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A DYNAMIC MODEL FOR THICK ELASTIC PLATES

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In this paper the general equations of motion for thick elastic plates with arbitrary shape are derived by using a variational principle. In addition to the influences of the bending, the transverse shear deformation and the rotatory inertia, the proposed theory also contains the effects of the transverse normal stress and the membrane forces. The equations presented in this paper can be reduced to those deduced by E. Reissner and R. D. Mindlin. Some numerical results are compared with those obtained from the Reissner–Mindlin plate theory and the classical plate theory. It is found that for thick plates there exists a dense region of frequencies and the position of frequencies is shifted, so that the influence of the transverse normal stress must be considered.

1. INTRODUCTION

The classical plate theory is based on the Kirchhoff–Love assumptions [1]: (a) the plate is thin; (b) the deflections of the plate are small; (c) the normal stresses perpendicular to the middle surface can be neglected in comparison with the other stresses; (d) straight lines normal to the undeformed middle surface remain straight and normal to the deformed middle surface.

In view of these assumptions, the classical plate theory cannot be expected to hold for plates whose thickness is large with respect to the span. Neither can it be applied to describe the dynamic behavior of plates when the wave numbers are large. Therefore, in order to adequately describe the motion of plate-type structures, various improved theories of plates have been developed and established [2–12]. For example, the Reissner–Mindlin plate theory replaces the normal line assumption by the straight line assumption and contains the influences of the transverse shear deformation and the rotatory inertia. It is able to describe a wider range of phenomena than the classical plate theory which includes only bending effects. However, the Reissner–Mindlin plate theory is restricted to the analysis of only moderately thick plates because the transverse normal stress is neglected. In addition, the Reissner–Mindlin plate theory neglects the thickness change of the plate (as does the classical plate theory) which means that the transverse normal strain is also neglected. Such approximations can be removed by using either the three-dimensional theory or improved theories where the thickness change is also considered. In several papers (e.g., references [11, 12]) stability and vibration problems of laminated rectangular plates are analyzed by applying the finite element method to such a refined plate theory.

This paper presents a consistent plate theory which contains not only the influences of the bending, the transverse shear deformation and the rotatory inertia but also the effects of the transverse normal stress and the membrane forces. Thus, it has a wider range of application than the moderately thick plate theories. All in all, six independent variables are used. To point out the influence of the transverse normal stress, we introduce a fairly simple analytical model. It assumes constant normal stress and normal strain across the thickness co-ordinate. However, even this simple model is capable of showing the influence of the transverse normal stress on the eigenmodes of the plates. In special cases the solution of the problem to be investigated can be given in closed form.

It is noted that the use of six independent variables is not a new idea; see, e.g., reference [13]. A comprehensive review of this matter is beyond the scope of this paper. For more information please refer to reference [14].

2. KINEMATIC AND CONSTITUTIVE EQUATIONS

The salient features of the plate geometry are shown in Figures 1(a) and (b). The plate is referred to by a curvilinear orthogonal co-ordinate system x_1, x_2 and x_3, see Figure 1(a). The axes x_1 and x_2 lie in the middle surface of the plate, x_3 points into the direction of the normal to the middle surface, forming a right-hand co-ordinate system. The arc lengths are denoted by s_1 and s_2. One has the relations

$$\mathrm{d}s_1 = A_1\,\mathrm{d}x_1, \qquad \mathrm{d}s_2 = A_2\,\mathrm{d}x_2. \tag{1}$$

The quantities A_1 and A_2 are the coefficients of the first fundamental form of the middle surface of the plate. When the metric in tensorial notation is used these quantities are also referred to as g_{11}, g_{22} and g_{12} in the literature. Since we restrict ourselves to an orthogonal co-ordinate system, the coefficient $g_{12} = A_{12} = 0$. For example, $A_1 = A_2 = 1$ for rectangular plates and $A_1 = 1$, $A_2 = r$ for circular plates (r denotes the radius).

We introduce the dimensions of the plate with $x_1 \in [0, a]$, $x_2 \in [0, b]$, $x_3 \in [-H/2, H/2]$, where H is the thickness of the plate. The six faces ① to ⑥ are referred to by ①: $x_1 \equiv 0$, ②: $x_1 \equiv a$, ③: $x_2 \equiv 0$, ④: $x_2 \equiv b$, ⑤: $x_3 \equiv -H/2$, and finally ⑥: $x_3 \equiv H/2$. This notation implies that the respective co-ordinate is held at the given value while the other co-ordinates vary within their bounds. Figure 1(b) shows a rectangular plate. This plate is used later for the numerical analysis.

In considering the bending, the transverse shear deformation, the transverse normal strain and the membrane strains, the displacement vector can be taken as

$$\underline{u}^*(x_1, x_2, x_3, t) = \underline{u}(x_1, x_2, t) + x_3\underline{\varphi}(x_1, x_2, t), \tag{2}$$

with

$$\underline{u}^* = [u_1^*, u_2^*, u_3^*]^{\mathrm{T}}, \qquad \underline{u} = [u_1, u_2, u_3]^{\mathrm{T}}, \qquad \underline{\varphi} = [\varphi_1, \varphi_2, \varphi_3]^{\mathrm{T}}, \tag{3–5}$$

