·5402	25·536
0·7	45·414	44·043	43·143	43·015	43·001	42·9800	42·970
0·8	103·682	100·448	97·445	93·019	92·917	92·8154	92·773
0·9	428·727	420·303	411·876	361·597	361·137	352·9534	359·549

[a] Using reference [5].

observed that the azimuthal variation contained in equation (2f) improves the results drastically.

The observation of the fundamental mode obtained by means of the FE method reveals that for $b/a = 0{\cdot}7$ the transverse displacements along the x-axis are somewhat smaller than the displacements along the y-axis. The trend increases considerably for $b/a = 0{\cdot}8$, the displacements along the x-axis being practically non-existent. As a consequence of this phenomenon, if one computes the eigenvalue corresponding to a mode antisymmetric with respect to x and symmetric with respect to y, one obtains the frequency coefficient 76·3510 which practically coincides with the value of Ω_1; see Table 3. A similar analysis for

TABLE 3

Orthrotropic, annular plate with a free edge, clamped at the outer boundary

b/a	Values of Ω_1 determined using: (2a)	(2e)	(2f)	Values of Ω_1 calculated by means of FE[a]	FE[b]
0	9·236	9·213	9·213	–	–
0·1	9·301	9·293	9·293	9·125	9·10
0·2	9·642	9·485	9·485	9·314	9·29
0·3	10·502	10·313	10·313	10·212	10·19
0·4	12·297	12·226	12·221	12·159	12·13
0·5	15·904	15·899	15·855	15·811	15·80
0·6	23·412	23·047	22·792	22·750	22·72
0·7	40·936	38·761	37·403	37·319	37·24
0·8	93·458	83·754	77·213	76·339	76·01
0·9	386·450	325·524	289·020	275·746	274·5

[a] Using reference [5].
[b] Using reference [6].

TABLE 4

Isotropic, annular plate with a free edge, simply supported at the outer boundary

	Values of Ω_1 determined using (2a)	(2e)	Ω_1 exact [9]	Ω_1 using FE[a]
0	4·993	4·984	4·9838	–
0·1	4·992	4·985	4·8903	4·890
0·2	5·019	4·855	4·7327	4·732
0·3	5·108	4·731	4·6593	4·659
0·4	5·295	4·780	4·7437	4·743
0·5	5·637	5·061	5·0432	5·042
0·6	6·244	5·672	5·6630	5·662
0·7	7·382	6·868	6·8644	6·863
0·8	9·858	9·456	9·4550	9·452
0·9	17·742	17·511	17·5107	17·499

[a] Using reference [5].

$b/a = 0{\cdot}9$ yields for a mode, also antisymmetric with respect to x and symmetric with respect to y, a frequency coefficient of 275·7468 which now agrees with Ω_1.‡

Tables 4 and 5 depict comparisons of fundamental frequency coefficients for isotropic and orthotropic plates, respectively, when the outer edge is simply supported. For the isotropic situation the agreement between analytical and finite element results is excellent and it is quite good in the case of the hypothetical

‡ Use of references [5, 6] lead to the same conclusions, from a practical viewpoint.

TABLE 5

Orthotropic, annular plate with a free edge, simply supported at the outer boundary

	Values of Ω_1 determined using				Ω_1 determined using:	
b/a	(2a)	(2e)	(2d)	(2f)	FE[a]	FE[b]
0	4·501	4·492	4·493	4·492	–	–
0·1	4·500	4·494	4·500	4·494	4·400	4·40
0·2	4·524	4·376	4·407	4·376	4·254	4·25
0·3	4·605	4·264	4·279	4·264	4·187	4·18
0·4	4·773	4·310	4·315	4·310	4·262	4·26
0·5	5·081	4·571	4·565	4·571	4·530	4·53
0·6	5·628	5·127	5·114	5·127	5·082	5·08
0·7	6·654	6·207	6·192	6·207	6·147	6·15
0·8	8·886	8·524	8·524	8·523	8·440	8·44
0·9	15·993	15·784	15·784	15·777	15·565	15·57

[a] Using reference [5].
[b] Using reference [6].

orthotropic material for $0{\cdot}2 \leqslant b/a \leqslant 0{\cdot}9$ and reasonably acceptable for $b/a = 0{\cdot}1$. It is concluded that the use of equations (2e) or (2f) is practically equivalent.

It is rather remarkable that when dealing with clamped orthotropic annular plates the analytical approximations is not as good as in the case of the simply supported edge, although for the latter, only one satisfies exactly the essential boundary condition at the outer edge.

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