Journal of Sound and Vibration (1999) **221**(1), 49–66
Article No. jsvi.1998.1971, available online at http://www.idealibrary.com.on

LOCALIZATION OF LONGITUDINAL WAVES IN BI-PERIODIC ELASTIC STRUCTURES WITH DISORDER

C. CETINKAYA

Department of Mechanical and Aeronautical Engineering, Clarkson University, P.O. Box 5725, *Potsdam, NY* 13699–5800, *U.S.A.*

(*Received* 14 *May* 1998, *and in final form* 9 *September* 1998)

A formulation for studying the effect of random variation in the transfer matrix on the attenuation behavior of disordered one-dimensional bi-periodic layered structures is developed. This formulation, however, can be used for both stochastically and deterministically disordered systems. The mean and variance of localization factors for the disordered systems are numerically evaluated under the assumption that the source of disorder is the variance in the Young's modulus of the first layer of a set, and this variation is modelled by a random variable with uniform probability density function. In the calculations for the mean localization factor and its variance for two layered systems with different elasticity properties are considered for four different levels of disorder. The results presented in the current work is used to explain how the attenuation zones of a perfect system expand into adjacent propagation zones due to disorder. It is found that while the existence of disorder affects the structures of all propagation zones, its significance is most predominant in the first propagation zones of both systems. However, as the frequency increases, the disorder level and the effect of elastic coupling become less significant. Therefore, the wave components corresponding to higher propagation zones penetrate deeper into a structure. It is observed that the right boundary of the propagation zones is the mean localization factor asymptote. The behavior of the mean Lyapunov curves near the right propagation zone boundaries explains how the disorder level governs the expansion of attenuation zones into propagation zones. This expansion is strongest in the first propagation zone. Since the power density of motion induced by an external force is high in the first propagation zone, the structure and size of first propagation zone are of a particular interest in transient analysis. The structures of the variance curves exhibit similar behavior to those for the mean localization factor.

1. INTRODUCTION

A system consisting of uniformly connected identical subsystems is said to be periodic, and the nature of the relationship between the co-ordinates (degrees of freedom) of the subsystems of a periodic system defines its coupling type. If two adjacent subsystems are related through a single co-ordinate, the periodic system

0022–460X/99/110049 + 18 $30.00/0

is mono-coupled. Systems with more than one coupling co-ordinate are said to be multi-coupled. The dynamics of periodic systems has been the subject of many studies since a large number of physical systems in a wide variety of applications from electronics to engineering structures can be modelled and investigated by using the generic framework developed for periodic systems.

One-dimensional wave propagation in structures has been extensively studied in the context of periodic structures. The existence of a bay-wise dispersion relation for a periodic structure and the properties of attenuation and propagation zones in the frequency domain have been known for over half a century [1]. An extensive review of contributions in mono- and multi-coupled wave propagation problems is reported in reference in which works are carefully surveyed and numerous contributions made at Southampton in the period 1964–1995 are explained. In developing a generic framework for periodic structures, references [3] and [4] introduce a receptance formulation to study the structures of propagation and attenuation zones in infinite, mono- and multi-coupled periodic structures. Other methods used in the area include the transfer matrix approach [5] and modal analysis [6]. Some other methods and techniques adopted for periodic structures are outlined in reference [7].

The periodic system approach is an idealization since, in practice, the manufacture of identical subsystems is impossible. The effect of the deviation from the perfect periodicity becomes important when a large periodic system, in which additional successive reflections and mode conversions at the subsystem interfaces affect the structure of propagation and attenuation zones, is the focal point of a stress wave propagation investigation. Reference [8] illustrates localization of electron diffusion in an atomic lattice and triggered a large amount of research work in solid state physics. Mostly independently, stress localization in disordered periodic structures has been studied in the structural analysis context. In many studies the nature of disorder is assumed to be deterministically known (e.g. [9]). References [10] and [11] examine the effect of disorder in infinite and finite periodic structures, respectively, and conclude that a single order can reduce transmission of flexural waves in a beam even for a propagating frequency component in a propagation zone. Reference [12], the first work in which the effect of a generic stochastic disorder in a mono-coupled periodic system is studied, provides the Lyapunov exponents and shows that the largest Lyapunov exponent is the localization factor, which is the rate of amplitude decay in the spacial co-ordinate. Many techniques based on this result are generated as reported in references [13–15]. In reference [16], the work on mono-coupled systems is generalized to mutli-coupled systems. In references [15] and [17], procedural techniques based on reference [16] to calculate Lyapunov exponents are presented along with a body of simulation results generated by adopting Monte-Carlo methods.

In this study, one-dimensional wave localization in disordered bi-periodic elastic layered structures is investigated. The periodic system is mono-coupled and the elasticity problem is plane stress. First, by utilizing the results of application of the Floquet theorem to periodic structures and a transfer matrix formulation, the propagation problem is represented as an algebraic eigenvalue problem. From this formulation, it is shown how variations in the transfer matrix of a disordered periodic system affect the propagation constant. Using the expression derived for the equivalent propagation constant, the effects of disorder in localization characteristics of the system are examined. The mean and variance of the localization factor for two systems with four disorder levels are calculated. The effect of elastic coupling as well as the disorder level on the overall dynamics is discussed.

This current study can be considered as an extension of a previous work [18], in which a condition for shifting the last natural frequency of the first propagation zone into the first attenuation zone is presented for a finite structure, and the link between the frequency shift and its effect on the transient response of the system is studied in detail. Here, it is shown how disorder destroys the propagation zones of a perfectly periodic infinite system for longitudinal wave motion.

2. LONGITUDINAL MOTION LOCALIZATION

The transfer matrix formulation of a single bi-periodic isotropic set can be obtained by setting the radial wave number to zero in the axisymmetric formulation given in reference [19]. This results in two decoupled wave propagation problems; longitudinal and transverse. In this study, only the localization of longitudinal stress waves is considered. The transfer function for a bi-periodic set can be expressed as

$$[T_{set}]_k = \begin{bmatrix} \cosh(g_B H_B) & \dfrac{-\sinh(g_B H_B)}{g_B(\lambda_B + 2\mu_B)} \\ -g_B(\lambda_B + 2\mu_B)\sinh(g_B H_B) & \cosh(g_B H_B) \end{bmatrix}_k$$

$$\times \begin{bmatrix} \cosh(g_A H_A) & \dfrac{-\sinh(g_A H_A)}{g_A(\lambda_A + 2\mu_A)} \\ -g_A(\lambda_A + 2\mu_A)\sinh(g_A H_A) & \cosh(g_A H_A) \end{bmatrix}_k,$$

where ω is the radial frequency for the scaled time (cf. reference [19]), H_A and H_A are, respectively, the layer thicknesses for *Layer A* (the first layer of the bi-periodic set) and *Layer B* (the second layer of the set). The scaled time, t_s, is given as $t_s = (c_T/H)_A t$, and the frequency, ω, in the current formulation corresponds to t_s (see below for the parameter c_T). The terms g_A and g_B in equation (1) are given as

$$g_A^2 = -\omega^2/a_A^2, \qquad g_B^2 = -\omega^2/a_B^2,$$

where

$$a_A = c_{L_A}/c_{T_A}, \qquad c_{L_A} = \sqrt{(\lambda_A + 2\mu_A)/\rho_A}, \qquad c_{T_A} = \sqrt{\mu_A/\rho_A},$$

$$a_B = c_{L_B}/c_{T_B}, \qquad c_{L_B} = \sqrt{(\lambda_B + 2\mu_B)/\rho_B}, \qquad c_{T_B} = \sqrt{\mu_B/\rho_B},$$

where λ, μ and ρ with an appropriate subscript are the Lamè coefficients and material density of a layer material indicated by its subscript.