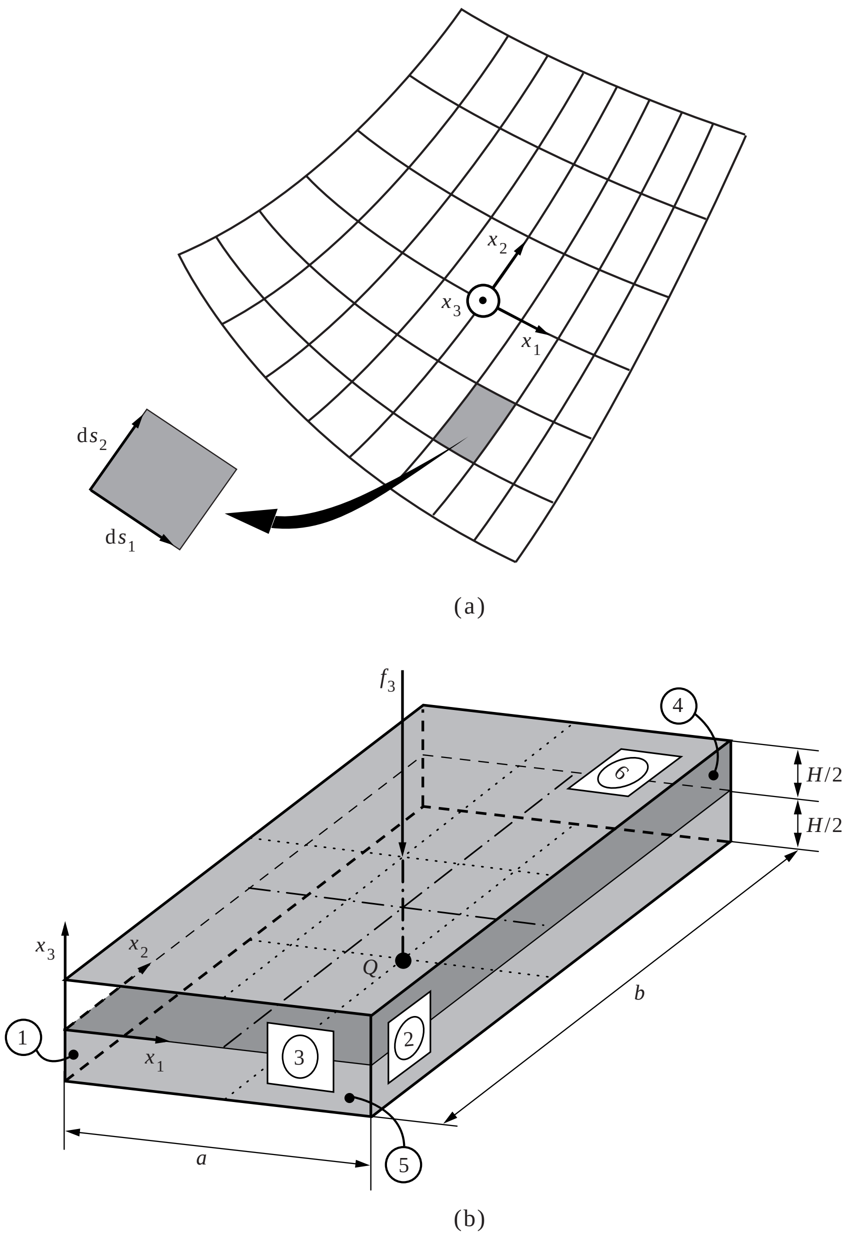

Figure 1. (a) Curvilinear co-ordinate system; (b) rectangular plate used in the numerical analysis.

where u_i ($i = 1, 2, 3$) are the displacement components of the middle surface of the plate, φ_1 and φ_2 are the angles of rotation of the transverse normal in the x_1–x_3 and x_2–x_3 planes, φ_3 is the transverse normal strain, t is the time and superscript T denotes transposition.

Using equation (2), the kinematic equations are written as

$$\underset{\sim}{e}^* = \begin{bmatrix} \underset{\sim}{E}_{c11} & \underset{\sim}{E}_{c12} \\ \underset{\sim}{E}_{c21} & \underset{\sim}{E}_{c22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{u} \\ \underset{\sim}{\varphi} \end{bmatrix}, \tag{6}$$

with

$$\underset{\sim}{E}_{c11} = \begin{bmatrix} \dfrac{1}{A_1}\dfrac{\partial}{\partial x_1} & \dfrac{1}{A_1 A_2}\dfrac{\partial A_1}{\partial x_2} & 0 \\ \dfrac{1}{A_1 A_2}\dfrac{\partial A_2}{\partial x_1} & \dfrac{1}{A_2}\dfrac{\partial}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{7}$$

$$\underset{\sim}{E}_{c12} = \begin{bmatrix} \dfrac{x_3}{A_1}\dfrac{\partial}{\partial x_1} & \dfrac{x_3}{A_1 A_2}\dfrac{\partial A_1}{\partial x_2} & 0 \\ \dfrac{x_3}{A_1 A_2}\dfrac{\partial A_2}{\partial x_1} & \dfrac{x_3}{A_2}\dfrac{\partial}{\partial x_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{8}$$

$$\underset{\sim}{E}_{c21} = \begin{bmatrix} \left(\dfrac{1}{A_2}\dfrac{\partial}{\partial x_2} - \dfrac{1}{A_1 A_2}\dfrac{\partial A_1}{\partial x_2}\right) & \left(\dfrac{1}{A_1}\dfrac{\partial}{\partial x_1} - \dfrac{1}{A_1 A_2}\dfrac{\partial A_2}{\partial x_1}\right) & 0 \\ 0 & 0 & \dfrac{1}{A_1}\dfrac{\partial}{\partial x_1} \\ 0 & 0 & \dfrac{1}{A_2}\dfrac{\partial}{\partial x_2} \end{bmatrix}, \tag{9}$$

$$\underset{\sim}{E}_{c22} = \begin{bmatrix} x_3\left(\dfrac{1}{A_2}\dfrac{\partial}{\partial x_2} - \dfrac{1}{A_1 A_2}\dfrac{\partial A_1}{\partial x_2}\right) & x_3\left(\dfrac{1}{A_1}\dfrac{\partial}{\partial x_1} - \dfrac{1}{A_1 A_2}\dfrac{\partial A_2}{\partial x_1}\right) & 0 \\ 1 & 0 & \dfrac{x_3}{A_1}\dfrac{\partial}{\partial x_1} \\ 0 & 1 & \dfrac{x_3}{A_2}\dfrac{\partial}{\partial x_2} \end{bmatrix}, \tag{10}$$

$$\underset{\sim}{e}^* = [\underset{\sim}{e}^*_{11}, \underset{\sim}{e}^*_{22}, \underset{\sim}{e}^*_{33}, \underset{\sim}{e}^*_{12}, \underset{\sim}{e}^*_{13}, \underset{\sim}{e}^*_{23}]^{\mathrm{T}}, \tag{11}$$

where $\underset{\sim}{e}^*$ is the generalized strain vector of the plate.

