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ON THE EXISTENCE OF A NON-VIBRATORY MODE SHAPE FOR FREE–FREE STRAIGHT BEAMS UNDER COMPRESSIVE LOADS

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1. INTRODUCTION

A great variety of structural problems may be treated with idealized straight beams subjected to appropriately defined boundary conditions. An efficient solution method then consists in separating the time and (longitudinal) space variables, and to expand the space function as a linear combination of mode shapes (e.g., references [1, 2]). To be correct, the technique of modal expansion must involve the complete set of mode shapes; its efficiency is greater when mode shapes are orthogonal.

Free–free boundary conditions, where the ends of the beams are free from shear forces and bending moments, are probably less commonly used than more restrictive boundary conditions, such as pinned or clamped. Recent work on free–free straight beams under axial loads, however, revealed that one mode shape hitherto had been neglected in the technical literature [3]. A subsequent continuity analysis of the corresponding boundary-value problem confirmed the existence and the nature of the "missing" mode shape [4]. In these studies, only vibratory mode shapes were considered, that is when the corresponding time function is trigonometric, since non-vibratory mode shapes usually do not exist or are very sparse (e.g., rigid-body modes, buckling mode). A peculiarity of the "missing" (vibratory) mode shape for free–free straight beams, however, is that it only exists when the axial load is tensile. From a continuity standpoint, when considering the case of vanishing axial loads, this situation was deemed unsatisfactory [4]: it was argued that, perhaps, the "missing" mode shape also exists for compressive loads, but might be non-vibratory (with an exponential time function).

Thus, the present study is a brief investigation of non-vibratory mode shapes for free–free straight beams under axial loads. It aims at verifying the intuition expressed earlier that one non-vibratory mode shape exists when the axial load is compressive. From the point of view of modal expansion techniques, the discovery of such a mode shape would theoretically be useful.

2. MATHEMATICAL ANALYSIS

The boundary-value problem for the transverse motions $y(x, t)$ of a free–free straight beam under axial tension, in the absence of other external loading, is

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recalled below (e.g., reference [5]):

$$m\frac{\partial^2 y}{\partial t^2}+\frac{EI}{L^4}\frac{\partial^4 y}{\partial x^4}-\frac{T}{L^2}\frac{\partial^2 y}{\partial x^2}=0, \qquad \frac{\partial^2 y}{\partial x^2}(0)=\frac{\partial^2 y}{\partial x^2}(1)=0, \tag{1a, b}$$

$$\frac{\partial^3 y}{\partial x^3}(0)-\frac{TL^2}{EI}\frac{\partial y}{\partial x}(0)=\frac{\partial^3 y}{\partial x^3}(1)-\frac{TL^2}{EI}\frac{\partial y}{\partial x}(1)=0. \tag{1c}$$

The beam length L has been used to non-dimensionalize the axial co-ordinate x, and transverse displacement function $y(x, t)$. m represents the mass per unit length, EI the bending stiffness and T a uniform axial load taken as positive when tensile. Evidently, t is the time variable. End conditions (1b) represent vanishing bending moments, whereas equation (1c) express vanishing end shear forces.

The traditional method of solution of the partial-differential equation (1a) is to separate variables x and t by seeking solutions of the form $y(x, t) = y(x)f(t)$, where the notation y has been extended to the spatial function, or mode shape, with no loss of generality. The ordinary-differential modal equation thus obtained is given below:

$$\frac{\partial^4 y}{\partial x^4}-\pi^2\gamma\frac{\partial^2 y}{\partial x^2}-\Omega y=0, \tag{1d}$$

where γ is a non-dimensional axial load parameter defined as $TL^2/(EI\pi^2)$. Ω is a real constant, and y must also satisfy equations (1b) and (1c).

The temporal function $f(t)$ is non-vibratory if $\Omega \leqslant 0$. When $\Omega = 0$, $f(t)$ is a linear function of time, and the mode-shape boundary-value problem admits the normalized solutions $y_c = 1$ for all values of γ, $y_0 = 1 - 2x$ for $\gamma = 0$, and the buckling function $y(x) = \sin(\pi\ x)$ for $\gamma = -1$. Leaving aside that well-known case, we restrict ourselves to strictly negative values of Ω, when $f(t)$ is an exponential function of time. The notation $\beta = (-\Omega)^{1/4}$ is then introduced.

