Journal of Sound and Vibration (1999) **221**(1), 117–132
Article No. jsvi.1998.2009, available online at http://www.idealibrary.com.on

VIBRATION ANALYSIS OF ARBITRARILY SHAPED MEMBRANES USING NON-DIMENSIONAL DYNAMIC INFLUENCE FUNCTION

S. W. KANG, J. M. LEE AND Y. J. KANG

School of Mechanical and Aerospace Engineering, Seoul National University, San 56-1 Shinlim-Dong, Kwanak-Ku, Seoul 151-742, Korea

(*Received* 16 *May* 1998, *and in final form* 28 *September* 1998)

In the present study, a theoretical formulation based on the collocation method is presented for the vibration analysis of arbitrarily shaped membranes. The mathematical relation between the two points of selected collocation points is given by a special function, the so-called non-dimensional dynamic influence function. Unlike the collocation methods in the literature, approximate functions used in this paper are simple, one-dimensional functions of which the only independent variable is the distance between the two points. The function is also a wave-type solution that satisfies exactly the given governing differential equation and physically describes the displacement response of a point in an infinite membrane due to a unit displacement excited at another point. The approximate solution is obtained by linear superposition of non-dimensional dynamic influence functions, and then boundary conditions are applied at the discrete points. The system matrix is always symmetric regardless of the boundary shape of the membrane, and the calculated eigenvalues rapidly converge to the exact values thanks to the special function employed in this study. Moreover, the method gives the associated mode shapes successfully without using interpolation functions between the boundary nodes. The validity and efficiency of the method proposed in this paper are illustrated through several numerical examples.

1. INTRODUCTION

Although many engineering applications deal with plates, vibration problems of membranes have long been of great interest because it is known that a simply supported polygonal plate has the same natural frequencies and associated mode shapes as those of a membrane with identical geometry and fixed rectilinear edges [1, 2]. Exact solutions for the transverse vibration of a membrane of simple geometry such as rectangle, circle, ellipse are by now well established [3–5]. Mazumdar [6] used a concept of constant-deflection contours to find only the first natural frequency of membranes of arbitrary shape. Nagaya [7, 8] proposed an analytical method to obtain higher order modes and dynamic responses of arbitrarily shaped membranes by using the Fourier expansion method for

0022–460X/99/110117 + 16 $30.00/0

boundary conditions. Later, Ding [9] extended Nagaya's method to the study of a membrane carrying elastically supported masses. More recently, Kim and Kang [10] have made free vibration analysis of convex polygonal membranes by using plane wave-type solutions that satisfy exactly the governing differential equation, propagate in one direction and vary sinusoidally in the other direction. In their work, it has been illustrated that care must be taken in choosing functionally independent base functions of a simple form. However, there are still limitations in applying these analytical methods directly to the free vibration problem of membranes of any arbitrary shape.

Because there exists an analogy between the membrane vibration problems and the acoustical waveguide problems in the theoretical treatments, analysis methods for waveguides of general cross-sectional shapes are briefly surveyed. A variety of approximate methods, for example, finite difference method [11, 12], finite element method [13], collocation [14, 15], point matching [16], Rayleigh–Ritz [17], and Galerkin methods [18] have been applied to find cutoff frequencies of a waveguide problem. Most of these methods are based on the superposition procedure of independent, approximate solutions that are two-dimensional functions of two independent variables and satisfy either the governing differential equation or boundary condition. A large amount of numerical calculation is required and its accuracy is limited as the number of independent solutions increases for higher order modes.

In this paper, an alternative superposition method based on the collocation method is proposed for the free and forced vibration analyses of arbitrarily shaped membranes. In order to simplify a large amount of numerical calculation that may be caused by the two-dimensional independent solutions, a special function called "non-dimensional" dynamic influence (Green's) function is employed, which exactly satisfies the governing differential equation of a membrane. The non-dimensional dynamic influence function is a wave-type function that propagates omni-directionally from a point when an infinite domain is considered. Physically, it represents the displacement response at a point due to a unit displacement excitation at another point in the infinite region. Thus, it is a one-dimensional function, as the only independent variable, with the distance between the two collocation points in the boundary of the membrane. The solution of the given problem is found by linearly superposing the non-dimensional dynamic influence functions, and then by applying boundary conditions at discrete points.

The method presented in this paper may appear to be similar to the boundary element method [19, 20] in a sense that boundaries of a problem domain of interest are discretized, but the two methods use different discretization schemes. While the boundary element method divides the whole boundary into boundary elements in which insides are represented by appropriate interpolation functions and often has a difficulty in evaluating singular value integration, the proposed method does not require any interpolation function since it divides the whole boundary only into nodes. As a result, the proposed method has no numerical integration because it uses the non-dimensional dynamic influence functions that satisfy a governing differential equation only at boundary nodes.

It is well known that convergence of the collocation technique has never been proved and increasing the number of collocation points does not necessarily improve the value of the calculated eigenvalues [21]. Nevertheless, the method in this study has much less difficulty than conventional collocation methods in solving the eigenfrequency equation thanks to the one-dimensional, omni-directional wave-type function and the symmetry of the system matrix regardless of boundary shapes (note that the system matrix in either the collocation method or the boundary element method is generally asymmetric). Example studies presented here show that eigenvalues and their associated eigenmodes obtained from the method are found to be very accurate. Besides, the eigenvalues converge to the exact values even when a small number of the boundary nodes are used.

Although only the free vibration analysis of membranes of arbitrary shape are presented in this study, the method may be extended to study acoustic cavity problems as well as free vibration problems of arbitrarily shaped plates.