Considering the three-dimensional stress–strain relations and the kinematic equations, we obtain the constitutive equations

$$\underline{n}_a = \begin{bmatrix} \underline{N}_{1u} & \underline{N}_{1\varphi} \\ \underline{N}_{2u} & \underline{N}_{2\varphi} \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{\varphi} \end{bmatrix}, \qquad \underline{m}_a = \begin{bmatrix} \underline{M}_{1u} & \underline{M}_{1\varphi} \\ \underline{M}_{2u} & \underline{M}_{2\varphi} \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{\varphi} \end{bmatrix}. \tag{12, 13}$$

Here,

$$\underline{n}_a = [N_{11}, N_{22}, N_{33}, N_{12}, N_{21}, N_{13}, N_{23}]^{\mathrm{T}}, \tag{14}$$

$$\underline{m}_a = [M_{11}, M_{22}, M_{12}, M_{21}, M_{13}, M_{23}]^{\mathrm{T}} \tag{15}$$

are the stress and moment resultant vectors defined by

$$N_{ij} = \int_{-H/2}^{H/2} \sigma_{ij}^* \, \mathrm{d}x_3 \quad (i, j = 1, 2, 3), \tag{16}$$

$$M_{ij} = \int_{-H/2}^{H/2} x_3 \sigma_{ij}^* \, \mathrm{d}x_3 \quad (i = 1, 2 \quad \text{and} \quad j = 1, 2, 3), \tag{17}$$

and the generalized stress vector comprising the independent components of the stress tensor is introduced as

$$\underline{\sigma}^* = [\sigma_{11}^*, \sigma_{22}^*, \sigma_{33}^*, \sigma_{12}^*, \sigma_{13}^*, \sigma_{23}^*]^{\mathrm{T}}. \tag{18}$$

Finally, $\underline{N}_{iu}$, $\underline{N}_{i\varphi}$, $\underline{M}_{iu}$, $\underline{M}_{i\varphi}$, $(i = 1, 2)$ are differential operator matrices which contain the modulus of elasticity, E, Poisson's ratio ν and the transverse shear correction factor κ. Compared with the displacement field proposed by Kant Mallikarjuna [12], the displacement field defined in equation (2) neglects the higher order expansion terms in the thickness as well as in the in-plane co-ordinates. As a consequence, the transverse shear strain and stress fields violate the boundary conditions at the upper and the lower faces of the plate so that the shear correction factor κ is still needed to compensate the simplified stress field. Moreover, the transverse normal strain and stress are constant across the thickness of the plate. These quantities should be treated as the mean transverse normal strain and stress, respectively. The proposed model has a wider range of application than the moderately thick plate theories. However, it will still fail to model very thick plates correctly. In such cases a more elaborate displacement field—like the one proposed in reference [12]—should be used.

3. VARIATIONAL PRINCIPLE AND EQUATIONS OF MOTION

The strain–energy density is introduced by

$$W^* = \tfrac{1}{2} \underline{\sigma}^{*\mathrm{T}} \underline{e}^*. \tag{19}$$

The strain energy of the plate is obtained by integration over the whole volume of the plate which yields

$$W = \frac{1}{2}\int_0^a \int_0^b \int_{-H/2}^{H/2} \underline{\sigma}^{*\mathrm{T}} \underline{e}^* A_1 A_2 \, \mathrm{d}x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_1. \tag{20}$$

The kinetic energy density is given by

$$T^* = \frac{1}{2}\rho\left(\frac{\partial}{\partial t}\underline{u}^*\right)^{\mathrm{T}}\left(\frac{\partial}{\partial t}\underline{u}^*\right), \tag{21}$$

ρ being the material mass density. It is used to obtain the kinetic energy

$$T = \int_0^a \int_0^b \int_{-H/2}^{H/2} \frac{1}{2}\rho\left(\frac{\partial}{\partial t}\underline{u}^*\right)^{\mathrm{T}}\left(\frac{\partial}{\partial t}\underline{u}^*\right) A_1 A_2 \, \mathrm{d}x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_1, \tag{22}$$

whereby the influence of the rotatory inertia is considered.

Let f_i ($i = 1, 2, 3$) denote the x_i-component of the external forces per unit area acting at the surface of the plate. The work done by these forces is

$$W_e = \iint_S \underline{f}^{\mathrm{T}} \underline{u}^* \, \mathrm{d}S \tag{23}$$

with

$$\underline{f} = [f_1, f_2, f_3]^{\mathrm{T}}, \tag{24}$$

the integration being extended over the whole surface S of the plate.

The work done by the body force per unit volume

$$\underline{q} = [q_1, q_2, q_3]^{\mathrm{T}}, \tag{25}$$

where q_i ($i = 1, 2, 3$) denotes the x_i-component of the body force, is

$$W_b = \iiint_V \underline{q}^{\mathrm{T}} \underline{u}^* \, \mathrm{d}V, \tag{26}$$

the integration being extended over the whole volume V of the plate.

Hamilton's principle

$$\delta \int_{t_1}^{t_2} L \, \mathrm{d}t = 0, \tag{27}$$

where the Lagrange function is given by

$$L = T - [W - (W_e + W_b)], \tag{28}$$

can be applied to deduce the equations of motion as well as the appropriate boundary conditions. One obtains the general equations of motion for thick elastic plates with arbitrary shape

$$
\begin{aligned}
-J_0 \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial}{\partial x_1}(N_{1i}A_2) + \frac{\partial}{\partial x_2}(N_{2i}A_1) + \tilde{N}_i + (N_{3i}^* + \bar{N}_i)A_1A_2 = 0,\\
-J_2 \frac{\partial^2 \varphi_i}{\partial t^2} + \frac{\partial}{\partial x_1}(M_{1i}A_2) + \frac{\partial}{\partial x_2}(M_{2i}A_1) + \tilde{M}_i + (N_{i3} + M_{3i}^* + \bar{M}_i)A_1A_2 = 0,
\end{aligned} \tag{29}
$$

with the abbreviations

$$J_0 = \rho A_1 A_2 H, \qquad J_2 = \rho A_1 A_2 H^3/12, \tag{30}$$

$$[N_{1i}^*, M_{1i}^*] = \int_{-H/2}^{H/2} [1, x_3] f_i \, \mathrm{d}x_3 |_{①+②}, \qquad [N_{2i}^*, M_{2i}^*] = \int_{-H/2}^{H/2} [1, x_3] f_i \, \mathrm{d}x_3 |_{③+④}, \tag{31}$$

$$[N_{3i}^*, M_{3i}^*] = [1, x_3] f_i |_{⑤+⑥}, \tag{32}$$

$$[\bar{N}_i, \bar{M}_i] = \int_{-H/2}^{H/2} [1, x_3] q_i \, \mathrm{d}x_3 \quad (i = 1, 2, 3), \tag{33}$$

$$
\begin{gathered}
\tilde{N}_i = N_{ij} \frac{\partial A_i}{\partial x_j} - N_{jj} \frac{\partial A_j}{\partial x_i}, \qquad \tilde{M}_i = M_{ij} \frac{\partial A_i}{\partial x_j} - M_{jj} \frac{\partial A_j}{\partial x_i}\\
(i = 1, j = 2 \quad \text{and} \quad i = 2, j = 1),\\
\tilde{N}_3 = \tilde{M}_3 = 0.
\end{gathered} \tag{34}
$$

The notation ⓚ + ⓛ in the equations (31) and (32) should be simply read as . . . *evaluated at the faces k and l and added*.

