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THE GREATEST NUMBER OF LIMIT CYCLES OF THE GENERALIZED RAYLEIGH–LIÉNARD OSCILLATOR

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The limit cycles of the generalized Rayleigh–Liénard oscillator $\ddot{X} + AX + 2BX^3 + \varepsilon(z_3 + z_2X^2 + z_1X^4 + z_4\dot{X}^2)\dot{X} = 0$, for $A > 0$, $B > 0$, and $A < 0$, $B > 0$ are studied by using the Jacobian elliptic functions with the generalized harmonic balance method. For given values of the parameters $z_i \neq 0$, $i = 1, 2, 3, 4$, the values of A and B for which limit cycles exist are found as functions of a single parameter. There is one limit cycle in the region where the Hamiltonian, E say, is positive, i.e., a solution of type cn, and six limit cycles, three double values, in the region where E is negative, solution type dn.

1. INTRODUCTION

Recently, six papers [1–6] have appeared on the limit cycles of the generalized mixed Rayleigh–Liénard oscillator, i.e., on the number of limit cycles of the equation:

$$\ddot{X} + AX + 2BX^3 + \varepsilon(z_3 + z_2X^2 + z_1X^4 + z_4\dot{X}^2)\dot{X} = 0. \qquad (1)$$

In this equation $2B$ is written in the expression for the force so as to have the potential in the simple form $V(X) = AX^2 + BX^4$, which is the most important for non-linear systems. The total energy is therefore $E = \dot{X}^2 + V(X)$, where $\dot{X} = \mathrm{d}X/\mathrm{d}t$.

Equations of the type (1) with $\varepsilon = 0$ are called generating equations, and their solutions generating solutions. The generating equation may be rewritten as $E = \dot{X}^2 + AX^2 + BX^4$, or $E = \dot{X}^2 + V(X)$, with the family of potentials $V(X) = AX^2 + BX^4$.

Special cases of equation (1) are the generalized Rayleigh and van der Pol differential equations given, respectively, by the expressions

$$\ddot{X} + AX + 2BX^3 + \varepsilon(z_3 + z_4\dot{X}^2)\dot{X} = 0,$$

and

$$\ddot{X} + AX + 2BX^3 + \varepsilon(z_3 + z_2X^2 + z_1X^4)\dot{X} = 0.$$

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Lynch [2–5] estimates the maximum number of small-amplitude limit cycles for certain classes of the generalized Liénard system of the form

$$\dot{X} = h(y) - F(X): \qquad \dot{y} = -g(X), \tag{2}$$

where F, g, h are polynomials, for the degree of F from 2 to 12 and the degree of g from 1 to 11. He finds that in the case of equation (1) at most three such limit cycles can exist.

The letter of Burnette and Mickens [1] using the KBM method [7–9] and fixing the coefficient A at $+1$ finds that the corresponding equation cannot have more than two limit cycles. From this result they deduce that the generalized Rayleigh–Liénard oscillator differential equation can have, in the case of equation (1), a maximum of two limit cycles.

In a previous paper, the present authors using the Jacobian elliptic functions with the generalized harmonic balance method found for the case with $A > 0, \mathrm{B} > 0$ a maximum of two limit cycles [6]. The transitory motion, and consequently the limit cycles and their stability were also studied quantitatively with a generalized approximation of the Krylov–Bogoliubov slowly varying amplitude and phase type, giving the radius, frequency and energy of the limit cycles. Whether there are zero, one or two limit cycles, and their stability, depends on the values of the parameters z_i, $i = 1, 2, 3, 4$ in the equations. These solutions are interesting because they do not depend on the value of ε.

For the simple cases of the generalized Rayleigh and van der Pol oscillators with only two non-zero z parameters, plots of the universal functions were given as well as the limit cycle (l.c.) radii.

In the present paper the limit cycles of the generalized Rayleigh, van der Pol and the equations of the mixed (Rayleigh–Liénard) oscillators are studied for the cases with $A < 0, B > 0$ following the same methods as in the authors' previous paper. In these cases a maximum of seven limit cycles is found. Although these methods are quite good for small ε, the examples show that they provide an excellent approximation for $\varepsilon = 1{\cdot}0$.