2. THEORETICAL FORMULATION

2.1. DISCRETIZATION OF BOUNDARY CONDITION

Consider a uniform membrane of arbitrary shape, having mass per unit area ρ, as shown in Figure 1. It is assumed that in the equilibrium position the membrane lies entirely in one plane under a uniform tension T per unit length, and also that its boundary Γ is harmonically excited by a small displacement of $u(\mathbf{r}_\Gamma, t)$ where $\mathbf{r}_\Gamma$ is the position vector of a point on the boundary. If the transverse displacement of the membrane is denoted by $w(\mathbf{r}, t)$ where $\mathbf{r}$ is the position vector of a point in the membrane, the governing differential equation and boundary condition may be written as, respectively,

$$\nabla^2 w - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = 0, \qquad w(\mathbf{r}_\Gamma, t) = u(\mathbf{r}_\Gamma, t), \qquad (1, 2)$$

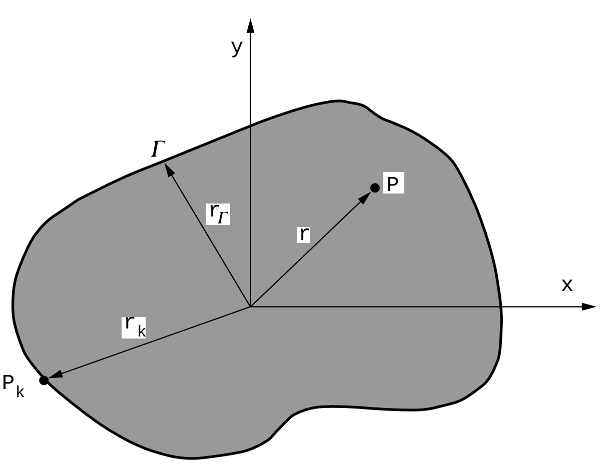

Figure 1. Arbitrarily shaped membrane with continuous boundary Γ.

where $c = \sqrt{T/\rho}$ is the speed of sound. If a harmonic time dependence of the form $e^{j\omega t}$ is assumed: i.e., $w(\mathbf{r}, t) = W(\mathbf{r})\,e^{j\omega t}$ and $u(\mathbf{r}_\Gamma, t) = U(\mathbf{r}_\Gamma)\,e^{j\omega t}$, equations (1) and (2) become:

$$\nabla^2 W + \Lambda^2 W = 0, \qquad W(\mathbf{r}_\Gamma) = U(\mathbf{r}_\Gamma), \tag{3, 4}$$

where $\Lambda = \omega/c$.

When the boundary is represented by discrete points P_i, $(i = 1, 2, \ldots, n_b)$, the discretized boundary conditions corresponding to equation (4) take the form

$$W(\mathbf{r}_i) = U(\mathbf{r}_i), \qquad i = 1, 2, \ldots, n_b. \tag{5}$$

Note that the discrete boundary conditions (5) converge to the continuous boundary condition (4) as the number of the points, n_b, increases to infinity.

2.2. DYNAMIC RESPONSE OF INFINITE MEMBRANE WITH POINT SOURCES

To evaluate dynamic response and eigenmode of the membrane of finite size shown in Figure 1, the non-dimensional dynamic influence function is introduced to a membrane of infinite lateral extent shown in Figure 2. In the present case, a unit harmonic displacement is assumed to be applied at a point P_k along the fictitious boundary which is actually the same as that of the finite-sized membrane as illustrated by a dotted line in Figure 2. For simple harmonic vibration of frequency ω, one form of the non-dimensional dynamic influence function for displacement at a point P in the infinite membrane produced by the displacement at the point P_k may be represented by a Bessel function of the first kind of order zero $J_0(\Lambda|\mathbf{r} - \mathbf{r}_k|)$, in which $\mathbf{r}$ and $\mathbf{r}_k$ are, respectively, the position vectors of the points P and P_k. Since there is no reflection returned away from the infinite boundary of the membrane, only the zero order of Bessel functions of the first kind is involved. Note also that the argument of the Bessel function used in this study is dimensionless.

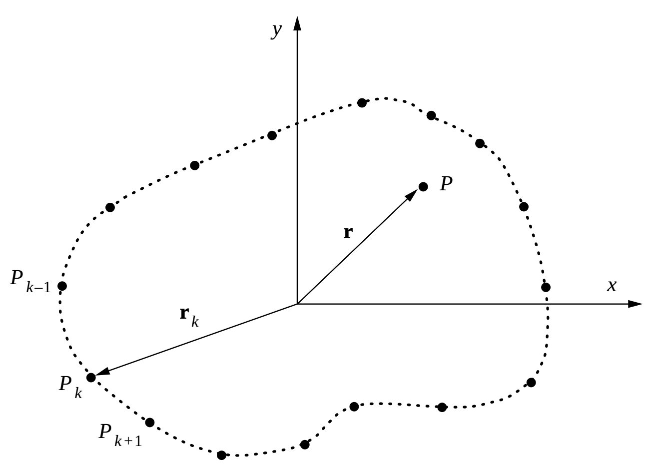

Figure 2. Infinite membrane with harmonic excitation points that are located on the identical position of the boundary Γ in Figure 1.

Assuming that harmonic displacements of amplitudes $A_1, A_2, \ldots, A_{n_b}$ are, respectively, generated at points $P_1, P_2, \ldots, P_{n_b}$ along the fictitious boundary, the response at the point P of the membrane may be obtained by superposing displacements that have resulted from the harmonic displacements at each

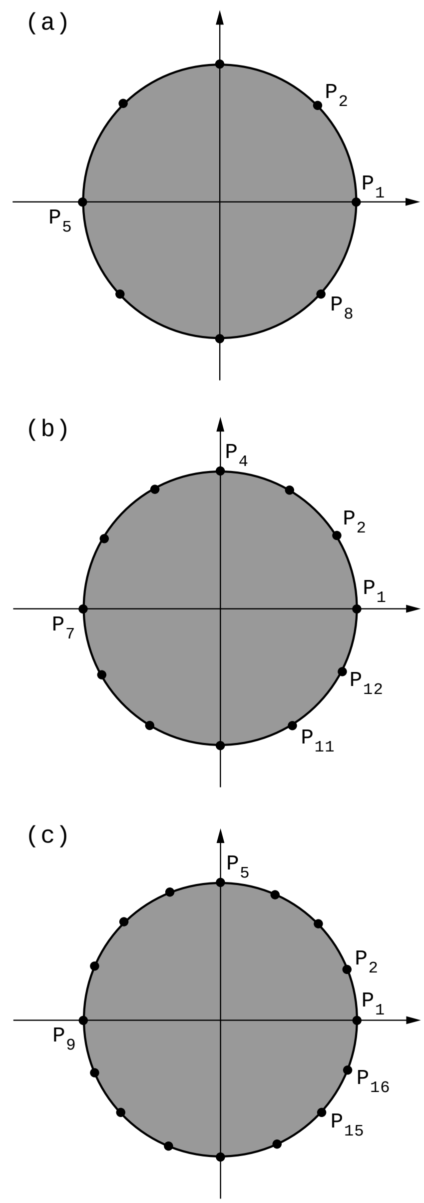

Figure 3. Discrete boundary nodes of the circular membrane when (a) $n_b = 8$, (b) $n_b = 12$, (c) $n_b = 16$.