With the generalized displacement vector

$$\underset{\sim}{v} = [\underset{\sim}{u}^{\mathrm{T}}, \underset{\sim}{\varphi}^{\mathrm{T}}] = [u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3]^{\mathrm{T}}, \tag{35}$$

the equations of motion, including damping, can be written compactly in the form

$$\underset{\sim}{M}\ddot{\underset{\sim}{v}} + \underset{\sim}{C}\dot{\underset{\sim}{v}} + \underset{\sim}{K}\underset{\sim}{v} = \underset{\sim}{p}. \tag{36}$$

Here, $\underset{\sim}{M}$, $\underset{\sim}{C}$, $\underset{\sim}{K}$ are the mass matrix, the damping matrix and the differential operator stiffness matrix, while dots denote differentiation with respect to the time t. To obtain equation (36), we made use of the Rayleigh assumption, namely $\underset{\sim}{C} = c_a \underset{\sim}{M} + c_b \underset{\sim}{K}$, c_a and c_b being constants. The generalized load vector $\underset{\sim}{p}$ acting at the middle surface is determined by the external forces acting at the surface of the plate and by the body force. It is expressed by

$$\underset{\sim}{p} = [p_1, p_2, p_3, p_4, p_5, p_6]^{\mathrm{T}}, \tag{37}$$

with

$$p_i = N_{3i}^* + \bar{N}_i \quad \text{and} \quad p_{i+3} = M_{3i}^* + \bar{M}_i \quad (i = 1, 2, 3). \tag{38}$$

From equations (32) and (38) it can be seen that the component f_3 of the external force contributes not only to the component p_3 of the generalized load vector but also to the component p_6 which causes the transverse normal stress and the transverse normal strain. Similarly, the components f_1 and f_2 contribute not only to the components p_1 and p_2 but also to the components p_4 and p_5 which cause the bending. Because there exist elastic coupling terms in $\underline{K}$, p_1 and p_2 cause not only the membrane stresses but also the transverse normal stress and/or the transverse normal strain.

It is evident that equation (36) can be reduced to the equation deduced by Reissner and Mindlin if the effects of the transverse normal stress and the membrane forces are neglected. The improved theory with six independent variables, the Reissner–Mindlin plate theory with three independent variables (u_3, φ_1 and φ_2) and the classical plate theory with one independent variable (u_3) are denoted in the following by IT6, MT3 and CT1, respectively.

The boundary conditions are

$$u_i = U_i \quad \text{or} \quad N_{ji} = N_{ji}^*, \qquad \varphi_i = \Phi_i \quad \text{or} \quad M_{ji} = M_{ji}^*$$

$$(i = 1, 2, 3) \text{ for the boundaries } x_j = \text{const.} \quad (j = 1, 2), \tag{39}$$

where U_i and Φ_i are given quantities. In addition, the initial displacements and velocities

$$\underline{v}(x_1, x_2, t)|_{t=0} = \underline{v}_0(x_1, x_2), \qquad \dot{\underline{v}}(x_1, x_2, t)|_{t=0} = \dot{\underline{v}}_0(x_1, x_2) \tag{40, 41}$$

with

$$\underline{v}_0 = [\underline{u}_0^{\mathrm{T}}, \underline{\varphi}_0^{\mathrm{T}}]^{\mathrm{T}}, \qquad \dot{\underline{v}}_0 = [\dot{\underline{u}}_0^{\mathrm{T}}, \dot{\underline{\varphi}}_0^{\mathrm{T}}]^{\mathrm{T}} \tag{42, 43}$$

must be specified.

4. ORTHOGONALITY CONDITIONS FOR THE NATURAL MODE FUNCTIONS

An expression for the generalized displacement vector in any mode of free vibrations may be written in the following form for arbitrary wave numbers m and n,

$$\underline{v}(x_1, x_2, t) = \sum_m \sum_n \underline{v}_{mn}(x_1, x_2) \sin \omega_{mn} t, \tag{44}$$

with

$$\underline{v}_{mn} = [U_{1mn}, U_{2mn}, U_{3mn}, \Phi_{1mn}, \Phi_{2mn}, \Phi_{3mn}]^{\mathrm{T}}. \tag{45}$$

Here U_{1mn} to Φ_{3mn} denote test functions in x_1 and x_2 for the respective degree of freedom (DOF). Substituting equation (44) into the free vibration equation without damping,

$$\underline{M}\ddot{\underline{v}} + \underline{K}\underline{v} = 0, \tag{46}$$

and using the homogeneous boundary conditions and the constitutive equations leads to the orthogonality conditions for the natural mode functions

$$I_{mnkl}\left\{\begin{array}{ll} =0 & \text{for } m \neq k \cup n \neq l \\ \neq 0 & \text{for } m = k \cap n = l \end{array}\right\}, \tag{47}$$

with

$$I_{mnkl} = \int_0^a \int_0^b \underline{v}_{mn}^{\mathrm{T}} \underline{M} \underline{v}_{kl} \, \mathrm{d}x_2 \, \mathrm{d}x_1. \tag{48}$$

It has to be kept in mind that for the six DOF system under consideration there exist in general six eigenfrequencies (branches) for every combination of the wave numbers m and n. Hence, equation (44) implies a summation over $p = 1 \ldots 6$ for every term with index mn; see section 6.