In a series of recent papers Chen and colleagues [10, 11] have presented two elliptic function methods, the elliptic perturbation method (EPM) [10] and the elliptic Linstedt–Poincaré method (ELP) [11], to calculate higher-order approximations for equations of the type

$$\ddot{X} + AX + 2BX^3 = \epsilon f(X, \dot{X}).$$

With respect to the general question of periodic solutions, it should be noted that periodic solutions also exist outside the potential wells. These periodic solutions, being elliptic functions, have finite period. In the case $A > 0, B < 0$ considered in reference [12], to every limit cycle inside the potential well, there corresponds a periodic solution outside.

2. THE METHODS

For a mixed type of oscillator (1), two types of solutions are considered, and the generalized harmonic balance method [13] used.

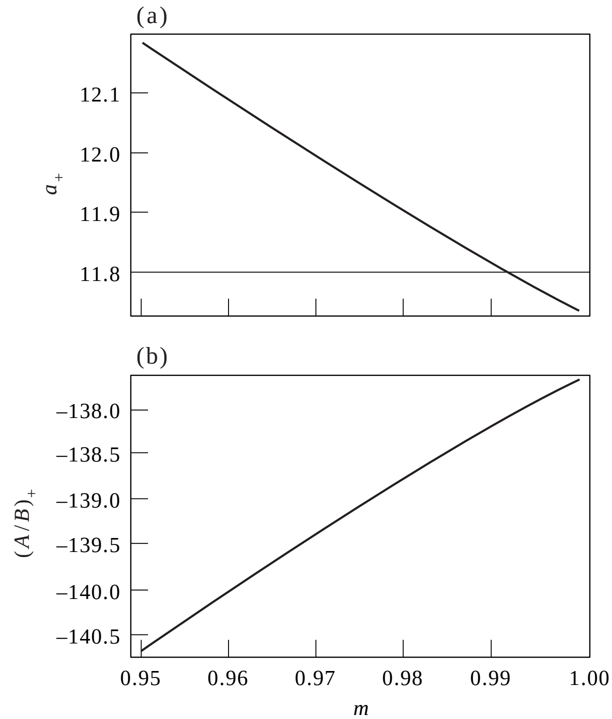

Figure 1. The limit cycle for equation (15) in the region $E > 0{\cdot}0$. (a) Maximum amplitude, a_+, versus m; (b) the ratio A/B versus m for the maximum amplitude.

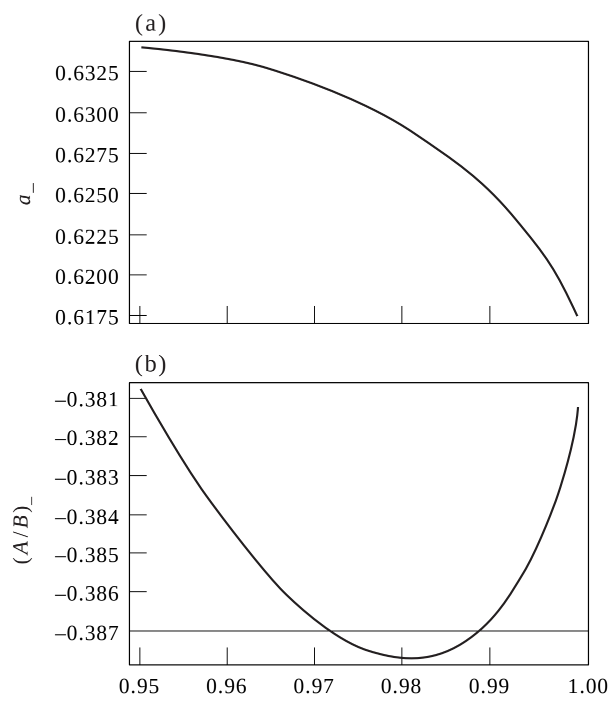

Figure 2. The limit cycle for equation (15) in the region $E > 0{\cdot}0$. (a) Minimum amplitude, a_-, versus m; (b) the ratio A/B versus m for the minimum amplitude.