As reported in reference [3], the locations of natural frequencies in a propagation zone of an n-set periodic system are calculated by solving the following equation for ω:

$$\cosh(\mu(\omega)) = \cos(\pi p/n),$$

where $\mu(\omega)$ is the propagation constant of the infinite system, n is the number of periodic bays in a finite system, and $p = 1, 2, \ldots, n-1$. This expression implies that the last frequency along with others moves closer to boundaries between the propagation and attenuation zones as the ratio $(n-1)/n$ increases. This observation and the algebraic condition given in reference [18] for a generic weakly coupled finite bi-periodic system with a single disorder explain why long periodic systems may be prone to stress localization in presence of even small disorder. For large n, even a very small shift of natural frequency locations due to disorder could cause one or more natural frequencies to pass into the next attenuation zone. This shift affects the global dynamics of a long chain.

The transfer matrix in equation (1) relates the stress and displacement states on the two interfaces of bi-periodic sets of a system shown in Figure 1. The relation for a set becomes

$$\{u\}_k = [T]_k\{u\}_{k-1}, \tag{2}$$

where $[T]_k$ and $\{u\}_k$ denote the transfer matrix for the kth periodic set and the stress and displacement state at the left face of the set k, respectively. Similarly, from the Floquet theorem for the linear ordinary differential equations with periodic coefficients, the states on the both faces of a set can be expressed as

$$\{\pi u\}_k = e^{\mu^k}\{u\}_{k-1}, \tag{3}$$

where μ^k is multi-varied complex function, the propagation constant. Equation (3) dictates that if the value of μ^k is purely imaginary, the mode for the corresponding frequency extents. A frequency interval for which μ^k is purely imaginary is said to be a propagation zone (PZ). If μ^k has a real part, the corresponding wave motions exponentially decay in the propagation direction. A frequency interval for which μ^k is of real part is said to be an attenuation zone (AZ). Since the conservation of energy is implicitly stated, it can be concluded that if the imaginary part of μ^k at a frequency is negative, the corresponding wave mode propagates

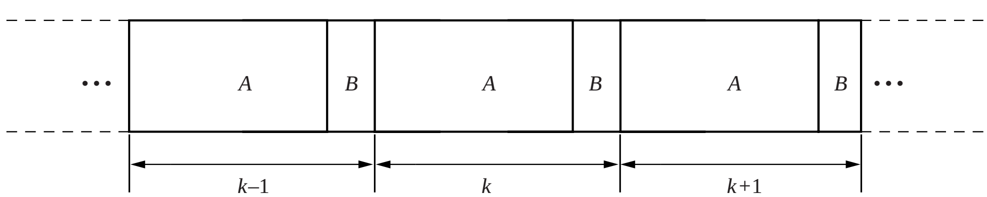

Figure 1. Three sets of a bi-periodic system consisting of *LayerA* and *LayerB* and the state of displacement and stress components at the interfaces. For plane stress case, layers extent to infinity in the direction parallel to interfaces.

rightward. It it is positive, the corresponding wave mode moves leftward. It is noteworthy that the motion becomes vibrational (standing waves) when $\mu^k = 0$. From equations (2) and (3), as one should expect, the problem can be expressed in eigenvalue problem form,

$$[T]_k\{u\}_{k-1} = \mathrm{e}^{\mu^k}\{u\}_{k-1}. \tag{4}$$

For a $2n \times 2n$ transfer matrix for a generic multi-coupled set, the eigenvalues are denoted as

$$\{\mathrm{e}^{\mu_1^k}, \dots, \mathrm{e}^{\mu_n^k}, \mathrm{e}^{\mu_{(n+1)}^k}, \dots, \mathrm{e}^{\mu_{2n}^k}\}$$

and the corresponding eigenvectors of the transfer matrix are given in the same order,

$$\{v_1^k, \dots, v_n^k, v_{n+1}^k, \dots, v_{2n}^k\}.$$

Above the eigenvalues and corresponding eigenvectors are ordered is such a way that the first n propagation constant corresponds to rightward waves and the rest is for leftward waves, i.e.,

$$\mathrm{Im}\,(\mu_j) < 0, \qquad \text{for } j = 1, \dots, n;$$

$$\mathrm{Im}\,(\mu_j) > 0, \qquad \text{for } j = n+1, \dots, 2n;$$

where the function Im (·) returns the imaginary part of its argument. Thus, any stress–displacement state at an interface can be decomposed in terms of the eigenvectors,

$$\{u\}_{k-1} = \sum_{j=1}^{n} \alpha_j^k\{v_j^k\} + \sum_{j=n+1}^{2n} \alpha_j^k\{v_j^k\}, \tag{5}$$

where the first sum corresponds to rightward travelling waves while the second sum is for leftward, and the terms α_j are expansion coefficients. From equations (2) and (3), the state $\{u\}_k$ can be written as

$$\{u\}_k = [T]_k\{u\}_{k-1} = \sum_{j=1}^{n} \alpha_j^k\, \mathrm{e}^{\mu_j^k}\{v_j^k\} + \sum_{j=n+1}^{2n} \alpha_j^k\, \mathrm{e}^{\mu_j^k}\{v_j^k\}. \tag{6}$$

As can be seen in equation (6), μ_j^k is the propagation constant for the infinite system consisting of the periodic set $[T]_k$. The imaginary part of μ_j^k plays the role of the bay-wise wave number in the direction of propagation. If a wave state at the left interface of a set travelling rightward is taken as

$$\{u\}_k = \alpha_1^k\{v_1^k\} \tag{7}$$

then no reflection at this interface occurs for this mode of motion. However, by no means this statement implies that there are no waves travelling leftward. Similarly, the same state can be expressed in terms of the bases of the set $k+1$:

$$\{u\}_k = [v_1^{k+1} \vdots v_2^{k+1}] \begin{Bmatrix} \alpha_1^{k+1} \\ \alpha_2^{k+1} \end{Bmatrix}. \tag{8}$$

Thus, from equations (7) and (8), the following can be written:

$$[v_1^k \vdots \{0\}] \begin{Bmatrix} \alpha_1^k \\ \cdots \\ 0 \end{Bmatrix} = [v_1^{k+1} \vdots v_2^{k+1}] \begin{Bmatrix} \alpha_1^{k+1} \\ \cdots \\ \alpha_2^{k+1} \end{Bmatrix}, \tag{9}$$

where the expansion coefficients α_1^{k+1} correspond to the transmitted wave whereas α_2^{k+1} is for the reflection. The rightward propagating component of the state $\{u\}_{k+1}$ can be expressed as $\{v_1^{k+1}\}\, e^{\mu_1^{k+1}} \alpha_1^{k+1}$ from equation (3). By repeating this procedure, the entire wave field for the rightward propagating motion can be created. For example, after solving equation (9) for α_1^{k+1} and α_1^{k+2}, the state at the set $k+2$ is represented as

$$[v_1^{k+1} \vdots \{0\}] \begin{Bmatrix} e^{\mu_1^{k+1}} \alpha_1^{k+1} \\ \cdots \\ 0 \end{Bmatrix} = [v_1^{k+2} \vdots v_2^{k+2}] \begin{Bmatrix} \alpha_1^{k+2} \\ \cdots \\ \alpha_2^{k+2} \end{Bmatrix}. \tag{10}$$