It is a simple matter to show that the general solution of equation (1d) is a linear combination of four fundamental functions:

$$y(x) = A\exp(\alpha_1 x)\cos(\alpha_2 x) + B\exp(\alpha_1 x)\sin(\alpha_2 x) + C\exp(-\alpha_1 x)\cos(\alpha_2 x) + D\exp(-\alpha_1 x)\sin(\alpha_2 x), \tag{2}$$

where $\alpha_1 = \beta\{1/2 + \pi^2\gamma/(4\beta^2)\}^{1/2}$ and $\alpha_2 = \beta\{1/2 - \pi^2\gamma/(4\beta^2)\}^{1/2}$. For α_1 and α_2 to be real, the condition $\beta^2 \geqslant |\gamma|\pi^2/2$ must be satisfied, or equivalently, $-\Omega \geqslant \gamma^2\pi^4/4$. It can be shown that the boundary-value problem defined by equations (1b)–(1d), has no solution when $-\gamma^2\pi^4/4 \leqslant \Omega < 0$; therefore, it will implicitly be assumed that $\beta^2 \geqslant |\gamma|\pi^2/2$ holds in what follows.

Three of the unknown coefficients A, B, C and D in the general solution (2) can be determined as a function of the fourth one (for example, B, C, and D as functions of A) by enforcing three of the four boundary conditions (1b) and (1c). The remaining boundary condition must be redundant to avoid the trivial solution $A = B = C = D = 0$, with a condition of zero determinant called the characteristic equation. Solving the characteristic equation allows β to be known, which completely specifies the solution. Algebraically, the above steps, are straightforward but rather cumbersome, and details will be omitted.

The two boundary conditions at $x = 0$ lead to relatively simple relationships between the coefficients A, B, C, and D, that is,

$$A\gamma\pi^2 + 4B\alpha_1\alpha_2 + C\gamma\pi^2 - 4D\alpha_1\alpha_2 = 0, \qquad -A\alpha_1 + B\alpha_2 + C\alpha_1 + D\alpha_2 = 0;$$

in turn, these equalities can be written as

$$C = (A\beta^2 - 4B\alpha_1\alpha_2)/(\beta^2 + \gamma\pi^2), \qquad D = (A\alpha_1\gamma\pi^2 + B\alpha_2\beta^2)/\{\alpha_2(\beta^2 + \gamma\pi^2)\}. \tag{3, 4}$$

Expressing the zero bending moment condition at $x = 1$, and substituting C and D by means of equations (3) and (4), yields

$$\begin{aligned} B = {} & -A[\alpha_2(\beta^2 + \gamma\pi^2)\exp(\alpha_1)\{\gamma\pi^2\cos(\alpha_2) - 4\alpha_1\alpha_2\sin(\alpha_2)\} + \alpha_2\beta^2\exp(-\alpha_1) \\ & \times\{\gamma\pi^2\cos(\alpha_2) + 4\alpha_1\alpha_2\sin(\alpha_2)\} + \alpha_1\gamma\pi^2\exp(-\alpha_1) \\ & \times\{\gamma\pi^2\sin(\alpha_2) - 4\alpha_1\alpha_2\cos(\alpha_2)\}]/[\alpha_2(\beta^2 + \gamma\pi^2)\exp(\alpha_1) \\ & \times\{\gamma\pi^2\sin(\alpha_2) + 4\alpha_1\alpha_2\cos(\alpha_2)\} + \alpha_2\beta^2\exp(-\alpha_1) \\ & \times\{\gamma\pi^2\sin(\alpha_2) - 4\alpha_1\alpha_2\cos(\alpha_2)\} - 4\alpha_1\alpha_2^2\exp(-\alpha_1) \\ & \times\{\gamma\pi^2\cos(\alpha_2) + 4\alpha_1\alpha_2\sin(\alpha_2)\}]. \end{aligned} \tag{5}$$