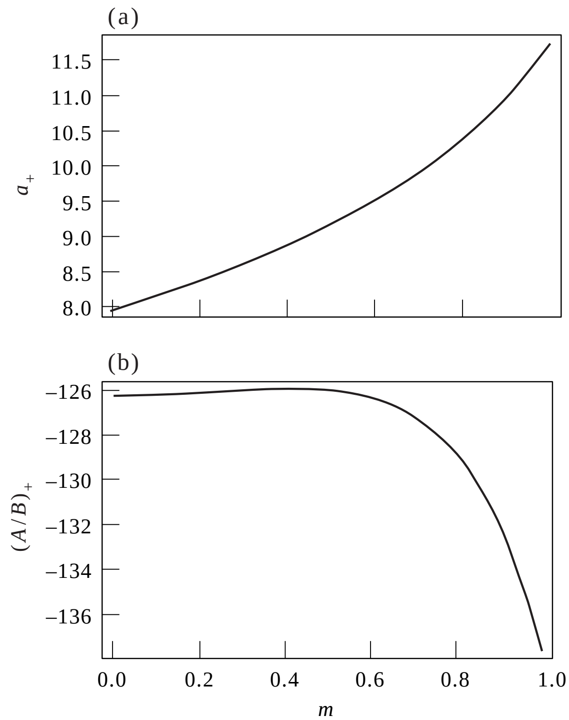

Figure 3. The limit cycle for equation (15) in the region $E < 0{\cdot}0$. (a) Maximum amplitude, a_+, versus m; (b) the ratio A/B versus m for the maximum amplitude.

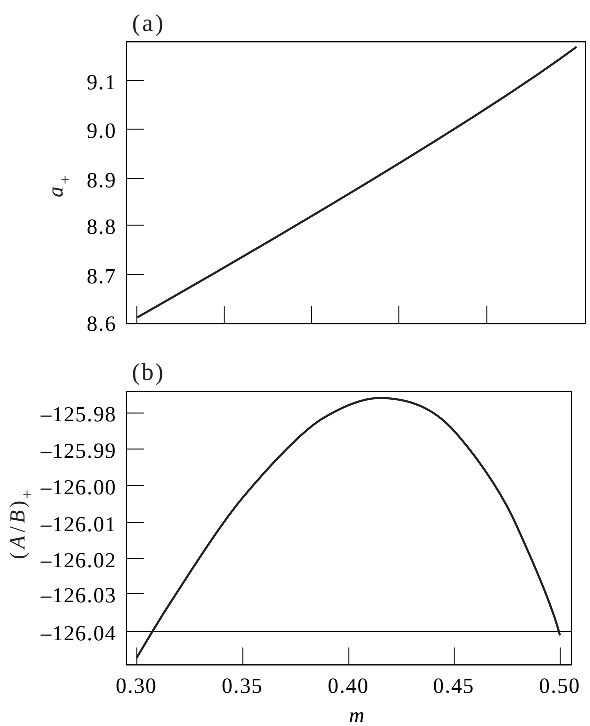

Figure 4. Figure 3 enlarged in the region $0{\cdot}3 \leqslant m \leqslant 0{\cdot}5$.

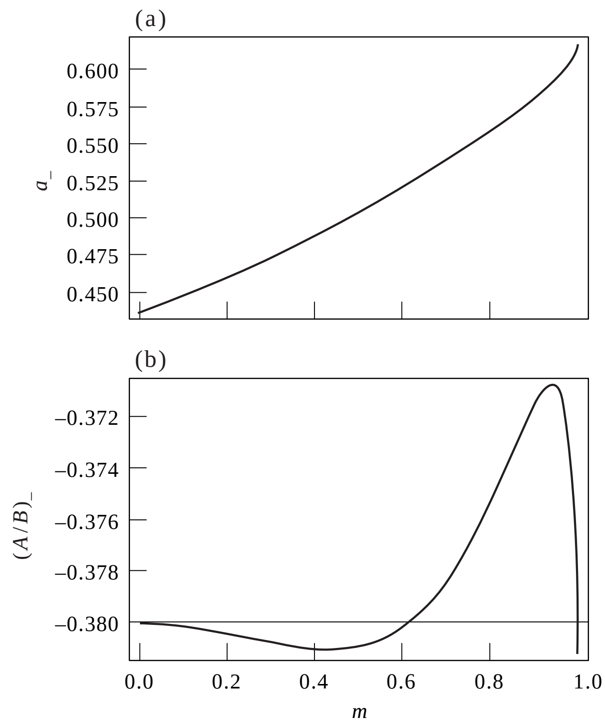

Figure 5. The limit cycle for equation (15) in the region $E < 0{\cdot}0$. (a) Minimum amplitude, a_-, versus m; (b) the ratio A/B versus m for the minimum amplitude.