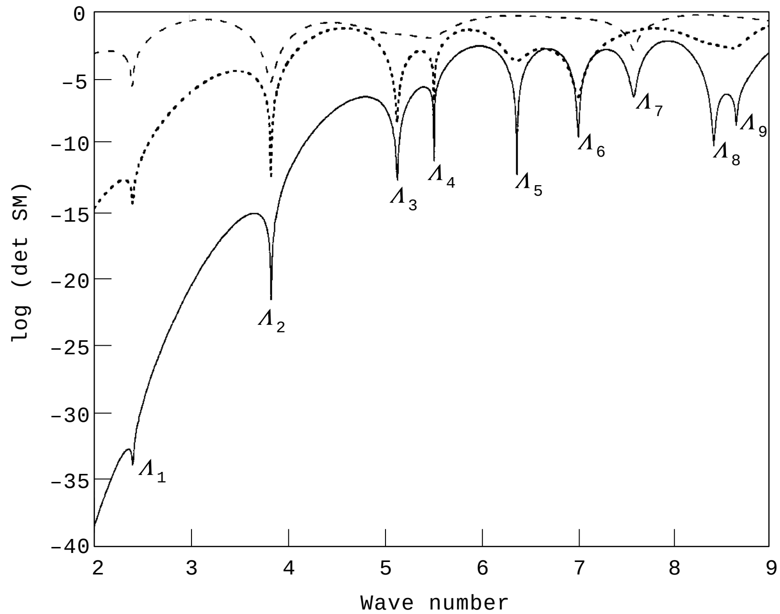

Figure 4. Determinant of the system matrix versus wave number for the circular membrane. – – –, $n_b = 8$; - - - -, $n_b = 12$; ——, $n_b = 16$.

boundary point. Thus, the displacement response at the position **r** in the infinite membrane is

$$\mathbf{W}(\mathbf{r}) = \sum_{k=1}^{n_b} A_k J_0(\Lambda|\mathbf{r} - \mathbf{r}_k|). \tag{6}$$

In equation (6), A_k's indicate the strength of participation of each displacement at the point P_k in the total response at the point P, and therefore may be defined as the participation factor. When the boundary nodes of the membrane shown in Figure 1 are assumed to be excited by the harmonic displacements having a proper set of participation factors to satisfy the boundary conditions, equation (6) can also be considered as the dynamic response of the finite membrane in the approximate sense.

2.3. DYNAMIC RESPONSE AND EIGENMODE ANALYSIS OF THE FINITE MEMBRANE

The participation factors A_k's have to be determined by applying boundary conditions (5). Substituting equation (6) into equation (5) gives

$$W(\mathbf{r}_i) = \sum_{k=1}^{n_b} A_k J_0(\Lambda|\mathbf{r}_i - \mathbf{r}_k|) = U_i, \qquad i = 1, 2, \ldots, n_b. \tag{7}$$

Equation (7) may be written in a simple form:

$$\mathbf{SM}(\Lambda)\mathbf{A} = \mathbf{U}, \tag{8}$$

where the $n_b \times n_b$ matrix $SM(\Lambda)$ is given by $SM_{ik} = J_0(\Lambda|\mathbf{r}_i - \mathbf{r}_k|)$ and the $n_b \times 1$ vectors $\mathbf{A}$ and $\mathbf{U}$ represent the participation factors and harmonic displacements at boundary points respectively. Note that $\mathbf{SM}(\Lambda)$ is a function of wave number $\Lambda = \omega/c$. From equation (8), the participation factors A_k can, therefore, be determined as follows

$$\mathbf{A} = \mathbf{SM}(\Lambda)^{-1}\mathbf{U}. \tag{9}$$

By substituting equation (9) into equation (6), the dynamic response of the membrane can then be achieved: i.e.,

$$W(\mathbf{r}) = \mathbf{J}(\Lambda)\mathbf{SM}(\Lambda)^{-1}\mathbf{U}, \tag{10}$$

where the kth element of the $1 \times n_b$ row vector $\mathbf{J}(\Lambda)$ is given by $J_0(\Lambda|\mathbf{r} - \mathbf{r}_k|)$.

For free vibration analysis, eigenvalues, and thus natural frequencies of the arbitrarily shaped membrane whose edges are simply supported can be determined by letting $\mathbf{U} = \mathbf{0}$ in equation (8): i.e.,

$$\mathbf{SM}(\Lambda)\mathbf{A} = \mathbf{0}. \tag{11}$$

For equation (11) to have a non-trivial solution,

$$\det(\mathbf{SM}(\Lambda)) = 0. \tag{12}$$

The eigenvalues can be calculated from equation (12), and the participation factors are obtained as the eigenvector of equation (11). The mode shapes associated with the eigenvalues can be determined from equation (10).

