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# INVERSE FINITE ELEMENT VIBRATION PROBLEMS

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The paper concerns the reconstruction of a consistent FEM model of an in-line system of 2-dof elements, fixed at one end and free at the other. Such a system has tridiagonal stiffness and mass matrices, **K**, **M**. Because each element has one rigid body mode, **K** has negative codiagonal and is constrained to have a particular form. **M** has positive codiagonal. It is shown how to construct (an infinite family of) such models so that each has a specified undamped frequency response at the free end, and how to construct a system with a damper at the free end so that the system has specified (complex) eigenvalues.



## 1. INTRODUCTION

The term *inverse vibration problem* is used to denote a class of problems in which it is required to construct a vibrating system from specified vibratory behaviour. There are different kinds of inverse vibration problems depending on the type of system which is being sought, the vibratory behaviour which is being modelled, and the way the problem is being viewed: as an engineering problem with incomplete and inaccurate data, or as a mathematical problem with complete and accurate data.

The engineering types of inverse vibration problems are often called *finite element model updating problems*, for a review of which see Mottershead and Friswell [1] or Friswell and Mottershead [2]. The essence of these problems is that there is a finite element method (FEM) model of a vibrating system, its predictions do not match some experimental behavioural data, and it is required to update the model to improve the match between prediction and experimental data.

This paper is concerned with mathematical inverse vibration problems, specifically problems for FEM models, undamped or damped.

For an undamped FEM model the time-reduced equation governing the natural frequencies ($\omega$) and principal modes (*eigenvectors*) of free vibration is

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{u} = \mathbf{0}, \qquad \lambda = \omega^2. \tag{1}$$

We call $\lambda$ an *eigenvalue*. For a general conservative FEM system, all that can be said about **K**, **M** is that they are symmetric, the stiffness matrix **K** is positive

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semi-definite (ps-d) (positive definite (p-d) if the system is anchored) and the mass matrix $\mathbf{M}$ is p-d. *Direct* problems relating to equation (1), i.e., given $\mathbf{K}$, $\mathbf{M}$ find the eigenvalues $\lambda$ and eigenvectors $\mathbf{u}$, are well understood; see for example references [3–5]. *Inverse* problems relating to equation (1) still pose many questions. There are two principal sources of difficulty. On the one hand, one *spectrum* $(\lambda_i)_1^n$ of eigenvalues of equation (1) is insufficient to construct two $n \times n$ matrices $\mathbf{K}$, $\mathbf{M}$; more information is needed, but what? On the other hand $\mathbf{K}$, $\mathbf{M}$ cannot be sought as arbitrary symmetric p-d (or ps-d) matrices, but must have the particular forms appropriate to the system being studied.

In fact, all the inverse eigenvalue problems that have been solved for equation (1) are variants of the one solved by Gantmakher and Krein [6]. The model that they considered was not a FEM model, but a taut spring with $n$ attached point masses which was vibrating transversely. However, this system is mathematically analogous to the simplest FEM model: $n$ elements each with two degrees of freedom, one at each end of an element, linked end to end, *and with the mass lumped at the nodes*. (e.g., a model of a thin rod in longitudinal or torsional vibration). If the system is fixed at the left end and free at the right then the generic form of the stiffness matrix is [7],

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & & \\ -k_2 & k_2 + k_3 & -k_3 & \\ \cdots & \cdots & \cdots & \cdots \\ & & -k_n & k_n \end{bmatrix}. \tag{2}$$

We will call such a matrix a $\mathbf{K}$-*matrix*. The mass matrix is diagonal, i.e.,

$$\mathbf{M} = \operatorname{diag}(m_1, m_2, \ldots, m_n). \tag{3}$$

Gantmakher and Krein showed that the $2n$ parameters $(k_r, m_r)_1^n$ specifying the system could be reconstructed uniquely from the following data: the eigenvalues $(\lambda_i)_1^n$ of equation (1); the eigenvalues $(\mu_i)_1^{n-1}$ of equation (1) subject to the condition $u_n = 0$; these are the eigenvalues of the system *fixed* at the right-hand end; a scaling factor, e.g., the total mass $m = \sum_{r=1}^{n} m_r$.