5. SOLUTIONS FOR FORCED VIBRATIONS

It is assumed that the mode functions $\underline{v}_{mn}$ form a complete set. Therefore, the generalized displacement vector, the generalized load vector as well as the initial displacement and velocity vectors may be expressed in the following form for any wave numbers m and n:

$$\underline{v}(x_1, x_2, t) = \sum_m \sum_n \underline{v}_{mn}(x_1, x_2) T_{mn}(t), \tag{49}$$

$$\underline{p}(x_1, x_2, t) = \sum_m \sum_n \underline{v}_{mn}(x_1, x_2) P_{mn}(t), \tag{50}$$

$$\underline{v}_0(x_1, x_2) = \sum_m \sum_n \underline{v}_{mn}(x_1, x_2) T_{mn}(0), \tag{51}$$

$$\dot{\underline{v}}_0(x_1, x_2) = \sum_m \sum_n \underline{v}_{mn}(x_1, x_2) \dot{T}_{mn}(0). \tag{52}$$

Considering the orthogonality conditions and using the Rayleigh damping assumption yields the equations for the generalized co-ordinates T_{mn} as

$$\ddot{T}_{mn}(t) + 2\zeta_{mn}\omega_{mn}\dot{T}_{mn}(t) + \omega_{mn}^2 T_{mn}(t) = P_{mn}(t), \tag{53}$$

where

$$\zeta_{mn} = \frac{c_a + c_b\omega_{mn}^2}{2\omega_{mn}}, \tag{54}$$

$$P_{mn}(t) = \frac{1}{I_{mnmn}} \int_0^a \int_0^b \underline{p}^{\mathrm{T}}(x_1, x_2, t) \underline{M} \underline{v}_{mn}(x_1, x_2) \, \mathrm{d}x_2 \, \mathrm{d}x_1, \tag{55}$$

$$\begin{bmatrix} T_{mn}(0) \\ \dot{T}_{mn}(0) \end{bmatrix} = \frac{1}{I_{mnmn}} \int_0^a \int_0^b \begin{bmatrix} \underline{v}_0^{\mathrm{T}}(x_1, x_2) \underline{M} \underline{v}_{mn}(x_1, x_2) \\ \dot{\underline{v}}_0^{\mathrm{T}}(x_1, x_2) \underline{M} \underline{v}_{mn}(x_1, x_2) \end{bmatrix} \mathrm{d}x_2 \, \mathrm{d}x_1, \tag{56}$$

while c_a and c_b denote the factors of the external and internal damping, respectively. The solution of equation (53) with the initial conditions (56) is

$$T_{mn}(t) = \mathrm{e}^{-\varepsilon_{mn} t} \left[T_{mn}(0) \cos \omega_{mn}^* t + \frac{\dot{T}_{mn}(0) + \varepsilon_{mn} T_{mn}(0)}{\omega_{mn}^*} \sin \omega_{mn}^* t \right]$$

$$+ \frac{1}{\omega_{mn}^*} \int_0^t \mathrm{e}^{-\varepsilon_{mn}(t-\tau)} P_{mn}(\tau) \sin \omega_{mn}^* (t - \tau) \, \mathrm{d}\tau \tag{57}$$

with

$$\varepsilon_{mn} = \zeta_{mn} \omega_{mn}, \qquad \omega_{mn}^* = \omega_{mn} \sqrt{1 - \zeta_{mn}^2}. \tag{58, 59}$$

Substituting equation (57) into equation (49) leads to the solution for forced vibrations with damping.

As a typical example we consider the concentrated load

$$\underline{p}(x_1, x_2, t) = P(t) \delta(x_1 - x_a, x_2 - x_b) [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{60}$$

where $\delta(x_1 - x_a, x_2 - x_b)$ is the two-dimensional Dirac function. The corresponding solution is

$$\underline{v}(x_1, x_2, t)$$

$$= \sum_m \sum_n \underline{v}_{mn}(x_1, x_2) \left\{ \mathrm{e}^{-\varepsilon_{mn} t} \left[T_{mn}(0) \cos \omega_{mn}^* t + \frac{\dot{T}_{mn}(0) + \varepsilon_{mn} T_{mn}(0)}{\omega_{mn}^*} \sin \omega_{mn}^* t \right] \right.$$

$$\left. + \frac{J_0 U_{3mn}(x_a, x_b)}{I_{mnmn} \omega_{mn}^*} \int_0^t \mathrm{e}^{-\varepsilon_{mn}(t-\tau)} P(\tau) \sin \omega_{mn}^* (t - \tau) \, \mathrm{d}\tau \right\}. \tag{61}$$

6. NUMERICAL RESULTS

Numerical computations were carried out for the undamped free and forced vibration analysis of rectangular plates, using the improved theory with six independent variables (IT6). These results are compared with those obtained from the Reissner–Mindlin plate theory (MT3) and the classical plate theory (CT1). The boundary conditions are given by

$$N_{11} = u_2 = u_3 = M_{11} = \varphi_2 = \varphi_3 = 0 \quad \text{(for the faces ①, ②)},$$

$$N_{22} = u_1 = u_3 = M_{22} = \varphi_1 = \varphi_3 = 0 \quad \text{(for the faces (③, ④)}.$$

TABLE 1

Non-dimensional frequency parameters and non-dimensional characteristic vectors; $a/b = 1,\ H/a = 0{\cdot}01$

Ω_{111}	Ω_{112}	Ω_{113}	Ω_{114}	Ω_{115}	Ω_{116}
0·9997	46·13	74·38	3262	3263	6079
$\underline{ev}^*_{111}$	$\underline{ev}^*_{112}$	$\underline{ev}^*_{113}$	$\underline{ev}^*_{114}$	$\underline{ev}^*_{115}$	$\underline{ev}^*_{116}$
0	**−0·7071**	**0·7071**	0	0	−1·57E-03
0	**0·7071**	**0·7070**	0	0	−1·57E-03
0·9998	0	0	ε	7·40E-03	0
−1·57E-02	0	0	**−0·7071**	**0·7070**	0
−1·57E-02	0	0	**0·7071**	**0·7070**	0
0	ε	6·67E-03	0	0	**1 − ε**
Ω_{121}	Ω_{122}	Ω_{123}	Ω_{124}	Ω_{125}	Ω_{126}
2·498	72·93	117·6	3263	3265	6080
$\underline{ev}^*_{121}$	$\underline{ev}^*_{122}$	$\underline{ev}^*_{123}$	$\underline{ev}^*_{124}$	$\underline{ev}^*_{125}$	$\underline{ev}^*_{126}$
0	**0·8944**	4·47E-01	0	0	−1·57E-03
0	−4·47E-01	**0·8944**	0	0	−3·14E-03
0·9994	0	0	ε	1·17E-02	0
−1·57E-02	0	0	**0·8944**	4·47E-01	0
−3·14E-02	0	0	−4·47E-01	**0·8944**	0
0	ε	1·05E-02	0	0	**1 − ε**
Ω_{221}	Ω_{222}	Ω_{223}	Ω_{224}	Ω_{225}	Ω_{226}
3·994	92·25	148·8	3263	3267	6080
$\underline{ev}^*_{221}$	$\underline{ev}^*_{222}$	$\underline{ev}^*_{223}$	$\underline{ev}^*_{224}$	$\underline{ev}^*_{225}$	$\underline{ev}^*_{226}$
0	**−0·7071**	**0·7071**	0	0	−3·14E-03
0	**0·7071**	**0·7070**	0	0	−3·14E-03
0·9990	0	0	ε	1·48E-02	0
−3·13E-02	0	0	**−0·7071**	**0·7070**	0
−3·13E-02	0	0	**0·7071**	**0·7070**	0
0	ε	1·33E-02	0	0	**1 − ε**

It should be emphasized, however, that different types of plates (e.g., circular plates) and different boundary conditions can just as well be chosen.