3. CASE STUDIES

To verify the method presented in this paper, free vibration analyses of circular, rectangular, and arbitrarily shaped membranes were performed. For each case, the eigenvalues obtained by the present method are compared with those obtained by

TABLE 1

Comparison of eigenvalues of the circular membrane obtained by the present method, the exact method, and FEM

Eigenvalues	Present $n_b = 8$	Present $n_b = 12$	Present $n_b = 16$	Exact solution	FEM $n_{nd} = 1024$	FEM $n_{nd} = 256$	FEM $n_{nd} = 144$
Λ_1	2·4048	2·4048	2·4048	2·4048	2·4166	2·4524	2·4905
Λ_2	3·8306	3·8317	3·8317	3·8317	3·8513	3·9109	3·9743
Λ_3	None	5·1356	5·1356	5·1356	5·1744	5·2929	5·4191
Λ_4	5·4969	5·5201	5·5201	5·5201	5·5515	5·6472	5·7489
Λ_5	None	6·3790	6·3802	6·3802	6·4610	6·7077	6·9655
Λ_6	None	7·0143	7·0156	7·0156	7·0592	7·1920	7·3335
Λ_7	7·5876	None	7·5876	7·5883	7·7445	8·2150	8·6513
Λ_8	None	None	8·4172	8·4172	8·4841	8·6887	8·9073

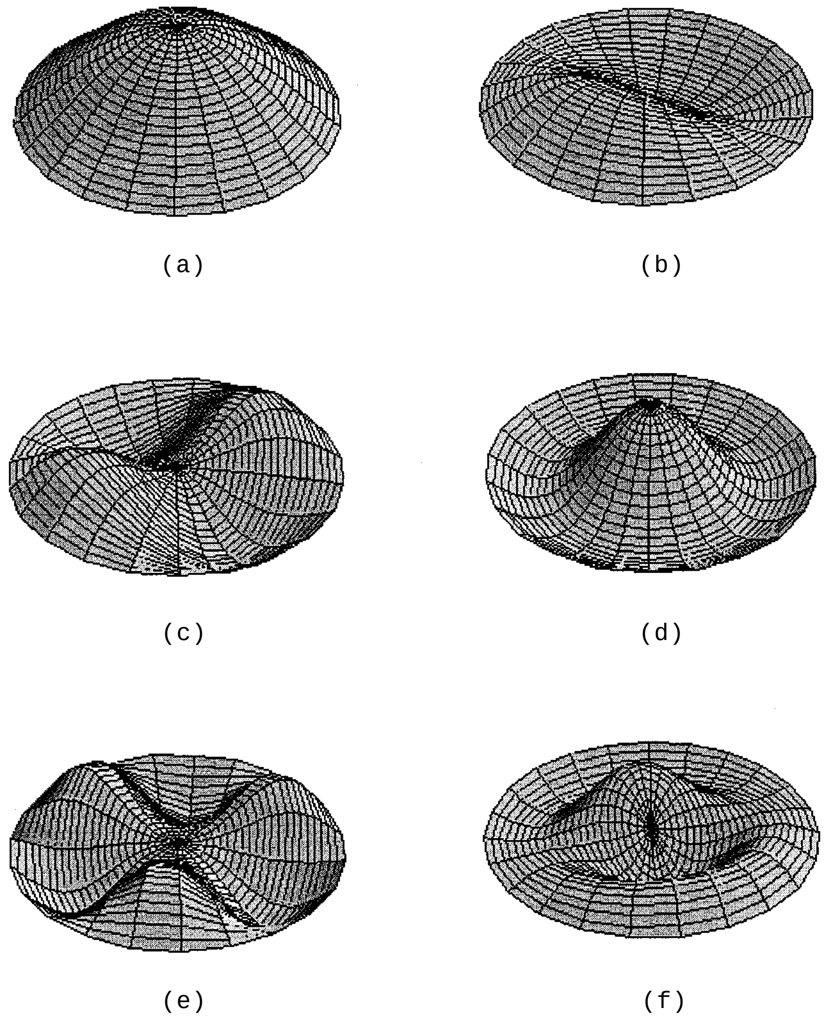

Figure 5. Mode shapes of the circular membrane obtained by the present method when $n_b = 16$: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, (f) 6th mode.

exact and numerical analyses. The mode shapes are also compared with those obtained by exact or numerical methods to ensure the validity of the method.

3.1. CIRCULAR MEMBRANE

To discretize the boundary condition as shown in equation (5), various numbers of nodes were chosen on the boundary of a uniform, circular membrane of unit radius. Equally spaced 8, 12, and 16 nodes were used respectively, as depicted in Figures 3(a)–(c). For $n_b = 8$, $n_b = 12$, and $n_b = 16$, logarithmic values of $\det(\mathbf{SM}(\Lambda))$ are plotted as a function of Λ in Figure 4 where the values of Λ corresponding to the troughs represent the eigenvalues of the membrane. In Table 1, the results that are obtained using the present method are compared with the exact solutions and FEM results. The eigenvalues by FEM approach those of the exact solutions when a large number of nodes and thus a significant amount of computation were used. On the other hand, only a small number of nodes on the boundary, in the present instance $n_b = 16$, were enough to yield accurate solutions for the geometry considered here.

Interestingly, some of the eigenvalues could not be predicted when eight and twelve nodes were used: i.e., the third eigenvalue was not found for $n_b = 8$, and the seventh and eighth eigenvalues for $n_b = 12$. Considering the modes of vibration that are associated with the third and fourth eigenvalues may be of help to explain these results when $n_b = 8$. From Figure 5, it may be seen that the third mode has

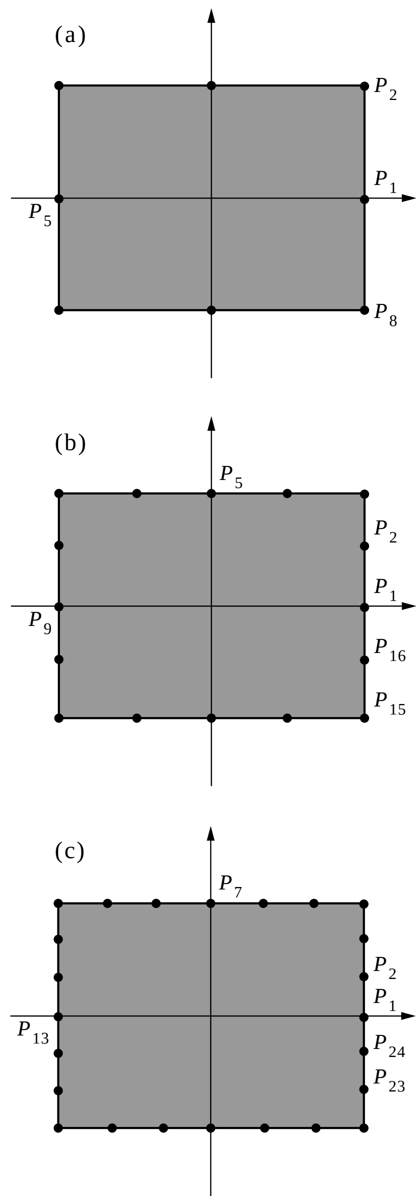

Figure 6. Discrete boundary nodes of the rectangular membrane when (a) $n_b = 8$, (b) $n_b = 16$, (c) $n_b = 24$.