2.1. THE SOLUTION "cn"

If $A < 0$, $B > 0$ and $1/2 \leqslant m \leqslant 1$, a solution of the form [14]

$$X(t) = a\,\mathrm{cn}\,(\omega t; m), \tag{3}$$

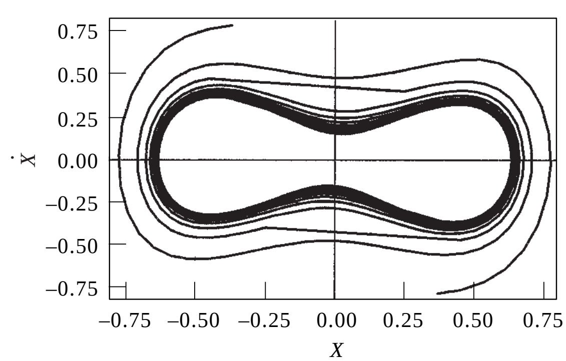

Figure 6. Phase portraits of equation (15) in the $E > 0{\cdot}0$ region for $A/B = -0{\cdot}381$, $X(t=0) = 0{\cdot}633963$, $\dot{X}(t=0) = 0{\cdot}0$, and $m = 0{\cdot}950547$.

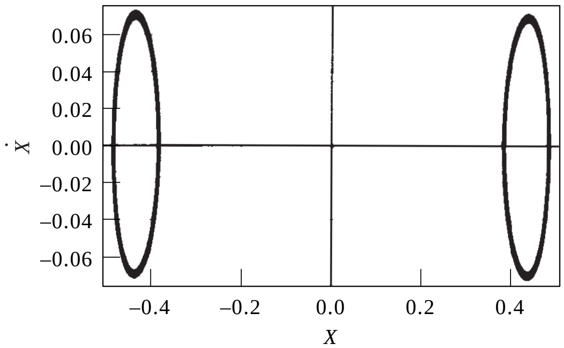

Figure 7. Phase portraits of equation (15) in the $E < 0{\cdot}0$ region for $A/B = -0{\cdot}381$, $X(t=0) = 0{\cdot}482434$, $\dot{X}(t=0) = 0{\cdot}0$, and $m = 0{\cdot}363000$.

is assumed, where a, ω and $m = \kappa^2$ are constants determined to first-order approximation. Substituting equation (3) into equation (1) and applying a generalized Fourier expansion limited to the first harmonic [6] gives

$$a^4 z_1 C_1 + a^2 z_2 C_2 + z_3 C_3 + a^2 \omega^2 z_4 C_4 = 0, \tag{4}$$

and

$$\omega^2 = Ba^2/m \quad \text{or} \quad \omega^2 = A/(1-2m), \tag{5a, b}$$

and from these two equations

$$A/B = (1-2m)a^2/m. \tag{6}$$

2.2. THE SOLUTION "dn"

If $A < 0$, $B > 0$ and $0 \leqslant m \leqslant 1$, a solution of the form [14]

$$x(t) = a \,\mathrm{dn}\,(\omega t; m), \tag{7}$$

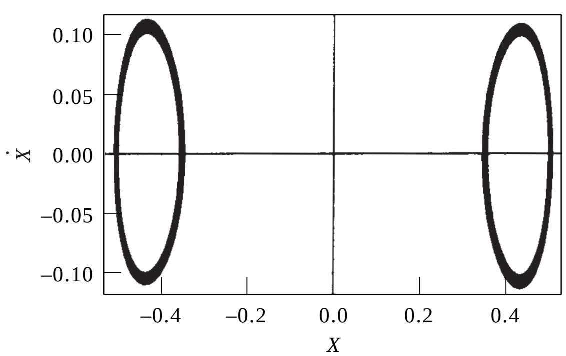

Figure 8. Phase portraits of equation (15) in the $E < 0{\cdot}0$ region for $A/B = -0{\cdot}381$, $X(t=0) = 0{\cdot}503146$, $\dot{X}(t=0) = 0{\cdot}0$, and $m = 0{\cdot}495000$.