The two spectra $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$ always *interlace*, i.e.,

$$0 < \lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n. \tag{4}$$

The operations

$$\mathbf{M} = \mathbf{D} \cdot \mathbf{D}, \qquad \mathbf{C} = \mathbf{D}^{-1}\mathbf{K}\mathbf{D}^{-1}, \qquad \mathbf{x} = \mathbf{D}\mathbf{u}, \tag{5}$$

where $\mathbf{D} = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, reduce equation (1) to the standard form

$$(\mathbf{C} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{6}$$

The matrix $\mathbf{C}$ has the form

$$\mathbf{C} = \begin{bmatrix} a_1 & -b_1 & & \\ -b_1 & a_2 & -b_2 & \\ \cdots & \cdots & \cdots & \cdots \\ & & -b_{n-1} & a_n \end{bmatrix}. \tag{7}$$

It is symmetric p-d, tridiagonal, with negative *codiagonal*. We will call such a matrix an *NJ-matrix* meaning *negative* (sign) Jacobi-matrix. (In linear algebra, such a matrix is called an M-matrix, but this is confusing here because $\mathbf{M}$ is used for mass matrix). We will call a symmetric p-d tridiagonal matrix with *positive* codiagonal a *J-matrix*.

It has long been known (see section 2) that for the matrix $\mathbf{C}$, the data $(\lambda_i)_1^n$, $(\mu_i)_1^{n-1}$ is equivalent to $(\lambda_i, x_n^{(i)})_1^n$, normalized so that $\mathbf{x}^{(i)T}\mathbf{x}^{(i)} = 1$. The (well-conditioned) procedure for reconstructing $\mathbf{C}$ (uniquely) from $(\lambda_i,\ x_n^{(i)})_1^n$ is called the *Lanczos* process; see references [8, 9].

Once we have found the NJ-matrix $\mathbf{C}$ we must find that K-matrix $\mathbf{K}$. A K-matrix is characterized by the property

$$\mathbf{K}\{1, 1, \ldots 1\} = \{k_1, 0, 0 \ldots 0\}. \tag{8}$$

(This symbolizes the fact that a static load $k_1$ at node 1 (next to the fixed end) will shift all the nodes $1, 2, \ldots n$ to the right by one unit). But $\mathbf{C} = \mathbf{D}^{-1}\mathbf{K}\mathbf{D}^{-1}$ means that $\mathbf{K} = \mathbf{DCD}$ and therefore

$$\mathbf{DCD}\{1, 1 \ldots 1\} = \mathbf{DC}\{d_1, d_2, \ldots d_n\} = \{k_1, 0, \ldots 0\}. \tag{9}$$

Thus, one must find $d_1, d_2, \ldots d_n$ so that

$$\mathbf{C}\{d_1, d_2, \ldots d_n\} = \{k_1/d_1, 0, \ldots 0\}. \tag{10}$$

This one can do by solving

$$\mathbf{C}\{d_1, d_2, \ldots d_n\} = \{1, 0, \ldots 0\}. \tag{11}$$

and then putting $k_1 = d_1$. It is known [10] that when $\mathbf{C}$ is an NJ-matrix, the $d_i$ obtained by solving equation (11) are positive. This procedure for finding a K-matrix $\mathbf{K}$ from an NJ-matrix $\mathbf{C}$ will be called the *stiffness transformation*. It is possible to generalize this transformation to produce stiffness matrices having a more general form than equation (2), for instance stiffness matrices that correspond to additional springs attached to the nodes of the FEM model.