For a mono-coupled system, solving equation (9) for the transmission and reflection coefficients, respectively, yields:

$$\alpha_1^{k+1} = (y_1^{k+1} x_1^k + y_2^{k+1} x_2^k)\alpha_1^k, \qquad \alpha_2^{k+1} = (y_3^{k+1} x_1^k + y_4^{k+1} x_2^k)\alpha_1^k. \tag{11}$$

Similarly, equation (10) results in

$$\begin{aligned} \alpha_1^{k+2} &= (y_1^{k+2} x_1^{k+1} + y_2^{k+2} x_2^{k+1})\, e^{\mu_1^{k+1}} \alpha_1^{k+1}, \\ \alpha_2^{k+2} &= (y_3^{k+2} x_1^{k+1} + y_4^{k+2} x_2^{k+1})\, e^{\mu_1^{k+1}} \alpha_1^{k+1}, \end{aligned} \tag{12}$$

where

$$\begin{bmatrix} y_1^{k+1} & y_2^{k+1} \\ y_3^{k+1} & y_4^{k+1} \end{bmatrix} = [v_1^{k+1}, v_2^{k+1}]^{-1}, \qquad \begin{bmatrix} x_1^k & 0 \\ x_2^k & 0 \end{bmatrix} = [v_1^k, 0].$$

The transmission coefficients for the kth and $(k+2)$th sets are then related as

$$\alpha_1^{k+2} = (y_1^{k+2} x_1^{k+1} + y_2^{k+2} x_2^{k+1})(y_1^{k+1} x_1^k + y_2^{k+1} x_2^k)\, e^{\mu_1^{k+1}} \alpha_1^k. \tag{13}$$

This expression is generalized to a relation between for kth and $(k+n)$th sets as follows:

$$\alpha_1^{k+n} = \left\{ \prod_{j=1}^{n} f(k+j-1, k+j) \right\} \left[\prod_{j=1}^{n-1} e^{\mu_1^{k+j}} \right\} \alpha_1^k, \tag{14}$$

where $F(k+1, k+2) \triangleq (y_1^{k+2} x_1^{k+1} + y_2^{k+2} x_2^{k+1})$. Since, for the system with disordered sets, an equivalent propagation constant, μ, is sought, the coefficient of α_1^k in equation (14) is expressed as

$$\alpha_1^{k+n} = \left\{ \prod_{j=1}^{n} f(k+j-1, k+j) \right\} \exp\left(\sum_{j=1}^{n-1} \mu_1^{k+j} \right) \alpha_1^k. \tag{15}$$

Consequently, expressing the coefficient of α_1^k in equation (15) in terms of an average exponential decay rate term, i.e., $\exp(n\mu_e)$, and solving it for μ_e result in an expression for μ_e result in the expression

$$\mu_e = \frac{1}{n} \left(\sum_{j=1}^{n} \log\left(\frac{\beta_1^j - \beta_2^{j+1}}{\beta_1^{j+1} - \beta_2^{j+1}} \right) + \sum_{j=1}^{n-1} \mu_1^{j+1} \right), \tag{16}$$

where β_1^k and β_2^k stem from the eigenvectors of the kth bi-periodic set:

$$[v_1^k \vdots v_2^k] = \begin{bmatrix} \beta_1^k & \beta_2^k \\ 1 & 1 \end{bmatrix}.$$

The terms in equation (16) consist of two terms; a correction term casted by the first sum and the sum of propagation constants for each bi-periodic set. It is worth to note that if the bi-periodic sets are identical, the correction terms in equation (16) vanish, and μ_e will be identical to the propagation contact of the corresponding wave mode. Since no requirement on the nature of disorder in the periodic structure is specified, equation (16) holds for both systems with deterministic and stochastic disorders.

The eigenvalues and the inversion of the eigenvectors matrix for a bi-periodic set are calculated to determine the quantities in equation (11). From equation (1), it can be shown that the algebraic forms of the eigenvalues of the transfer matrix of the kth bi-periodic set are

$$e^{\mu_1^k} = \lambda_1^k = \frac{1}{x_1^k}\left(x_2^k - \sqrt{x_3^k}\right), \qquad e^{\mu_2^k} = \lambda_2^k = \frac{1}{x_1^k}\left(x_2^k + \sqrt{x_3^k}\right), \tag{17}$$

where

$$x_1^k = 2E_A E_B g_A g_B (\cosh(g_B H_B) \sinh(g_A H_A) E_A g_A + \cosh(g_A H_A) \times \sinh(g_B H_B) E_B g_B),$$

$$x_2^k = \sinh(g_A H_A) \sinh(g_B H_B) E_A^2 g_A^2 + 2\cosh(g_A H_A)\cosh(g_B H_B) E_A E_B g_A g_B + \sinh(g_A H_A) \sinh(g_B H_B) E_B^2 g_B^2,$$

$$x_3^k = -(2E_A E_B g_A g_B)^2 + (\sinh(g_B H_B) E_A^2 g_A^2 + 2\cosh(g_A H_A) \times \cosh(g_B H_B) E_A E_B g_A g_B + \sinh(g_A H_A) \sinh(g_B H_B) E_B^2 g_B^2)^2.$$

TABLE 1

The elasticity, layer thicknesses, and non-dimensional coupling coefficients of the layered structures SystemI and SystemII

System I		System II	
$\tau = 13{\cdot}67$ $v = 0{\cdot}09$		$\tau = 3{\cdot}07$ $v = 0{\cdot}20$	
Layer A	*Layer B*	*Layer A*	*Layer B*
$E_A = 310{\cdot}3$ GPa	$E_B = 0{\cdot}69$ GPa	$E_A = 310{\cdot}3$ GPa	$E_B = 1{\cdot}38$ GPa
$\lambda_A = 124{\cdot}1$ GPa	$\lambda_B = 5{\cdot}59$ GPa	$\lambda_A = 124{\cdot}1$ GPa	$\lambda_B = 11{\cdot}18$ GPa
$\mu_A = 124{\cdot}1$ GPa	$\mu_B = 0{\cdot}24$ GPa	$\mu_A = 124{\cdot}1$ GPa	$\mu_B = 0{\cdot}47$ GPa
$H_A = 12{\cdot}7 \times 10^{-3}$ m	$H_B = 0{\cdot}25 \times 10^{-3}$ m	$H_A = 12{\cdot}7 \times 10^{-3}$ m	$H_B = 0{\cdot}25 \times 10^{-3}$ m
$\rho_A = 3248{\cdot}8$ kg/m^3	$\rho_B = 1068{\cdot}6$ kg/m^3	$P_A = 3248{\cdot}8$ kg/m^3	$\rho_B = 10686{\cdot}9$ kg/m^3
$v_A = 0{\cdot}25$	$v_B = 0{\cdot}48$	$v_A = 0{\cdot}25$	$v_B = 0{\cdot}48$
$c_{L_A} = 10705{\cdot}2$ m/s	$c_{L_B} = 2380{\cdot}5$ m/s	$c_{L_A} = 10705{\cdot}2$ m/s	$c_{L_B} = 1064{\cdot}6$ m/s
$c_{T_A} = 6180{\cdot}6$ m/s	$c_{T_B} = 446{\cdot}86$ m/s	$c_{T_A} = 6180{\cdot}6$ m/s	$c_{T_B} = 208{\cdot}8$ m/s