The following data are given: modulus of elasticity $E = 2{\cdot}06 \times 10^5$ N/mm^2; Poisson's ratio $\nu = 0{\cdot}3$; shear correction factor $\kappa = \pi^2/12$; length to width ratio $a/b = 1{\cdot}0, 2{\cdot}0$; thickness to length ratio $H/a = 0{\cdot}01, 0{\cdot}1, 0{\cdot}2, 0{\cdot}3$.

The following symbols are used: fundamental frequency of the plate ω_0 (classical theory); wave numbers m and n in x_1 and x_2-direction; branch number p of the frequency (six branches for IT6, three branches for MT3 and one branch for CT1); non-dimensional frequency parameter $\Omega_{mnp} = \omega_{mnp}/\omega_0$; characteristic vector $\underline{ev}_{mnp} = [A_{mnp}, B_{mnp}, C_{mnp}, D_{mnp}, E_{mnp}, F_{mnp}]^{\mathrm{T}}$ for IT6, where A_{mnp} to F_{mnp} correspond to the independent kinematic quantities u_1, u_2, u_3, $\varphi_1 H/2$, $\varphi_2 H/2$, $\varphi_3 H/2$; non-dimensional characteristic vector $\underline{ev}^*_{mnp} = \underline{ev}_{mnp}/|\underline{ev}_{mnp}|$.

First we consider free vibrations. Table 1 shows examples of the non-dimensional frequency parameters and the non-dimensional characteristic vectors for a thin square plate based on IT6 for the wave numbers

TABLE 2

As Table 1 but for $a/b = 2$ and $H/a = 0{\cdot}2$

Ω_{181}	Ω_{182}	Ω_{183}	Ω_{184}	Ω_{185}	Ω_{186}
9·269	10·46	10·95	11·02	17·83	18·09
$\underline{ev}^{*}_{181}$	$\underline{ev}^{*}_{182}$	$\underline{ev}^{*}_{183}$	$\underline{ev}^{*}_{184}$	$\underline{ev}^{*}_{185}$	$\underline{ev}^{*}_{186}$
0	**0·9981**	0	6·05E-03	5·99E-02	0
0	−6·24E-02	0	9·68E-02	**0·9580**	0
0·9754	0	ε	0	0	7·51E-02
−1·37E-02	0	**0·9923**	0	0	6·22E-02
−2·20E-01	0	−6·24E-02	0	0	**0·9952**
0	ε	0	**0·9953**	−2·81E-01	0
Ω_{281}	Ω_{282}	Ω_{283}	Ω_{284}	Ω_{285}	Ω_{286}
9·325	10·52	11·01	11·07	17·93	18·19
$\underline{ev}^{*}_{281}$	$\underline{ev}^{*}_{282}$	$\underline{ev}^{*}_{283}$	$\underline{ev}^{*}_{284}$	$\underline{ev}^{*}_{285}$	$\underline{ev}^{*}_{286}$
0	**0·9923**	0	1·19E-02	1·19E-01	0
0	−1·24E-01	0	9·55E-02	**0·9530**	0
0·9757	0	ε	0	0	7·47E-02
−2·72E-02	0	**0·9923**	0	0	1·24E-01
−2·18E-01	0	−1·24E-01	0	0	**0·9895**
0	ε	0	**0·9954**	−2·79E-01	0
Ω_{381}	Ω_{382}	Ω_{383}	Ω_{384}	Ω_{385}	Ω_{386}
9·419	10·62	11·11	11·15	18·1	18·35
$\underline{ev}^{*}_{381}$	$\underline{ev}^{*}_{382}$	$\underline{ev}^{*}_{383}$	$\underline{ev}^{*}_{384}$	$\underline{ev}^{*}_{385}$	$\underline{ev}^{*}_{386}$
0	**0·9829**	0	1·75E-02	1·77E-01	0
0	−1·84E-01	0	9·34E-02	**0·9449**	0
0·9761	0	ε	0	0	7·41E-02
−4·01E-02	0	**0·9829**	0	0	1·84E-01
−2·14E-01	0	−1·84E-01	0	0	**0·9802**
0	ε	0	**0·9955**	−2·75E-01	0

$[m, n] = [1, 1], [1, 2], [2, 2]$. Numbers of an absolute value less than 1×10^{-4} are replaced by ε for clarity. None of the characteristic vectors contains more than three elements. Elements with an absolute value of more than 0·5 which predominate this mode are shown in bold. From the characteristic vectors it is apparent that IT6 may be divided into two separate mechanical submodels.† The first submodel represents the Reissner–Mindlin plate theory (MT3), to which the third, fourth and fifth position in every characteristic vector for the branches $p = 1, 4, 5$ correspond. The second one contains the effects of the transverse normal stress and the membrane forces, to which the first, second and sixth position in every characteristic vector for the branches $p = 2, 3, 6$ correspond. The frequencies for the branches $p = 4$, 5 and 6 correspond to the angles of rotation of the transverse normal and to the transverse normal strain. They are very high and almost independent of the wave numbers m and n (the largest frequency

† In fact this can be shown by inspection of the equation of motion (36). For the system under consideration the submodels are neither statically nor dynamically coupled. However, coupling occurs through external loading; see below.