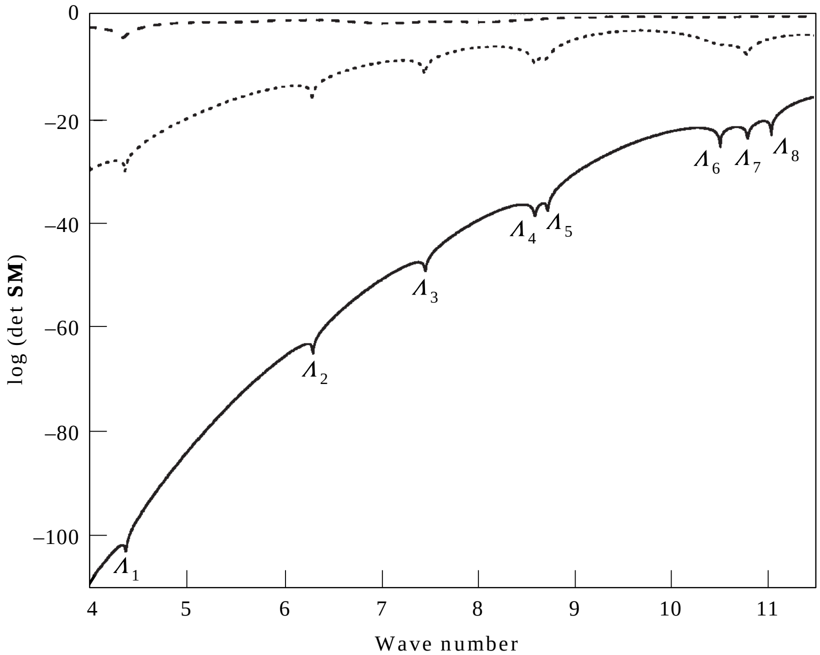

Figure 7. Determinant of the system matrix versus wave number for the rectangular membrane. – – –, $n_b = 8$; - - - -, $n_b = 16$; ——, $n_b = 24$.

two radial nodal lines and one azimuthal nodal circle, and the fourth mode two azimuthal nodal circles. It may be imagined that more boundary nodes are required to describe displacement variations in both the azimuthal and radial directions. On the other hand, note that the seventh eigenvalue for $n_b = 8$ is predicted in spite of the small number of boundary nodes. This phenomenon may happen because the seventh mode has four radial nodal lines, and because the locations of nodal points at which the nodal lines intersect with boundary are identical with those of the boundary nodes. Thus, it may also be said that the

TABLE 2

Comparison of eigenvalues of the rectangular membrane obtained by the present method, the exact method, and FEM

Eigenvalues	Present $n_b = 8$	Present $n_b = 16$	Present $n_b = 24$	Exact solution	FEM $n_{nd} = 1089$	FEM $n_{nd} = 289$	FEM $n_{nd} = 49$
Λ_1	4·3491	4·3633	4·3633	4·3633	4·3651	4·3703	4·4133
Λ_2	None	6·2927	6·2929	6·2929	6·3006	6·3240	6·5166
Λ_3	None	7·4549	7·4560	7·4560	7·4669	7·4996	7·7682
Λ_4	None	8·6001	8·5948	8·5947	8·6213	8·7013	9·1287
Λ_5	None	8·7101	8·7266	8·7266	8·7407	8·7828	9·3523
Λ_6	None	None	10·5083	10·5083	10·5370	10·6234	11·3284
Λ_7	None	10·7881	10·7943	10·7943	10·8313	10·9428	11·8467
Λ_8	None	None	11·0389	11·0384	11·1029	11·2974	12·7802