The corresponding eigenvectors are found in the form of

$$v_1^k = \begin{Bmatrix} \dfrac{1}{y_1^k}(y_2^k + \sqrt{y_3^k}) \\ 1 \end{Bmatrix}, \qquad v_2^k = \begin{Bmatrix} \dfrac{1}{y_1^k}(y_2^k - \sqrt{y_3^k}) \\ 1 \end{Bmatrix}, \tag{18}$$

where

$$y_1^k = 2E_A E_B g_A g_B (\cosh(g_B H_B)\sinh(g_A H_A) E_A g_A + \cosh(g_A H_A) \times \sinh(g_B H_B) E_B g_B),$$

$$y_2^k = -\sinh(g_A H_A)\sinh(g_B H_B) E_A^2 g_A^2 + \sinh(g_A H_A)\sinh(g_B H_B) E_B^2 g_B^2,$$

$$y_3^k = -(2E_A E_B g_A g_B)^2 + (\sinh(g_A H_A)\sinh(g_B H_B) E_A^2 g_A^2 + 2\cosh(g_A H_A)\cosh(g_B H_B) E_A E_B g_A g_B \sinh(g_A H_A)\sinh(g_B H_B) E_B^2 g_B^2)^2.$$

The calculation of the localization constant for a deterministic system involves evaluation of the sum in equation (16) using x_1^k, x_2^k, x_3^k, y_1^k, y_2^k, and y_3^k. For stochastic systems, ensemble averages are calculated after transformations of random variables by recognizing that the terms with the superscripts j and $j+1$ in the first sum of equation (16) are two independent random variables with identical statistical properties.

3. COMPUTATIONAL RESULTS

The effect of random variation in the transfer matrix of a bi-periodic set on the attenuation behavior of disordered one-dimensional structures is examined by calculating the mean localization factor and its variance. In the calculations, two layered systems with different elasticity properties, referred to as *SystemI* and *SystemII* in the following, are considered for four different levels of disorder. The elastic coupling between the layers (*LayerA* and *LayerB* in a bi-periodic set) of *SystemI* is stronger than that of *SystemII*. As given in Table 1, the non-dimensional coupling parameters for these two perfect systems are $v = 0{\cdot}09$ and $\tau = 13{\cdot}67$ for *SystemI* and $v = 0{\cdot}20$ and $\tau = 3{\cdot}07$ for *SystemII*. Consequently,

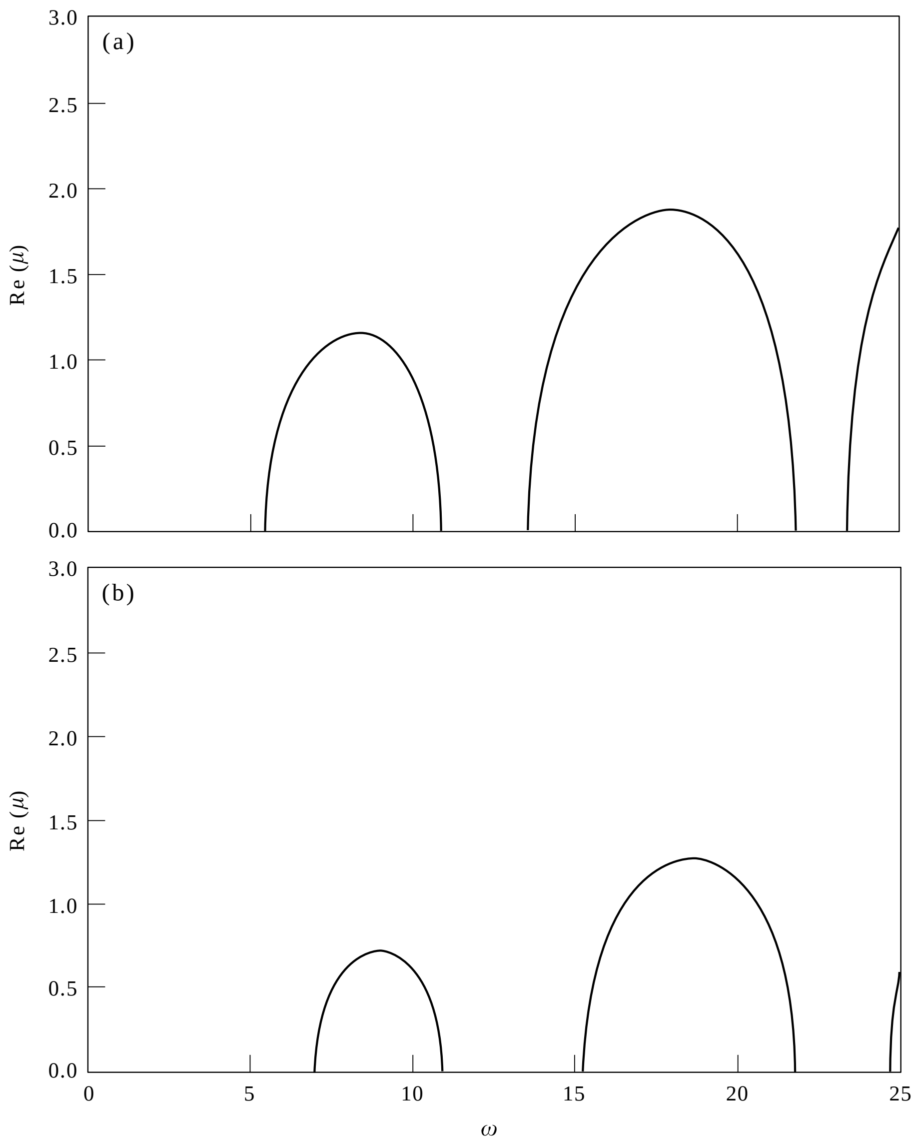

Figure 2. The real part of the propagation constant with respect to the frequency for (a) *SystemI* and (b) *SystemII* without disorder. The no-zero real part defines the attenuation zones for the systems under investigation.

TABLE 2

The variation intervals and the values of the standard deviation of the random variable $v(n)$ used in simulations

Simulation No.	Interval	σ^2
1	$(-0{\cdot}10, +0{\cdot}10)$	0·0033
2	$(-0{\cdot}20, +0{\cdot}20)$	0·0133
3	$(-0{\cdot}30, +0{\cdot}30)$	0·0300
4	$(-0{\cdot}40, +0{\cdot}40)$	0·0533

the propagation zones of *SystemI* are larger than those of *SystemII* [Figure 2(a, b)]. These parameters are calculated using the expressions

$$\tau = \frac{(\rho c_L)_A}{(\rho c_L)_B}, \qquad v = \frac{(H/c_L)_B}{(H/c_L)_A},$$

where τ is the rtio of mechanical impedances of the layers, and v is the ratio of travel times of the pressure waves in *LayerB* and *LayerA*. The effect of these non-dimensional parameters, τ and v, on the overall dynamics of a periodic layered periodic structure is examined in reference [18] in detail. The structure and the sizes of propagation and attenuation zones of a bi-periodic system can be determined by the values of these two parameters. In a perfectly periodic system, the real part of the propagation constant determines the attenuation rate of the corresponding propagating wave component. At this point, it is noteworthy that the one-dimensional problem under investigation possesses no complex modes. In Figure 2, the real part of the propagation constant as a function of the scaled frequency is plotted for the two systems without disorder. For a frequency inside a propagation zone, the value of the propagation constant, μ, is purely imaginary, thus in Figure 2 the propagation zones are where the real part of μ is zero. In an attenuation zone, the real part of μ is a non-zero real number, and its value at a certain frequency is the exponential attenuation rate for the wave component travelling at that frequency. As stated above, the imaginary part of μ is zero due to the lack of complex modes in the one-dimensional problem. *SystemI* has considerably smaller propagation zones than *SystemII* does; the first propagation zone of *SystemII* is approximately 28% larger than that of *SystemI*.