TABLE 3

As Table 1 but for $a/b = 2$ and $H/a = 0{\cdot}3$

Ω_{181}	Ω_{182}	Ω_{183}	Ω_{184}	Ω_{185}	Ω_{186}
6·257	6·804	6·972	7·121	11·83	11·91
$\underline{ev}^*_{181}$	$\underline{ev}^*_{182}$	$\underline{ev}^*_{183}$	$\underline{ev}^*_{184}$	$\underline{ev}^*_{185}$	$\underline{ev}^*_{186}$
0	3·71E-03	**0·9981**	0	6·14E-02	0
0	5·94E-02	−6·24E-01	0	**0·9825**	0
0·9881	0	0	ε	0	5·19E-02
−9·61E-03	0	0	**0·9981**	0	6·23E-02
−1·54E-01	0	0	−6·24E-02	0	**0·9967**
0	**0·9982**	ε	0	−1·76E-01	0
Ω_{281}	Ω_{282}	Ω_{283}	Ω_{284}	Ω_{285}	Ω_{286}
6·294	6·838	7·012	7·161	11·89	11·98
$\underline{ev}^*_{281}$	$\underline{ev}^*_{282}$	$\underline{ev}^*_{283}$	$\underline{ev}^*_{284}$	$\underline{ev}^*_{285}$	$\underline{ev}^*_{286}$
0	7·33E-03	**0·9923**	0	1·22E-01	0
0	5·86E-02	−1·24E-01	0	**0·9770**	0
0·9982	0	0	ε	0	5·16E-02
−1·90E-02	0	0	**0·9923**	0	1·24E-01
−1·52E-01	0	0	−1·24E-01	0	**0·9910**
0	**0·9983**	ε	0	−1·75E-01	0
Ω_{381}	Ω_{382}	Ω_{383}	Ω_{384}	Ω_{385}	Ω_{386}
6·356	6·895	7·08	7·226	12·01	12·09
$\underline{ev}^*_{381}$	$\underline{ev}^*_{382}$	$\underline{ev}^*_{383}$	$\underline{ev}^*_{384}$	$\underline{ev}^*_{385}$	$\underline{ev}^*_{386}$
0	1·08E-02	**0·9829**	0	1·82E-01	0
0	5·74E-02	−1·84E-01	0	**0·9681**	0
0·9884	0	0	ε	0	5·12E-02
−2·80E-02	0	0	**0·9829**	0	1·84E-01
−1·49E-01	0	0	−1·84E-01	0	**0·9816**
0	**0·9983**	ε	0	−1·73E-01	0

parameter is more than 6000 times the smallest one in the case $m = n = 1$). This leads to the well known conclusion that the effects of the transverse shear deformation, the rotatory inertia and the transverse normal stress are negligible for thin plates.

Similar results are obtained for thin rectangular plates and small values of the wave numbers m and n. If, however, the wave numbers and/or the ratio of thickness to span are increased, two interesting phenomena arise (see Tables 2 and 3). First, the ratio of the largest eigenfrequency to the smallest one becomes much smaller (in Tables 2 and 3, the largest frequency parameter is not even twice the smallest one). Hence, dense regions of frequencies arise in these cases. Second, the positions of the frequencies are shifted. For example, the frequency for the transverse normal strain (branch 6 in Table 1) is shifted to branch 4 (Table 2) or even to branch 2 (Table 3), respectively. We conclude that the transverse normal strain, negligible for thin plates, becomes increasingly important for thick plates.

Now we turn to forced vibrations. Figures 2–5 show the non-dimensional dynamic deflection $2\tilde{u}/\tilde{u}_{CT1}$ at the point Q of a rectangular plate subjected to the

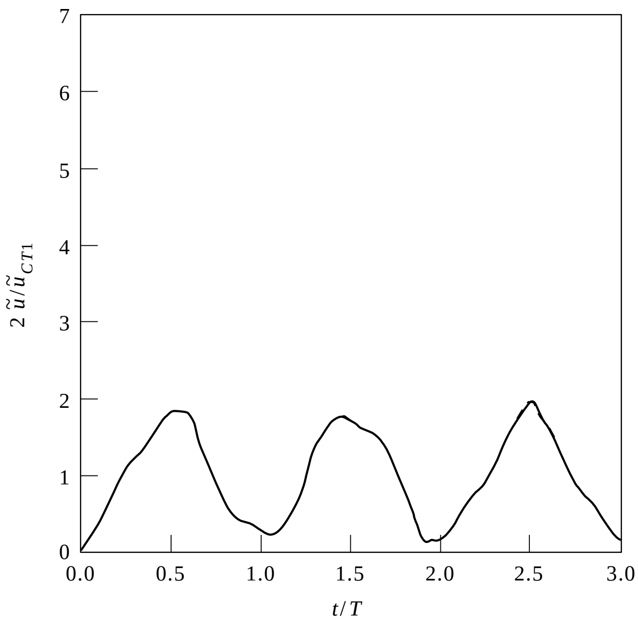

Figure 2. Normalized deflection versus time; $a/b = 2$, $H/a = 0{\cdot}01$. – –, CT1; · – · –, MT3; ——, IT6.

concentrated load

$$f_i(x_1, x_2, x_3, t) = 0 \quad (i = 1, 2),$$

$$f_3(x_1, x_2, x_3, t) = \begin{cases} -P_0 \mathrm{H}(t)\delta(x_1 - a/2, x_2 - b/2) & \text{for } x_3 = H/2 \\ 0 & \text{for } x_3 = -H/2 \end{cases}$$

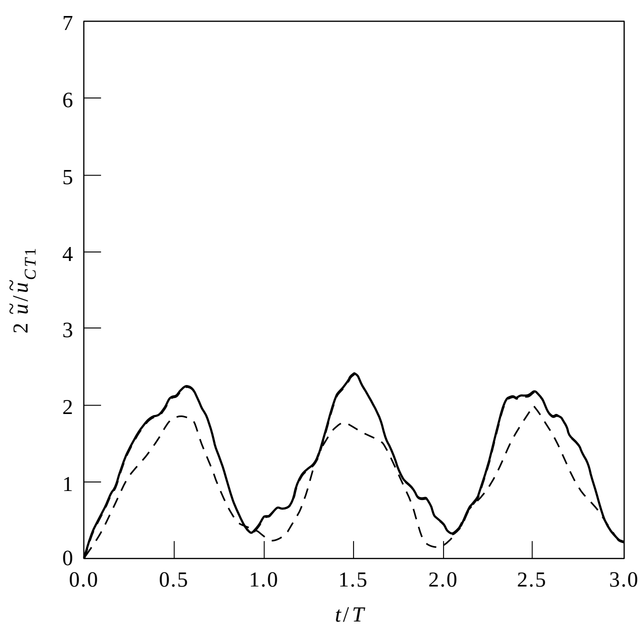

Figure 3. Normalized deflection versus time; $a/b = 2$, $H/a = 0{\cdot}1$. Key as Figure 2.