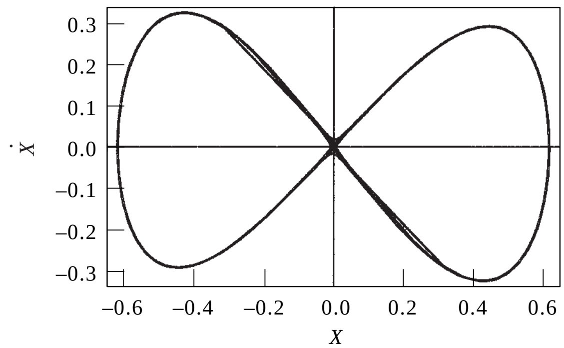

Figure 9. Phase portraits of equation (15) in the $E < 0{\cdot}0$ region for $A/B = -0{\cdot}381$, $X(t=0) = 0{\cdot}617205$, $\dot{X}(t=0) = 0{\cdot}0$, and $m = 0{\cdot}999850$.

is assumed where a, ω and $m = \kappa^2$ are constants determined to first-order approximation. Substituting equation (7) into equation (1), and applying a generalized Fourier expansion limited to the first harmonic [15], gives

$$a^4 z_1 B_1 + a^2 z_2 B_2 + z_3 B_3 + a^2 m^2 \omega^2 z_4 B_4 = 0, \qquad (8)$$

and

$$\omega^2 = Ba^2 \quad \text{or} \quad \omega^2 = A/(m-2), \qquad (9a, b)$$

and from these two equations

$$A/B = (m-2)a^2. \qquad (10)$$

3. ANALYTICAL RESULTS

Assuming always that $\varepsilon \cong 0$ and considering only the general cases, for all $z_i \neq 0$, $i = 1, 2, 3, 4$ equation (1) becomes

$$\ddot{X} + AX + 2BX^3 + \varepsilon(z_3 + z_2 X^2 + z_1 X^4 + z_4 \dot{X}^2)\dot{X} = 0.$$

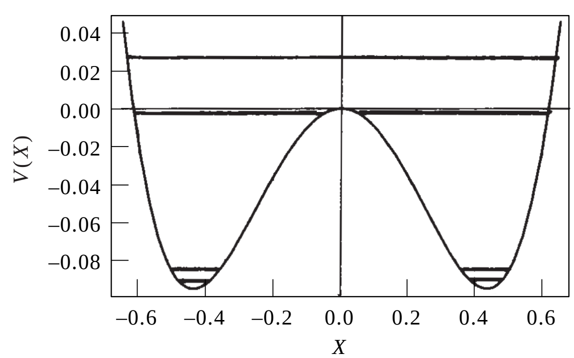

Figure 10. The seven limit cycles in the potentials of equation (15), for $A/B = -0{\cdot}381$.

3.1. FOR $A < 0, B > 0$ AND $1/2 \leqslant m \leqslant 1$, THE STATIONARY STATE IS GIVEN BY EQUATION (4)

For equation (5b) $\omega^2 = A/(1-2m)$, this equation may be rewritten as

$$y^2 b_c + y c_c + d_c = 0, \tag{11}$$

where $y = a^2$, $b_c = (1-2m)z_1 C_1$, $c_c = (1-2m)z_2 C_2 + A z_4 C_4$ and $d_c = (1-2m)z_3 C_3$. The roots of equation (11) are

$$y = \frac{-c_c \pm \sqrt{c_c^2 - 4b_c d_c}}{2b_c} \tag{12}$$

For given A, B and all z_i, if $c_c^2 > 4b_c d_c$ there are two roots, which depend on m, and both are positive. There are therefore two different possible limit cycles and, from equation (6), two values of A/B.