Over the years, many variants of Gantmakher and Krein's problem have been solved. The spectrum $(\mu_i)_1^{n-1}$ for the fixed end has bee replaced by the spectrum $(\lambda_i^*)_1^n$ for the system obtained by adding an extra mass at the free end, or additionally attaching the system to an anchor by a spring [12]. Instead of fixing the end, one may fix an interior node [13]. Further references may be found in Gladwell [4].

To this point all the systems we have considered have had a *diagonal* mass matrix. However, if the mass matrix is derived *consistently*, in the FEM sense, then the generic form of the mass matrix for an in-line system with $n$ degrees of freedom, one at each node, is J-matrix [7], i.e.,

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{11} & & \\ m_{21} & m_{22} & m_{23} & \\ \dots & \dots & \dots & \dots \\ & & m_{n,n-1} & m_{nn} \end{bmatrix}, \quad m_{ij} = m_{ji} > 0. \tag{12}$$

For such systems, the governing equation is thus equation (1), where $\mathbf{K}$ is an NJ-matrix (specifically a K-matrix), $\mathbf{M}$ a J-matrix. We studied inverse problems for such a system were studied in Gladwell [15]. We showed that the spectrum is again always simple, i.e.,

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \tag{13}$$

and that *if* one has one system which has the given spectrum $(\lambda_i)_1^n$, *then* one can construct various infinite families of systems with the same spectrum, so-called isospectral systems. However, we did not show how (because we did not know how!) to find *one*, starting, system. This we will now do, in section 3; moreover, we will show how to construct $\mathbf{K}$, $\mathbf{M}$, of the required form so that equation (1) has two spectra $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$ satisfying equation (4). The key to the solution of these problems is the recognition that in the reconstruction of $\mathbf{K}$ and $\mathbf{M}$ from one, or even two, spectra, there are many more parameters to be found than there are data values available: a K-matrix has $n$ parameters, an M-matrix $2n-1$. However, if we (appropriately) restrict our search, then we can find a solution, in fact an infinite family of solutions.

Until now we have been considering undamped systems. In two little-known papers, Vesilic [16, 17] considered the problem of constructing $\mathbf{M}$, $\mathbf{B}$, $\mathbf{K}$ such that the system

$$(\mathbf{M}\lambda^2 + \mathbf{B}\lambda + \mathbf{K})\mathbf{u} = \mathbf{0} \tag{14}$$

had specified eigenvalues. $\mathbf{M}$, $\mathbf{B}$, $\mathbf{K}$ were taken to be real and symmetric, so that the eigenvalues were either real or appeared as complex conjugate pairs. He took $\mathbf{K}$, $\mathbf{M}$ to be (effectively) of the form (2), (3), and supposed $\mathbf{B}$ to be a rank-one matrix $\mathbf{B} = \mathbf{b}\mathbf{b}^T$, where $\mathbf{b}$ is a column vector. It is noteworthy that he solved the problem by severely restricting the form of $\mathbf{B}$, to be of rank one, rather than as a general symmetric ps-d matrix. His analysis is adapted to find a J-matrix $\mathbf{M}$, a K-matrix $\mathbf{K}$ and a rank-one matrix $\mathbf{B}$ so that equation (14) has specified eigenvalues.

Ram and Elhay [18] have solved a different inverse problem for equation (14). They took $\mathbf{M} = \mathbf{I}$ and sought symmetric tridiagonal $\mathbf{B}$ and $\mathbf{K}$ so that equation (14) had specified complex eigenvalues, $(\lambda_i)_i^{2n}$, and the problem constrained so that $u_n = 0$ had specified complex eigenvalues $(\mu_i)^{2n-2}$. They were not able to use Vesilic's process, but developed an elegant numerical algorithm to solve this difficult non-linear problem. Their problem is beset with an unanswered question: what are the conditions on the two complex spectra which ensure that the solution **B**, **K** will be physically realistic, i.e., that **B**, **K** will be ps-d, and that **K** (and perhaps **B** also) will have negative codiagonal?