In the numerical evaluation of the mean and variance of the localization factor for the disordered systems, it is assumed that the source of disorder is the variance of Young's modulus of *LayerA*, and it varies bi-periodic set to bi-periodic set. This variation is modelled by a random variable with uniform probability density function, or $E_A = E_A^\circ(1 + v(n))$, where $v(n)$ is the random variable. The random variable $v(n)$ has zero mean, and its standard deviation is denoted by σ. The formulation given in equation (16) can be used for any type of probability density functions, and the uniform density function is chosen here for its simplicity. The four levels of disorder for the two systems are considered. Table 2 shows the values of four variances and corresponding variation intervals of the random number $v(n)$.

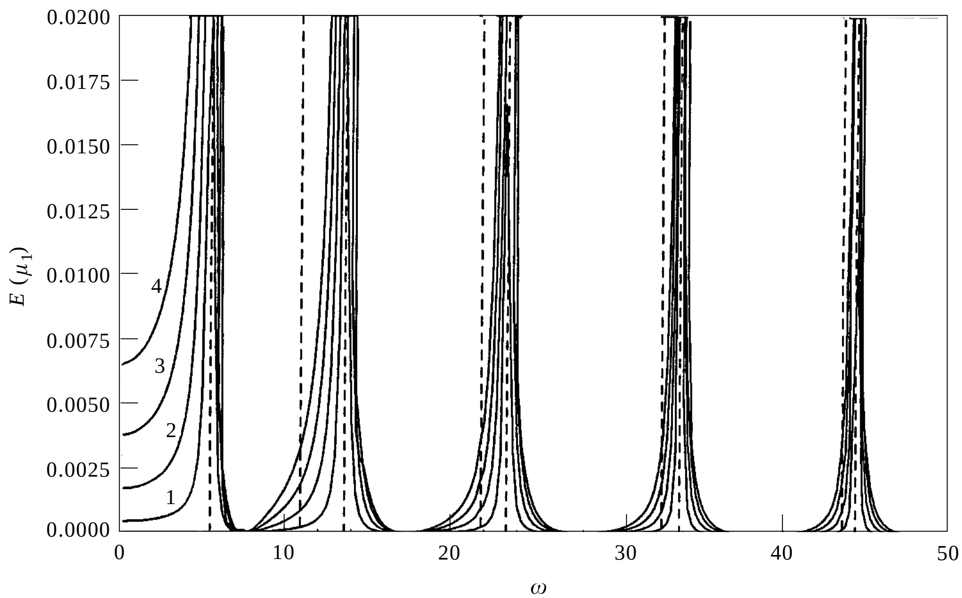

Figure 3. The mean localization factor coming from the first term of equation (16) is given for the disordered structure *SystemI* with the four levels of disorder. The disorders are in the Young's modulus of *LayerA*. The dashed lines are the boundary between attenuation and propagation zones.

A close examination of Figures 9, 10, 11, and 12 reveals that the second term in equation (16) plays no significant role in shifting the boundaries of the attenuation and propagation zones of the layered structures under consideration.

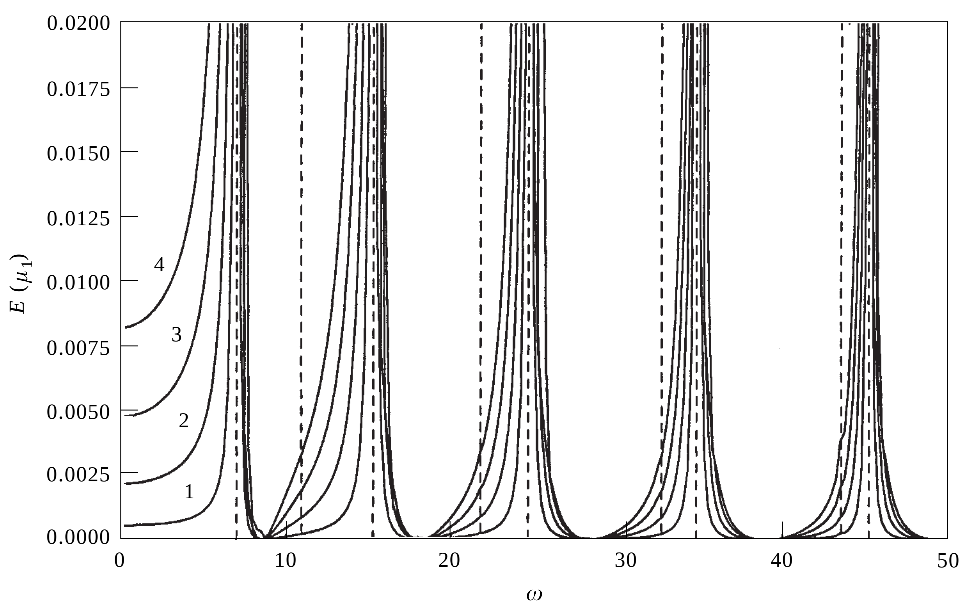

Figure 4. The mean localization factor coming from the first term of equation (16) is given for the disordered structure *SystemII* with the four levels of disorder. The disorders are in the Young's modulus of *LayerA*. The dashed lines are the boundary between attenuation and propagation zones.

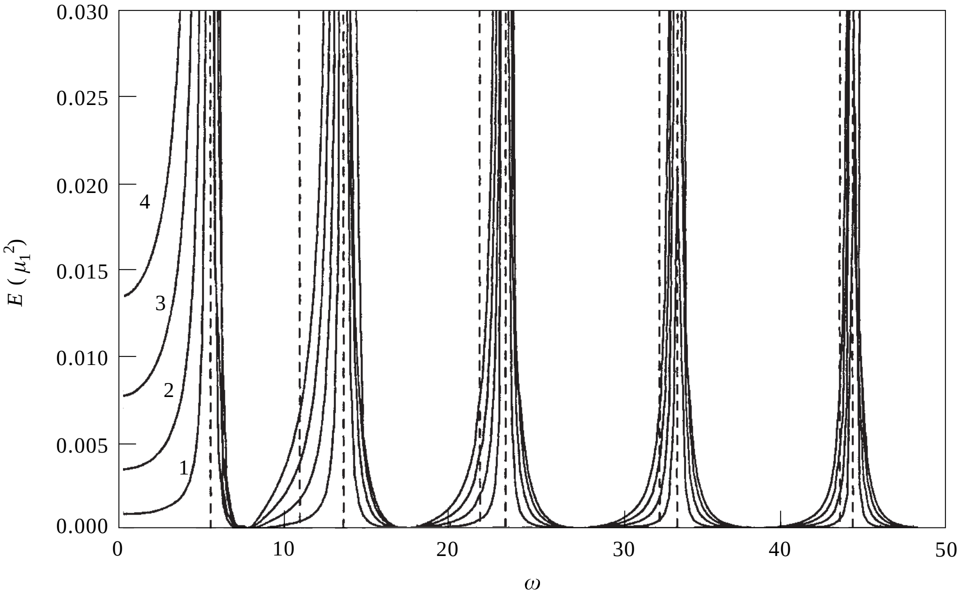

Figure 5. The variance of the first term in equation (16) for *SystemI* with four levels of disorder; $b = 0{\cdot}1$, $b = 0{\cdot}2$, $b = 0{\cdot}3$, and $b = 0{\cdot}4$.

It only changes the wave propagation behavior in attenuation zones and the imaginary part of μ in propagation zones. Therefore, the localization effect inside the propagation zones of the systems with disorder stems from the first term in equation (16) as in Figures 3, 4, 5, and 6.