The zero shear condition at $x = 1$ would result in another expression relating B to A at least as complicated as equation (5), and combining the two relationships theoretically would result in the characteristic equation. To circumvent such tedious algebraic manipulations, we will exploit and transform the well-known characteristic equation for vibratory mode shapes (e.g., reference [3]), written below with unambiguous notation:

$$2\delta^6\{1 - \cosh(\lambda_1)\cos(\lambda_2)\} - \gamma\pi^2(\gamma^2\pi^4 + 3\delta^4)\sinh(\lambda_1)\sin(\lambda_2) = 0, \tag{6}$$

where $\lambda_1 = \{(\gamma\pi^2/2) + \sqrt{\gamma^2\pi^4/4 + \delta^4}\}^{1/2}$ and $\lambda_2 = \{-(\gamma\pi^2/2) + \sqrt{\gamma^2\pi^4/4 + \delta^4}\}^{1/2}$; in this case, the constant in equation (1d), written Ω', is strictly positive and by definition, $\delta = (\Omega')^{1/4}$. Changing the sign of Ω' (i.e., considering Ω instead) results in an equation valid for non-vibratory modes, but involving complex numbers as δ^2 becomes $i\beta^2$:

$$-2i\beta^6\{1 - \cosh(\lambda_1)\cos(\lambda_2)\} - \gamma\pi^2(\gamma^2\pi^4 - 3\beta^4)\sinh(\lambda_1)\sin(\lambda_2) = 0, \tag{7}$$

where $\lambda_1 = \{(\gamma\pi^2/2) + i\sqrt{\beta^4 - \gamma^2\pi^4/4}\}^{1/2}$ and $\lambda_2 = \{-(\gamma\pi^2/2) + i\sqrt{\beta^4 - \gamma^2\pi^4/4}\}^{1/2}$; note that the condition $\beta^2 \geqslant |\gamma|\pi^2/2$ has been accounted for. Moreover, it can be seen by inspection that $\lambda_1 = \alpha_1 + i\alpha_2$ and $\lambda_2 = \alpha_2 + i\alpha_1$. This allows an expression of the products of trigonometric and hyperbolic functions in equation (7) as follows:

$$\cosh(\lambda_1)\cos(\lambda_2) = \{\cosh(2\alpha_1) + \cos(2\alpha_2)\}/2,$$

$$\sinh(\lambda_1)\sin(\lambda_2) = i\{\cosh(2\alpha_1) - \cos(2\alpha_2)\}/2.$$

When substituting the above results into equation (7), a real equation is obtained which is the characteristic equation for non-vibratory free–free mode shapes:

$$\cosh\{\sqrt{2\beta^2+\gamma\pi^2}\}(2\beta^6+3\beta^4\gamma\pi^2-\gamma^3\pi^6)+\cos\{\sqrt{2\beta^2-\gamma\pi^2}\}$$
$$\times(2\beta^6-3\beta^4\gamma\pi^2+\gamma^3\pi^6)-4\beta^6=0. \quad (8)$$

3. NUMERICAL RESULTS AND CONCLUSIONS

A formal study of the existence of solutions of equation (8) has not been attempted. Instead, when standard numerical equation solvers (e.g., reference [6]) failed to yield a solution, it was assumed that no solution exists. Recall that in general, non-vibratory ($\Omega < 0$) mode shapes do not exist, as opposed to the many solutions of the vibratory ($\Omega > 0$) problem. In the above case, it was shown, albeit numerically, that equation (8) has only one solution when the axial load is compressive ($\gamma < 0$).

Before presenting results, a study of equation (8) is performed in the limit when $|\gamma|$ tends to zero; the same *a priori* assumption as was used in a continuity analysis for vibratory modes [4] is made, i.e., that β^4 should be of order $|\gamma|\pi^2$. The cosine and hyperbolic cosine functions are expanded to order β^4, and the leading terms of the resulting equation yield

$$\beta^4=-12\gamma\pi^2. \quad (9)$$

TABLE 1

β and approximating expression

γ	β	γ	$\{\beta-\beta(9)\}/\beta$
0	0		
−0·01	1·0433	−0·01	0·00012
−0·02	1·2409	−0·02	0·00024
−0·03	1·3734	−0·03	0·00036
−0·04	1·4760	−0·04	0·00047
−0·05	1·5609	−0·05	0·00059
−0·06	1·6339	−0·06	0·00072
−0·07	1·6983	−0·07	0·00084
−0·08	1·7561	−0·08	0·00096
−0·09	1·8088	−0·09	0·00108
−0·1	1·8573	−0·1	0·0012
−0·2	2·2116	−0·2	0·0025
−0·3	2·4508	−0·3	0·0038
−0·4	2·6372	−0·4	0·0052
−0·5	2·7926	−0·5	0·0067
−0·6	2·9274	−0·6	0·0082
−0·7	3·0475	−0·7	0·0099
−0·8	3·1565	−0·8	0·0116
−0·9	3·2568	−0·9	0·0134
−1	3·3504	−1	0·0154