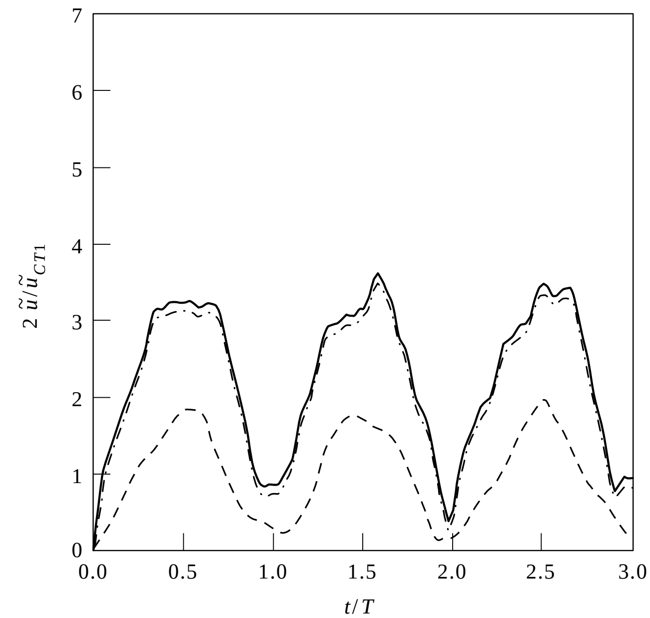

Figure 4. Normalized deflection versus time; $a/b = 2$, $H/a = 0{\cdot}2$. Key as Figure 2.

(see Figure 1(b)). In the equation above H(t) denotes the Heaviside function. Since all the results were scaled by one half of the *dynamic* deflection of the plates using CT1, $0{\cdot}5\tilde{u}_{CT1}$ (to achieve a peak value of 2 in order to reflect the dynamic load factor DLF) the absolute magnitude of P_0 is of no particular importance. The abscissa was scaled by T, the fundamental period of the plate.

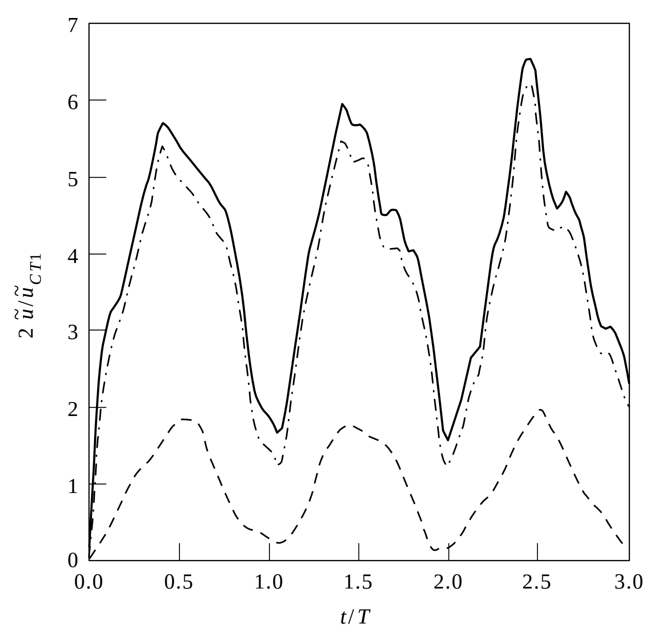

Figure 5. Normalized deflection versus time; $a/b = 2$, $H/a = 0{\cdot}3$. Key as Figure 2.

Figure 2 shows—as expected—that the three theories are equivalent for a thin plate. There is practically no difference between the three curves. This does not hold any more as the ratio of thickness to span H/a increases; see Figures 3–5. The most apparent change is the increase in the magnitude of the deflection. The higher order theories MT3 and IT6 describe a softer system in general. Therefore, the period of the vibration also increases. In addition, it can be seen that the peak deflections using the various theories do not coincide in time because there are now different modes which interchange energy. The increasing difference between the graphs for MT3 and IT6 is due to the influence of the change of thickness which MT3 neglects. It is apparent that for thick plates the influence of the transverse normal stress has to be accounted for.

7. CONCLUSIONS

In this paper a consistent plate theory with six independent variables is used, augmenting previously published theories [14]. The generalized displacement vector $\underline{v} = [u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3]^{\mathrm{T}}$ contains the displacement components of the middle surface of the plate, the angles of rotation of the transverse normal in the x_1–x_3 and x_2–x_3 planes as well as the transverse normal strain.

It is shown that the improved theory may be divided into two separate mechanical submodels: the first submodel represents the Reissner–Mindlin plate theory and the second one contains the effects of the transverse normal stress and the membrane forces. Because the thickness of the plate is taken into consideration these submodels are coupled by an external load. Inside each submodel the state variables influence each other by elastic coupling effects.

If an oblique external force is applied, each of its two components (perpendicular and tangential to the surface) induces both submodels to respond. In contrast, the classical plate theory and the moderately thick plate theory neglect the thickness of the plate and assume that the external force acts directly on the middle surface of the plate. As a result, each of the two components of an olique external force induces only one of the two submodels to respond and has no influence on the response of the other.

Numerical computations using the improved theory show that with the increases of the wave numbers as well as the ratio of thickness to span, dense regions of frequencies arise and the positions of the frequencies are shifted. For a thick rectangular plate and moderately large wave numbers, for example, the mode dominated by the transverse normal strain has the second lowest eigenfrequency while for a thin square plate, this mode possesses the highest eigenfrequency, three orders of magnitude larger than the lowest eigenfrequency for the combination of wave numbers considered.

Because the classical plate theory and the moderately thick plate theory neglect the influence of the transverse normal stress, the lower frequencies are lost in this case. The contributions of these frequencies on the dynamic response are also lost. Thus, a large error may be caused. Therefore, not only the influences of the transverse shear deformation and the rotatory inertia, but also the effects of the transverse normal stress must be included.

Consequently, the dynamic model for thick elastic plates presented in this paper has a wider range of application than the moderately thick plate theory. In addition, it provides estimates for the validity of simpler theories as well as being sometimes numerically more efficient than a FEM formulation.

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