## 2. SOME BACKGROUND ANALYSIS

Consider an undamped FEM system with K-matrix **K** and J-matrix **M**. The governing equation is equation (1). If $\mathbf{e}_n = \{0, 0, \ldots, 0, 1\}$, then the response **u** to a unit load at the free end, $u_n$, is given by

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{u} = \mathbf{e}_n. \tag{15}$$

Suppose that the eigenvalues and eigenvectors of equation (1) are $(\lambda_i, \mathbf{u}^{(i)})_1^n$. The eigenvalues satisfy equation (13). Normalize the eigenvectors so that

$$\mathbf{u}^{(i)}\mathbf{M}\mathbf{u}^{(j)} = \delta_{ij}, \qquad \mathbf{u}^{(i)T}\mathbf{K}\mathbf{u}^{(j)} = \lambda_i\delta_{ij}. \tag{16}$$

Expand the solution **u** of equation (15) in the form

$$\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{u}^{(i)}. \tag{17}$$

Then

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{u} = \sum_{i=1}^{n} (\lambda_i - \lambda)\alpha_i \mathbf{M}\mathbf{u}^{(i)} = \mathbf{e}_n. \tag{18}$$

Multiplying by $\mathbf{u}^{(i)T}$ and using equation (16), one finds

$$(\lambda_j - \lambda)\alpha_j = \mathbf{u}^{(j)T}\mathbf{e}_n = u_n^{(j)}. \tag{19}$$

When substituted into the last of equation (17), this gives

$$u_n = \sum_{i=1}^{n} \frac{[u_n^{(i)}]^2}{\lambda_i - \lambda} \equiv F(\lambda). \tag{20}$$

Thus, the $\lambda_i$ are the poles of the *response function* $F(\lambda)$; the zeros of $F(\lambda)$ are the value of $\lambda$ for which a load at $u_n$ yields no response there; these are the eigenvalues $(\mu_i)_1^{n-1}$ for the system fixed at $u_n$. Since $F(\lambda)$ has zeros $(\mu_i)_1^{n-1}$ one can write

$$F(\lambda) \equiv \sum_{i=1}^{n} \frac{[u_n^{(i)}]^2}{\lambda_i - \lambda} = \alpha \frac{\Pi_{i=1}^{n-1}(\mu_i - \lambda)}{\Pi_{i=1}^{n}(\lambda_i - \lambda)}. \tag{21}$$

On multiplying both sides by $\lambda_j - \lambda$ and then putting $\lambda = \lambda_j$ one finds

$$[u_n^{(j)}]^2 = \alpha \frac{\Pi_{i=1}^{n-1}(\mu_i - \lambda_j)}{\Pi_{i=1}^{n'}(\lambda_i - \lambda_j)}, \tag{22}$$

where $'$ denotes $i \neq j$. Comparing the sides of equation (21) for large $\lambda$, one finds

$$\alpha = \sum_{i=1}^{n} [u_n^{(n)}]^2. \tag{23}$$

We may draw the following conclusions. Suppose that one knows $(\lambda_i)_1^n$ for a system. If the $(u_n^{(i)})_1^n$ are known also, apart from an arbitrary common factor, then one can find the $(\mu_i)_1^{n-1}$ as the zeros of $F(\lambda)$. Conversely, if $(\mu_i)_1^{n-1}$ are known, then equation (22) gives the $(u_n^{(i)})_1^n$, again apart from a common factor. In the special case when the second matrix, $\mathbf{M}$, in equation (1) is the unit matrix $\mathbf{I}$, then the first of equations (16) states that the $\mathbf{u}^{(i)}$ are the columns of an orthogonal matrix; in this case the sum in equation (23) is always unity, i.e., $\alpha = 1$.