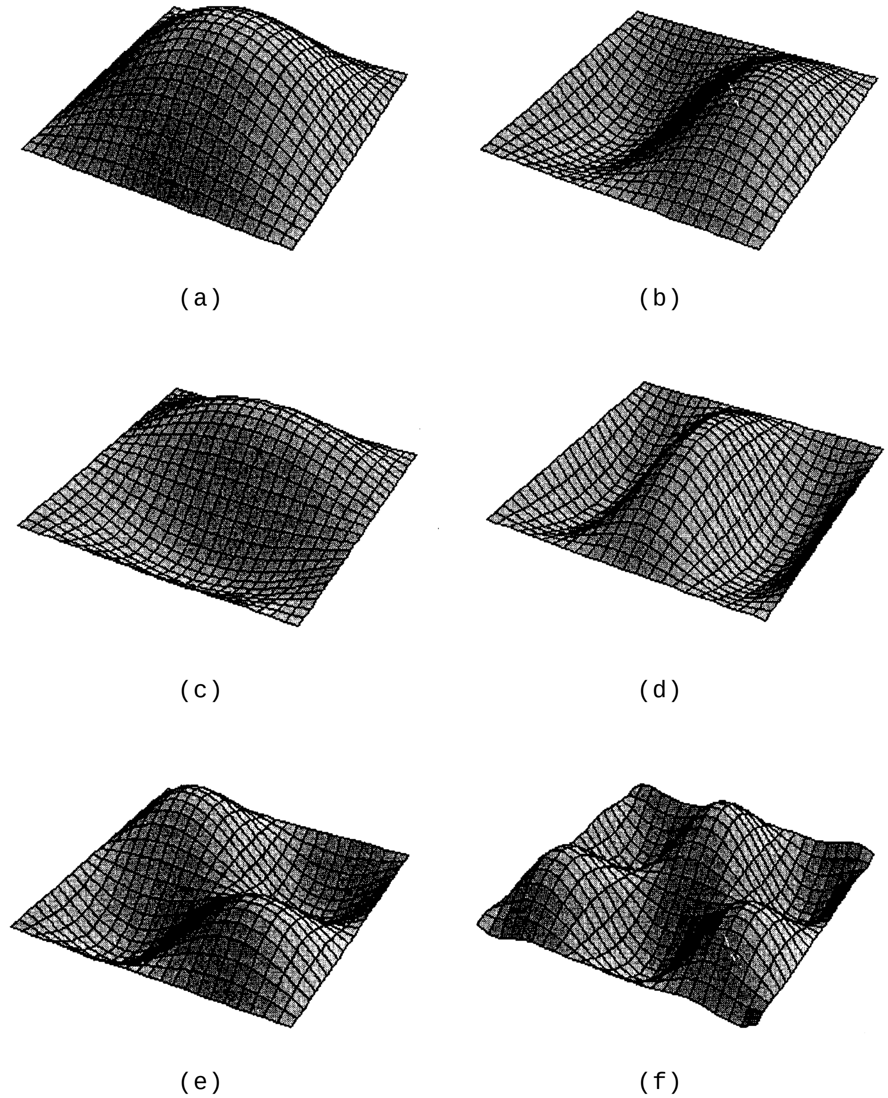

Figure 8. Mode shapes of the rectangular membrane obtained by the present method when $n_b = 24$: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, (f) 6th mode.

degree of convergence significantly depends on the modal behavior of membranes and the locations of boundary nodes.

3.2. RECTANGULAR MEMBRANE

As another verification example, a rectangular membrane whose dimension is 1·2 m by 0·9 m is considered. For $n_b = 8$, $n_b = 16$ and $n_b = 24$, the locations of nodes on the boundary of the membrane are illustrated in Figure 6, where in each case the equal number of nodes are located along the edges. In Figure 7 are shown logarithmic values of det $(\mathbf{SM}(\Lambda))$ as a function of Λ for three cases of discretized models to find the eigenvalues of the membrane. It may be seen in Table 2 that the present method yields more accurate and rapidly converging results with much less computational effort to the exact results than the FEM does. The fact that the sixth and eighth eigenvalues are not found for $n_b = 16$ may be explained by the same reasoning as done for the circular membrane. As may be seen from Figure 8, the five boundary nodes for each edge were not enough to describe the modal behavior of the sixth and eighth modes (i.e., (3, 2) and (4, 1) modes, respectively) in both the x- and y-directions. Thus, it may be concluded that the present method using the non-dimensional dynamic influence function can predict very accurate eigenvalues of membranes when a small but appropriate number of nodes along the boundary are used.

3.3. ARBITRARILY SHAPED MEMBRANE

Finally, free vibration analysis is carried out for an arbitrarily shaped membrane for which there exists no exact solution. The geometry and boundary node locations of the membrane considered in this section are shown in Figure 9. In

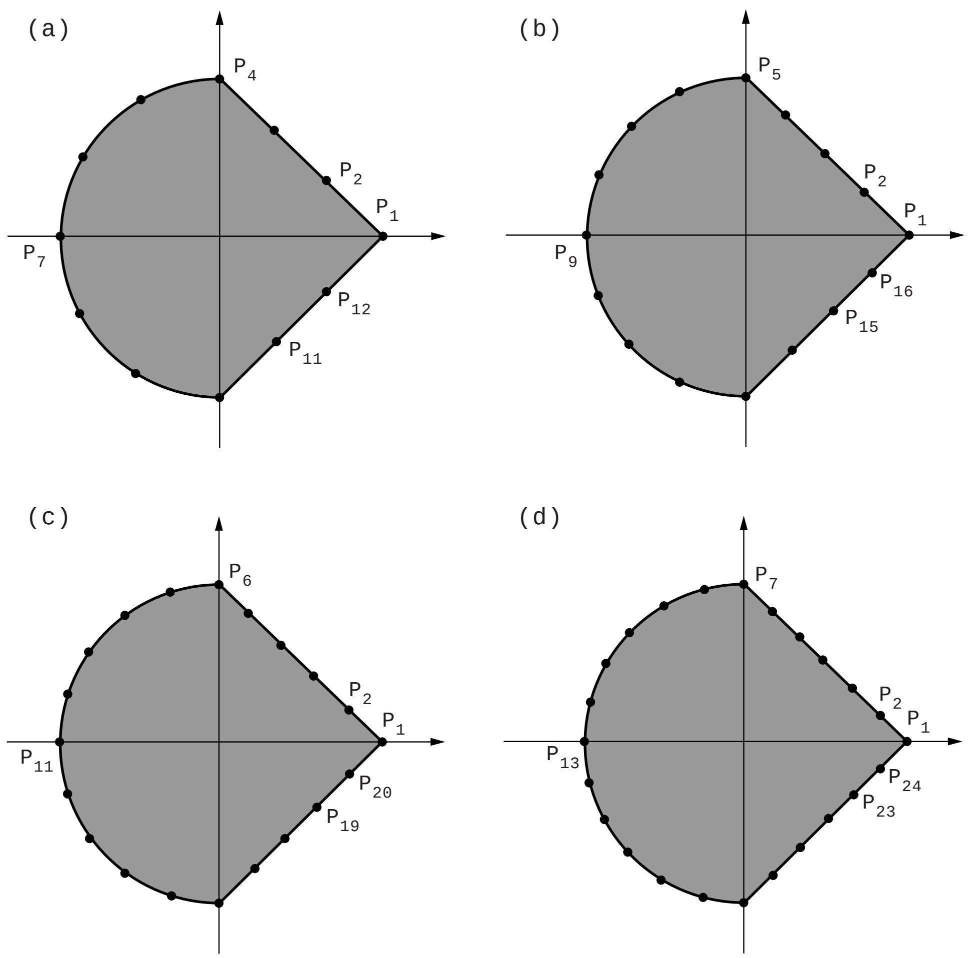

Figure 9. Discrete boundary nodes of the arbitrarily shaped membrane when (a) $n_b = 12$, (b) $n_b = 16$, (c) $n_b = 20$ (d) $n_b = 24$.