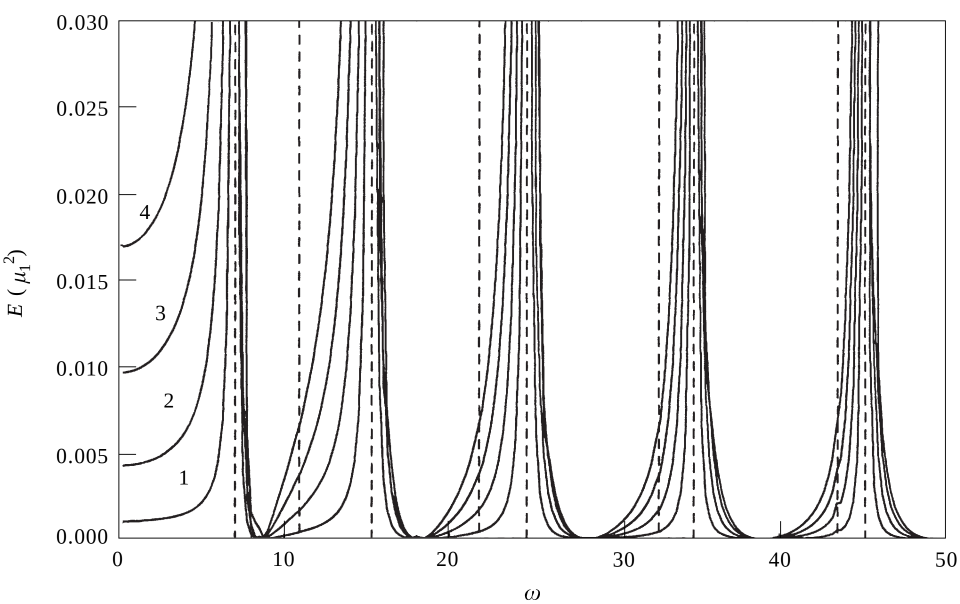

Figure 6. The variance of the first term in equation (16) for *SystemII* with four levels of disorder; $b = 0{\cdot}1$, $b = 0{\cdot}2$, $b = 0{\cdot}3$, and $b = 0{\cdot}4$.

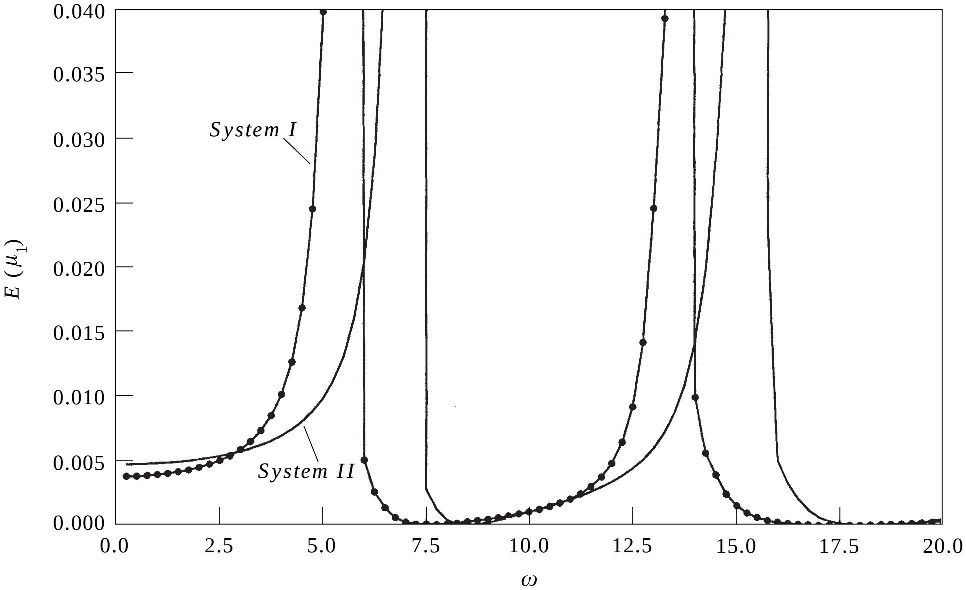

Figure 7. The mean of the first term in equation (16) for *SystemI* and *SystemII* with disorder ($b = 0{\cdot}2$) for the first two propagation zones.

Calculations for the mean localization factor based on the formulation given in the first term of equation (16) are performed to study the effect of this probabilistic disorder on the structures of propagation and attenuation. In Figure 3, the real part of the propagation constant, also referred to as the localization factor, is given

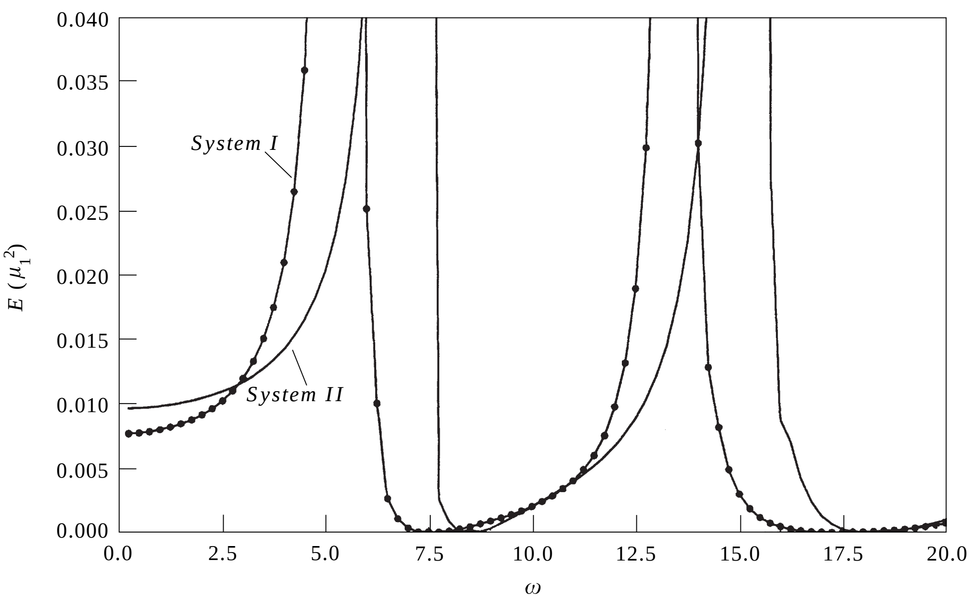

Figure 8. The variance of the first term in equation (16) for *SystemI* and *SystemII* with disorder ($b = 0{\cdot}2$) for the first two propagation zones.

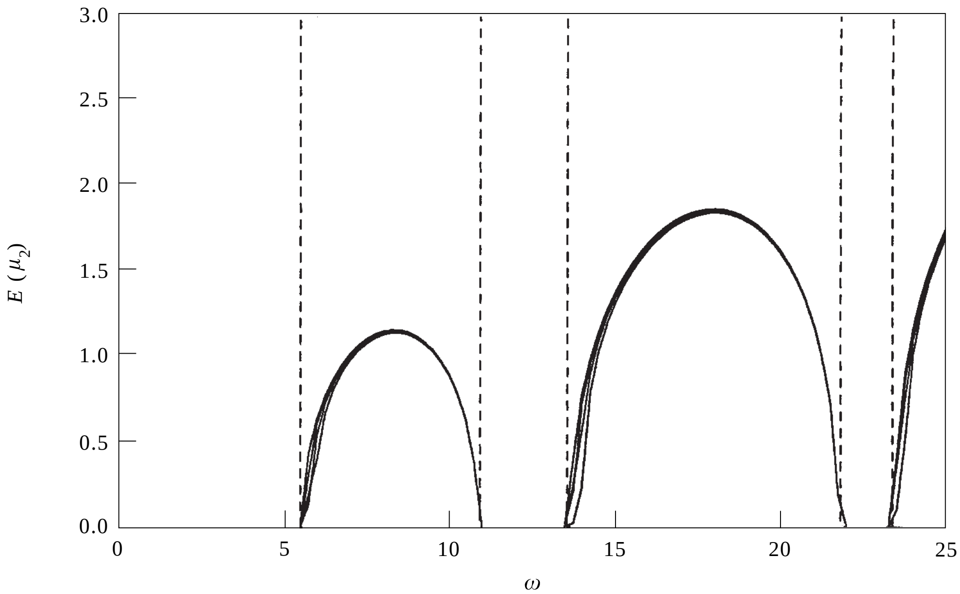

Figure 9. The mean localization factor coming from the second term of equation (16) is given for the disordered structure *SystemI* with the four levels of disorder.

for the disordered structure *SystemI* with the four levels of disorder. In Figure 4, the mean localization factor for *SystemII* with disorder is given.