3.2. FOR $A < 0, B > 0$ AND $0 \leqslant m \leqslant 1$, THE STATIONARY STATE IS GIVEN BY EQUATION (8)

$$a^4 z_1 \mathscr{B}_1 + a^2 z_2 \mathscr{B}_2 + z_3 \mathscr{B}_3 + a^2 m^2 \omega^2 z_4 \mathscr{B}_4 = 0.$$

From equation (9b), $\omega^2 = A(m-2)$, this equation may be rewritten as

$$y^2 b_d + y c_d + d_d = 0, \tag{13}$$

where $y = a^2$, $b_d = (m-2)z_1 \mathscr{B}_1$, $c_d = (m-2)z_2 \mathscr{B}_2 + A z_4 \mathscr{B}_4$ and $d_d = (m-2)z_3 \mathscr{B}_3$. The roots of equation (11) are

$$y = \frac{-c_d \pm \sqrt{c_d^2 - 4b_d d_d}}{2b_d} \tag{14}$$

For given A, B and all z_i, if $c_d^2 > 4b_d d_d$ in principle there are two roots, which depend on m, and both are positive. There are therefore two different possible limit cycles and, from equation (10), two values of A/B.

4. A CASE WITH SEVEN LIMIT CYCLES

Consider the case

$$\ddot{X} - X + 5{\cdot}2493X^3 + (0{\cdot}18 - 0{\cdot}95X^2 + 0{\cdot}015X^4 - 0{\cdot}545\dot{X}^2)\dot{X} = 0. \tag{15}$$

4.1. THE LIMIT CYCLES IN THE $E > 0$ REGION

The solutions are of the type cn. The stationary states are given by equation (11). Figures 1 and 2 show the roots (a_+) and (a_-) and the corresponding ratios A/B versus m.

4.2. THE LIMIT CYCLES IN THE $E < 0$ REGION

In this region the potential has a double well. The solutions are of the type dn, i.e., for every solution in the right-hand well, there corresponds a similar solution in the left-hand well. The stationary states are given by equation (13). Figure 3 shows the root (a_+) and the corresponding ratio A/B versus m. Figure 4 shows in more detail part of the region $0{\cdot}0 < m < 0{\cdot}50$, because in this region one can observe the presence of two solutions, i.e., four limit cycles.

Figure 5 shows the root (a_-) and the corresponding ratio A/B versus m. Now one can see that for every value of the ratio of A/B there are at least two values of m, i.e., four limit cycles, and for $(A/B) < -0{\cdot}381$ three values, i.e., six limit cycles.

From Figure 2 it is seen that for $(A/B) = -0{\cdot}381$ there is one l.c. for $m = 0{\cdot}951$ with $a = 0{\cdot}633963$. From Figure 5, for the same value of A/B there are three values of m, i.e., six limit cycles for $m = 0{\cdot}363$, $a = \pm 0{\cdot}482434$, for $m = 0{\cdot}495$, $a = \pm 0{\cdot}503146$ and for $m = 0{\cdot}99985$, $a = \pm 0{\cdot}617205$. Observe that for this value of the ratio A/B, and in the case being considered, one has a total of seven limit cycles.

5. COMPARISON WITH NUMERICAL INTEGRATION

It is instructive to compare these simple analytical approximations with a numerical integration of the equations. MATHEMATICA was used to illustrate the more important results with the value $\varepsilon = 1{\cdot}0$.

$A = +1{\cdot}000$ was selected and the limit cycle shown in the region $E > 0{\cdot}0$, i.e., the type cn, in Figure 6. The six limit cycles in the region $E < 0{\cdot}0$, i.e., those of type dn, are shown in Figures 7–9.

The analytical solutions are not shown because they are indistinguishable from the numerical estimates at the level of precision of the figures.

6. CONCLUSION

Equation (1) has been studied using the generalized method of harmonic balance to obtain the first-order approximations to the periodic solutions, using elliptic functions.

The method has been applied to the study of the limit cycles of equation (15). For $E > 0{\cdot}0$, (cn-type solution), and for $E < 0{\cdot}0$, (dn-type solution), for $z_i \neq 0{\cdot}0$, $i = 1, 2, 3, 4$, and for the ratio $A/B = -0{\cdot}381$, one has a total of seven limit cycles.

The method gives all the possible limit cycles and, in the case considered with a double potential well, yielded the following important results: (a) Inside the double wells, the method gives trivially the symmetrical solutions corresponding to a given energy of the cycle. (b) The method gives all the solutions in one step, avoiding the numerous pitfalls of numerical searches. (c) The method is especially well-suited to the case of the unstable cycles which are so difficult to find in numerical searches.

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