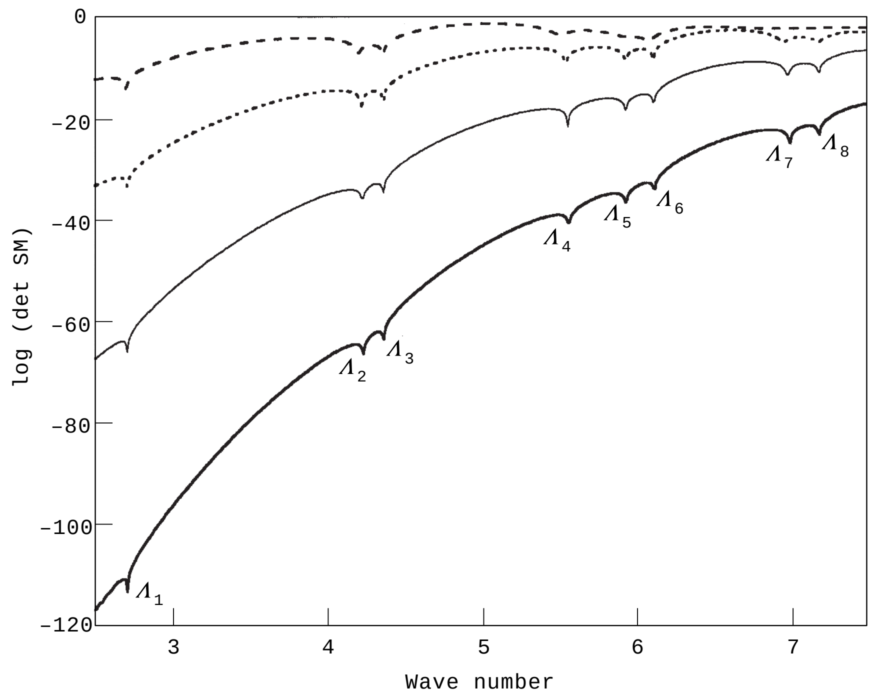

Figure 10. Determinant of the system matrix versus wave number for the arbitrarily shaped membrane. – – –, $n_b = 12$; - - - - -, $n_b = 16$; ——, $n_b = 20$; ——, $n_b = 24$.

Figure 10 are shown logarithmic values of det ($\mathbf{SM}(\Lambda)$) as a function of Λ for four cases. A comparison between the proposed and the numerical method is summarized in Table 3. As it may be observed from both Tables 1 and 2, the present method always yields lower eigenvalues and the FEM always yields higher eigenvalues compared to the exact method. Therefore, the exact eigenvalues of the membrane may exist between the eigenvalues that are obtained by the present method and FEM. It can also be found from Table 3 that the present and FEM

TABLE 3

Comparison of eigenvalues of the arbitrarily shaped membrane obtained by the present method and FEM

Eigenvalues	Present $n_b = 12$	Present $n_b = 16$	Present $n_b = 20$	Present $n_b = 24$	FEM $n_{nd} = 784$	FEM $n_{nd} = 576$	FEM $n_{nd} = 400$	FEM $n_{nd} = 256$
Λ_1	2·7038	2·7076	2·7089	2·7097	2·7230	2·7275	2·7349	2·7487
Λ_2	4·2027	4·2190	4·2253	4·2279	4·2598	4·2698	4·2864	4·3171
Λ_3	4·3585	4·3579	4·3579	4·3579	4·3786	4·3861	4·3987	4·4218
Λ_4	5·5037	5·5464	5·5593	5·5649	5·6336	5·6557	5·6924	5·7607
Λ_5	5·9199	5·9328	5·9336	5·9336	5·98460	6·0027	6·0324	6·0864
Λ_6	6·0859	6·1107	6·1143	6·1159	6·1641	6·1805	6·2077	6·2571
Λ_7	6·7746	6·9567	6·9849	6·9974	7·1334	7·1770	7·2495	7·3831
Λ_8	7·1607	7·1837	7·1858	7·1868	7·3002	7·3401	7·4057	7·5238