From Figures 3 and 4 it is apparent how the attenuation zones of a perfect system expand into adjacent propagation zones due to disorder. While the existence of disorder affects the structures of all propagation zones, its significance

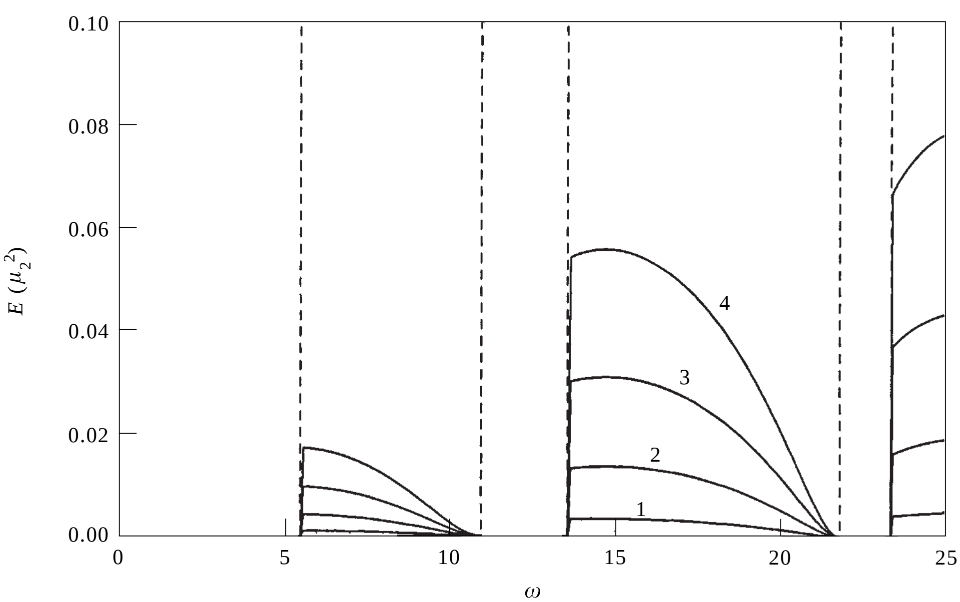

Figure 10. The variance of localization factor coming from the second term of equation (16) is given for the disordered structure *SystemI* with the four levels of disorder.

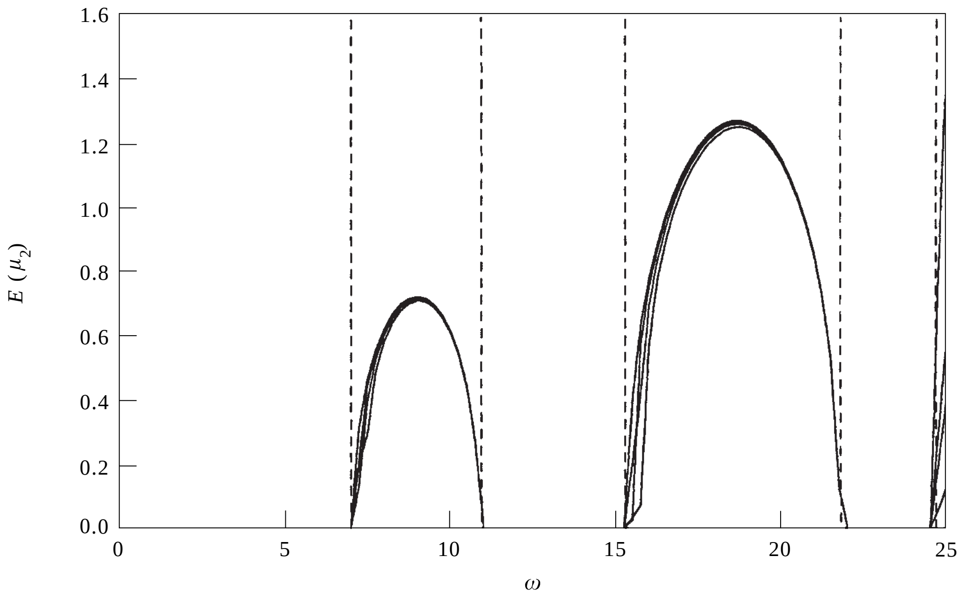

Figure 11. The mean localization factor coming from the second term of equation (16) is given for the disordered structure *SystemII* with the four levels of disorder.

is most visible in the first propagation zones of both systems. At $w = 0$, the mean localization factor of *SystemI* for $b = 0{\cdot}4$ is approximately 25% larger than that of *SystemII*. For the smaller values of b, the difference becomes less profound. In Figure 7, the mean Lyapunov exponents for *SystemI* and *SystemII* are compared for $b = 0{\cdot}2$. Considering that the wave components with smaller mean localization factor travels deeper into the structure, this observation highlights the relevance of weak coupling in stress wave propagation in disordered one-dimensional layered structures. However, as the frequency increases, not only the significance of the disorder level decreases, but also the effect of elastic coupling becomes less visible. Therefore, the wave components corresponding to higher propagation zones penetrate deeper into a structure and their effect is more pronounced in the far field. Another notable observation is that the right boundary of the propagation zones are the mean localization factor asymptotes. The behavior of the mean Lyapunov curves near the right propagation zone boundaries explains how the disorder level governs the expansion of attenuation zones into propagation zones. This expansion occurs from the right to left in a propagation zone as the level of disorder is increased, and it is strongest in the first propagation zone. Consequently, the corresponding wave component travelling at a frequency in the first propagation frequency interval has shorter localization distance. The structure of the first propagation zone in stress wave propagation is of a particular interest, since the power density of motion induced by an external force is maximum in the first propagation zone frequency interval.