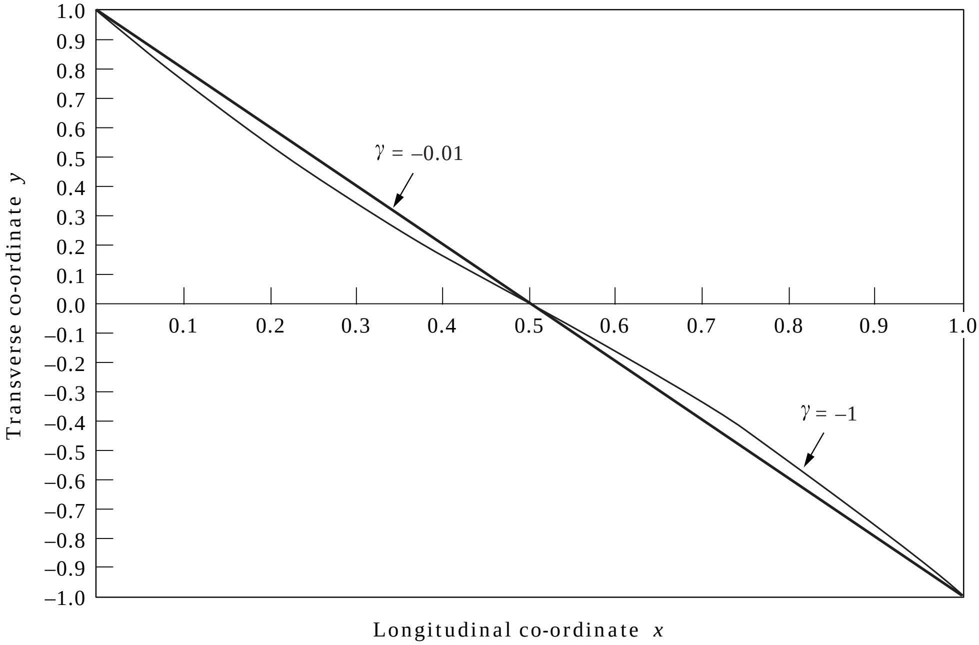

Figure 1. Non-vibratory mode shape for a free–free straight beam: $\gamma = -0{\cdot}01$ and $\gamma = -1$.

Equation (9) confirms the assumed order of magnitude for β^4; it also shows that β only exists for compressive loads ($\gamma < 0$) in the small axial load limit; finally, a symmetric result was obtained in the case of vibratory modes, but this small-β solution only existed for tensile loads (equation (7) in reference [4]). Table 1 shows that the solution of equation (8) is very well approximated by equation (9), for γ between 0 and the buckling limit -1: even when $\gamma = -1$, the relative error is only about 1·5%.

Figure 1 shows the mode shape corresponding to the solution of equation (8), and calculated by means of equations (2)–(5); since A is an arbitrary constant, the normalization condition $y(0) = 1$ was adopted by dividing equation (2) by $(A + C)$, for any value of A. When $\gamma = -0{\cdot}01$, the mode shape is undistinguishable from the (second) rigid-body mode shape obtained when $\Omega = 0$, $y_0 = 1 - 2x$. This is exactly the anticipated continuity result which motivated the present study: in the case of compressive loads ($\gamma < 0$), a non-vibratory mode shape exists that tends to $1 - 2x$ as γ vanishes; previously, it had been established that a vibratory mode shape degenerated to $1 - 2x$, but only for tensile loads [3, 4].

The above result completes the continuity study of Nihous [4]. In practical terms, the occurrence of free–free beams under compressive axial loads may not be very common. If, however, a modelling situation arose where free–free beams are subjected to a compressive axial load as well as to other forcing terms which would formally appear in the right-hand side of equation (1a), the solution technique of modal expansion would have to include the newly discovered non-vibratory mode shape in the set of orthogonal mode shapes.

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