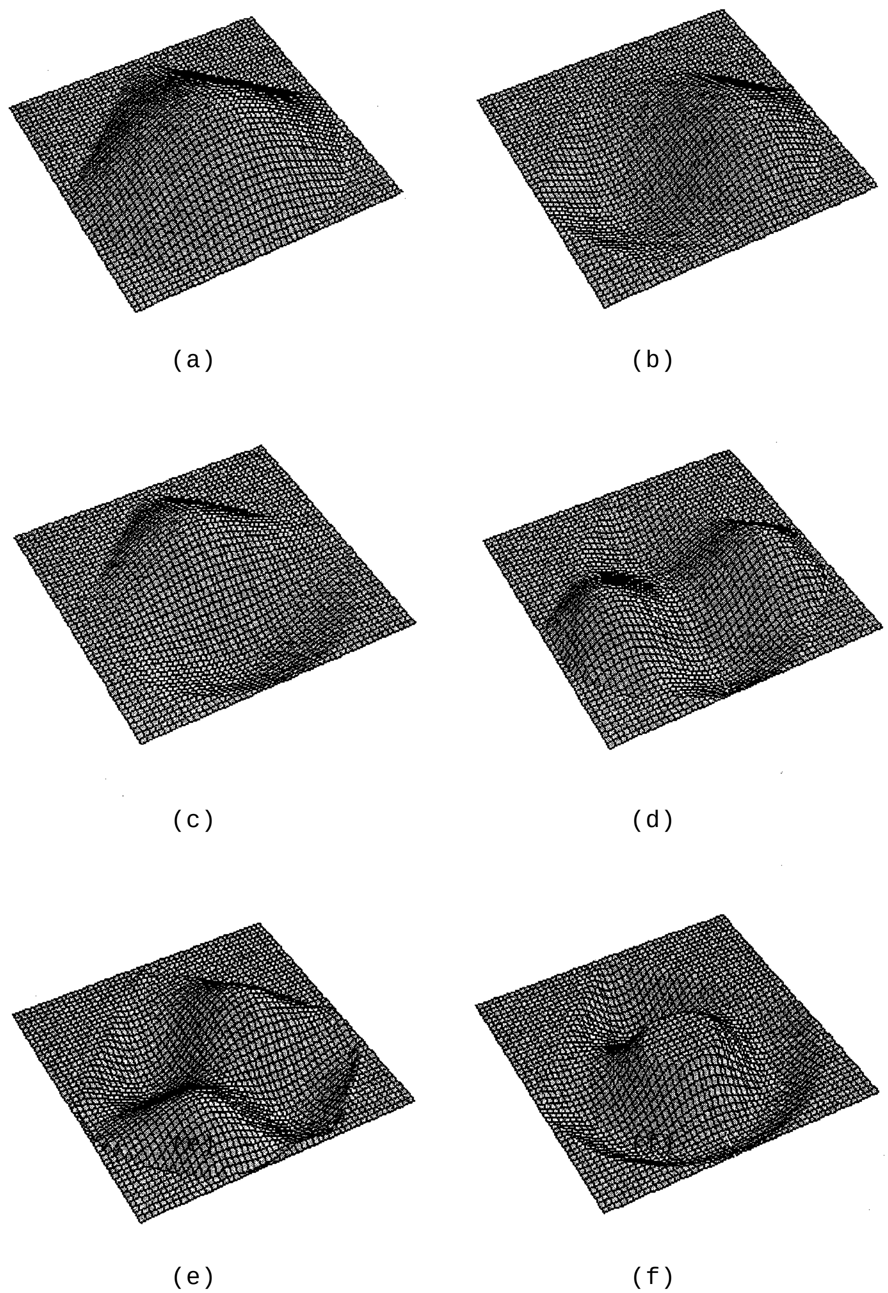

Figure 11. Mode shapes of the arbitrarily shaped membrane obtained by the present method when $n_b = 24$: (a) 1st mode, (b) 2nd mode, (c) 3rd mode, (d) 4th mode, (e) 5th mode, (f) 6th mode.

results converge to each other as the number of nodes used on the boundary and in the interior domain in the respective model increases. Mode shapes of the membrane that were obtained by the present method are shown in Figure 11, which were found to agree well with those by FEM.

4. CONCLUSION

In this paper, a method has been presented that can be applied to find free and forced responses of arbitrarily shaped membranes. Since the present method can be implemented without any integration procedures and the system matrix always becomes symmetric irrespective of boundary shape, complicated numerical calculations become very simple. It was also seen from the examples that the proposed method always gives the convergence as the number of collocation points increases step by step.

It is expected that the method presented in this work can be applied to analyze multiply-connected or elastically supported membranes. In addition, the basic theory used in the present study can be extended to the free vibration analysis of arbitrarily shaped plates and acoustic cavities with general boundary conditions.

REFERENCES

1. S. Timoshenko and S. Woinowsky-Krieger 1959 *Theory of Plates and Shells*. New York: McGraw-Hill; second edition.
2. H. D. Conway and K. A. Farnham 1965 *International Journal of Mechanical Sciences* **7**, 811–816. The free flexural vibration of triangular, rhombic and parallelogram plates and some analogies.
3. L. Rayleigh 1945 *Theory of Sound*. New York: Dover; second edition. See pp. 306–351.
4. N. W. McLachlan 1947 *Quarterly Applied Mathematics* **5**, 289–297. Vibrational problems in elliptical coordinates.
5. L. E. Kinsler, A. R. Frey, A. B. Coppens and J. V. Sanders 1982 *Fundamentals of Acoustics*. New York: Wiley; third edition.
6. J. Mazumdar 1973 *Journal of Sound and Vibration* **27**, 47–57. Transverse vibration of membranes of arbitrary shape by the method of constant-deflection contours.
7. K. Nagaya 1979 *Journal of Sound and Vibration* **65**, 1–9. Vibration of an arbitrarily shaped membrane with point supports.
8. K. Nagaya 1978 *American Society of Mechanical Engineers Journal of Applied Mechanics* **45**, 153–158. Vibrations and dynamic response of membranes with arbitrary shape.
9. Z. Ding 1992 *Computers and Structures* **42**, 389–394. Vibration of arbitrarily shaped membrane carrying elastically mounted masses.
10. Y. Y. Kim and J. H. Kang 1996 *Journal of the Acoustical Society of America* **99**, 2938–2946. Free vibration analysis of membranes using wave-types base functions.
11. J. B. Davies and C. A. Muilwyk 1966 *Proceedings of the Institution of Electrical Engineers* **113**, 277–284. Numerical solutions of uniform hollow waveguides with boundaries of arbitrary shape.
12. C. W. Steele 1968 *Journal of Computational Physics* **3**, 148–153. Numerical computation of electric and magnetic fields in a uniform waveguide of arbitrary cross section.
13. P. L. Arlett, A. K. Bahrani and O. C. Zienkiewicz 1968 *Proceedings of the Institution of Electrical Engineers* **115**, 1762–1766. Application of finite elements to the solution of Helmholtz's equation.
14. H. Y. Yee 1966 *Proceedings of the Institution of Electrical Engineers* **54**, 64. On determination of cutoff frequencies of waveguides with arbitrary cross-section.
15. H. Y. Yee and N. F. Audeh 1966 *Institute of Electrical and Electronic Engineers Transactions on Microwave Theory and Techniques* **MTT-14**, 487–493. Cutoff frequencies of eccentric waveguides.

16. R. H. T. Bates 1969 *Institute of Electrical and Electronic Engineers Transactions on Microwave Theory and Techniques* **MTT-17**, 294–301. The theory of the point-matching method for perfectly conducting waveguides and transmission lines.
17. R. M. Bulley and J. B. Davies 1969 *Institute of Electrical and Electronic Engineers Transactions on Microwave Theory and Techniques* **MTT-17**, 440–446. Computation of approximate polynomial solutions to TE modes in an arbitrary shaped waveguide.
18. P. A. Laura 1966 *Proceedings of the Institution of Electrical Engineers* **54**, 1495–1497. A simple method for determination of cut off frequencies of waveguides with arbitrary cross sections.
19. C. A. Brebbia, J. C. F. Telles and L. C. Wrobel 1984 *Boundary Element Techniques*. New York: Springer-Verlag.
20. C. A. Brebbia 1978 *The Boundary Element Method for Engineers*. New York: Wiley.
21. P. A. Laura 1965 *Proceedings of the Institution of Electrical Engineers* **53**, 1660–1661. Determination of cutoff frequencies of waveguides with arbitrary cross sections by point-matching.