In Figures 5 and 6, results of the variance of the localization factors are presented. The structures of these curves are quite similar to those for the mean localization factor. At $w = 0$, the variance of localization factor for *SystemI* for $b = 0{\cdot}4$ is

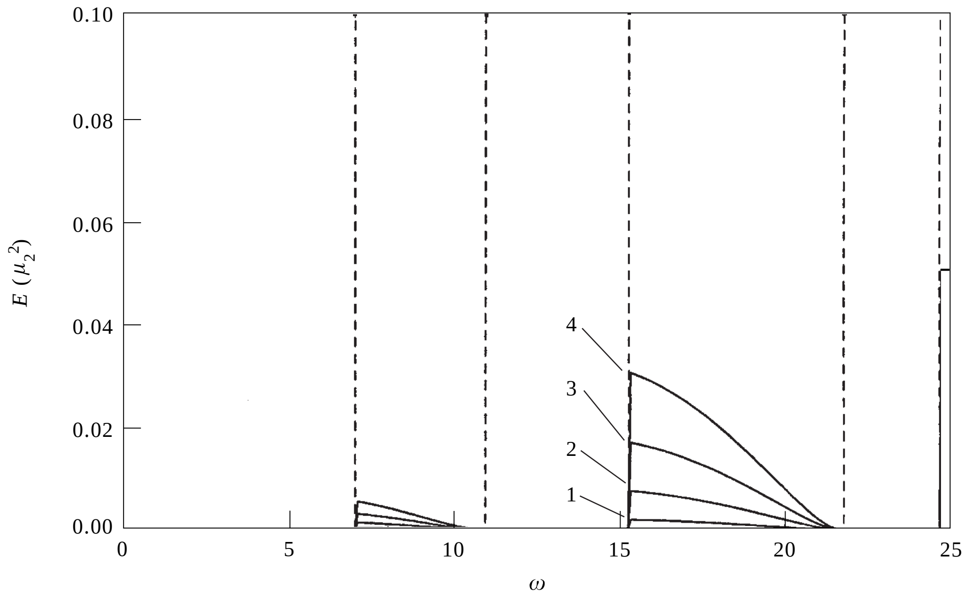

Figure 12. The variance of localization factor coming from the second term of equation (16) is given for the disordered structure *SystemII* with the four levels of disorder.

approximately 35% larger than that of *SystemII*. For the smaller values of b, the difference becomes less profound. In Figure 8, the variance of localization factors coming from the first term in equation (16) for *SystemI* and *SystemII* is compared for $b = 0{\cdot}2$. The asymptotes forming at the right boundaries between the propagation and attenuation zones indicate large variations in localization distances for the wave components at a frequency in the first propagation frequency interval. As for the mean localization factor, the propagation zones are destroyed from right to left and this type of degeneration is most visible in the first propagation zone.

4. CONCLUSIONS AND REMARKS

A formulation to study the effect of random variation in the transfer matrix of a bi-periodic set on the attenuation behavior of disordered one-dimensional structures is developed. In the calculations for the mean localization factor and its variance, two layered systems with different elasticity properties are considered for four different levels of disorder.

The mean and variance of localization factor for the disordered systems are numerically evaluated by assuming that the source of disorder is the variance of Young's modulus of *LayerA*, and it varies bi-periodic set to bi-periodic set. This variation is modelled by a random variable with uniform probability density function. The formulation developed in this work can be used with any type of probability density functions, and the uniform density function is chosen here for its simplicity. Moreover, the formulation can be used with deterministically disordered systems. In the numerical evaluations, four levels of stochastic disorder for the two systems are considered.

Utilizing the results presented in this investigation, it is explained how the attenuation zones of a perfect system expand into adjacent propagation zones due to disorder. It is found that while the existence of disorder affects the structures of all propagation zones, its significance is most visible in the first propagation zones of both systems. For instance, at $w = 0$, the mean localization factor of *SystemI* for the disorder level $b = 0{\cdot}4$ is approximately 25% larger than that of *SystemII*. For the smaller values of b, the difference becomes less profound. Considering that the wave components with smaller mean localization factor propagate deeper into the layered structure, this observation highlights the relevance of weak coupling in stress wave propagation in disordered one-dimensional layered structures. However, as the frequency increases, the significance of the disorder level and the effect of elastic coupling becomes less visible. Therefore, the wave components corresponding to higher propagation zones penetrate deeper into a structure and their effect is more visible in the far field.

It is observed that the right boundary of the propagation zones are the mean localization factor asymptotes. The behavior of the mean Lyapunov curves near the right propagation zone boundaries explains how the disorder level governs the expansion of attenuation zones into propagation zones. This expansion occurs from right to left in a propagation zone as the level of disorder is increased, and it is strongest in the first propagation zone. Consequently, the localization distance for the corresponding wave component travelling at a frequency in the first propagation frequency interval is considerably shorter. Since the power density of motion induced by an external force is maximum in the first propagation zone frequency interval, the structure and size of first propagation zone in stress wave propagation are of a particular interest in transient analysis.

The variance of the localization factors is also studied. The structures of the variance curves are quite similar to those for the mean localization factor. For the smaller values of b, the difference becomes less profound. The asymptotes forming at the right boundaries between the propagation and attenuation zones indicate large variations in localization distances for the wave components at a frequency in the first propagation frequency interval. It is observed that, like the mean localization factor, the propagation zones are destroyed from right to left, and this degeneration is most visible in the first propagation zone.

REFERENCES

1. L. Brillouin 1946 *Wave Propagation in Periodic Structures*. New York: Dover.
2. D. J. Mead 1996 *Journal of Sound and Vibration* **190**(3), 495–524. Wave propagation in continuous periodic structures: research contributions from Southampton, 1964–1995.
3. D. J. Mead 1975 *Journal of Sound and Vibration* **40**(1), 1–18. Wave propagation and natural modes in periodical systems: i-mono-coupled systems.
4. D. J. Mead 1975 *Journal of Sound and Vibration* **40**(1), 19–39. Wave propagation and natural modes in periodical systems: II-multi-coupled systems with and without damping.

5. M. G. Faulkner and D. P. Hong 1985 *Journal of Sound and Vibration* **99**(1), 29–42. Free vibrations of a mono-coupled periodic system.
6. R. C. Engels and L. Meirovitch 1978 *Journal of Sound and Vibration* **56**(4), 481–493. Response of periodic structures by modal analysis.
7. G. Sengupta 1980 *The Shock and Vibration Digest* **12**(3), 17–29. Vibration of periodic structures.
8. P. W. Anderson 1958 *Physical Review A*: *Journal of Experimental and Theoretical Physics* **109**(5), 1492–1505. Absence of diffusion in certain random lattices.
9. C. H. Hodges 1982 *Journal of Sound and Vibration* **82**, 411–424. Confinement of vibration by structural irregularities.
10. D. J. Mead and A. S. Bansal 1978 *Journal of Sound and Vibration* **61**(41), 481–515. Mono-coupled periodic systems with a single disorder: free wave propagation—response to convected loadings.
11. D. J. Mead and S. M. Lee 1984 *Journal of Sound and Vibration* **92**(3), 427–445. Receptance methods and the dynamics of disordered one-dimensional lattices.
12. G. J. Kissel 1988 *Ph.D. Thesis*, *M.I.T.* Localization in disordered periodic structures.
13. G. Q. Cai and Y. K. Lin 1991 *AIAA Journal* **29**(3), 450–456. Localization of wave propagation in disordered periodic structures.
14. S. T. Ariaratnam and W. C. Xie 1991 in *Nonlinear Stochastic Mechanics* (editors N. Bellomo and F. Casciati), 13–24. Berlin: Springer. On the localization phenomenon in randomly disordered engineering structures.
15. S. T. Ariaratnam and W. C. Xie 1995 *Journal of Sound and Vibration* **181**, 7–22. Wave localization in randomly disordered nearly periodic long continuous beams.
16. G. J. Kissel 1991 *Physical Review A* **44**, 1008–1014. Localization factor for multichannel disordered systems.
17. M. P. Castanier and C. Pierre 1995 *Journal of Sound and Vibration* **183**, 493–515. Lyapunov exponents and localization phenomena in multicoupled nearly periodic systems.
18. A. F. Vakakis, M. El-Raheb and C. Cetinkaya 1994 *Journal of Sound and Vibration* **172**(1), 23–46. Free and forced dynamics of a class of periodic elastic systems.
19. C. Cetinkaya, A. F. Vakakis and M. El-Raheb 1995 *Journal of Sound and Vibration* **182**(2), 283–302. Axisymmetric elastic waves in weakly coupled layered media of infinite radial extent.