## 3. THE BASIC IDEA

Suppose one wants to solve:

**Problem 1**. *Construct an NJ-matrix* $\mathbf{C}$ *and a J-matrix* $\mathbf{A}$ *such that*

$$(\mathbf{C} - \lambda\mathbf{A})\mathbf{x} = \mathbf{0} \tag{24}$$

*has specified spectrum* $(\lambda_i)_1^n$ *satisfying equation* (13).

Since there is an infinite family of pairs, we limit our search by taking

$$\mathbf{A} = \mathbf{I} - a\mathbf{C}, \qquad a > 0. \tag{25}$$

Now

$$(\mathbf{C} - \lambda\mathbf{A})\mathbf{x} = (\mathbf{C} - \lambda\mathbf{I} + \lambda a\mathbf{C})\mathbf{x} = (1 + \lambda a)(\mathbf{C} - v\mathbf{I})\mathbf{x} = \mathbf{0}, \tag{26}$$

where $v = \lambda/(1 + \lambda a)$. This means that we should construct the NJ-matrix $\mathbf{C}$ to have eigenvalues

$$v_i = \lambda_i/(1 + \lambda_i a). \tag{27}$$

Any such $\mathbf{C}$ will, when substituted into equation (25), give a J-matrix $\mathbf{A}$: $\mathbf{A}$ will have a positive codiagonal because $a > 0$, and will be p-d because it has eigenvalues $1 - av_i = 1/(1 + \lambda_i a) > 0$. One can construct $\mathbf{C}$, by the Lanczos process, from its eigenvalues $(v_i)_1^n$ and the end values $y_n^{(i)}$ of its eigenvectors, normalized so that $\mathbf{y}^{(i)T}\mathbf{y}^{(i)} = 1$. One can choose the $y_n^{(1)}$ to be arbitrary non-zero (e.g., positive) numbers satisfying

$$\sum_{i=1}^{n} [y_n^{(i)}]^2 = 1. \tag{28}$$

Having found one pair $\mathbf{A}$, $\mathbf{C}$, one may find an infinite family of other pairs $\mathbf{A}'$, $\mathbf{C}'$, by choosing any positive diagonal matrix $\mathbf{D}$ and taking

$$\mathbf{A}' = \mathbf{DAD}, \qquad \mathbf{C}' = \mathbf{DCD}. \tag{29}$$

In particular, one can find a K-matrix **K** and J-matrix **M** such that equation (1) has the given spectrum $(\lambda_i)_1^n$ by choosing **D** as in the *stiffness transformation* described in equations (8)–(11).

Now proceed to solve:

**Problem 2**. *Construct a solution* **A**, **C** *of Problem* 1, *such that the spectrum of equation* (24) *for* $x_n = 0$ *is* $(\mu_i)_1^{n-1}$.

The analysis of section 2, with **K**, **M** replaced by **C**, **A** respectively, allows one to compute the values of $[x_n^{(i)}]^2$ for the eigenvectors $\mathbf{x}^{(i)}$ of equation (24) apart from a common factor $\alpha$. As equation (26) shows, the eigenvectors $\mathbf{x}^{(i)}$ are eigenvectors of **C**, but they are normalized with respect to **A**, not **I**. Now, using equation (25) one sees that

$$1 = \mathbf{x}^{(i)T}\mathbf{A}\mathbf{x}^{(i)} = \mathbf{x}^{(i)T}\mathbf{x}^{(i)} - a\mathbf{x}^{(i)T}\mathbf{C}\mathbf{x}^{(i)} = \mathbf{x}^{(i)T}\mathbf{x}^{(i)} - a\lambda_i. \tag{30}$$

Thus, $\mathbf{x}^{(i)T}\mathbf{x}^{(i)} = 1 + a\lambda_i$, so that the I-normalized eigenvectors of **C** are

$$\mathbf{y}^{(i)} = \mathbf{x}^{(i)}/(1 + a\lambda_i)^{1/2}; \tag{31}$$

in particular therefore

$$\mathbf{y}_n^{(i)} = \mathbf{x}_n^{(i)}/(1 + a\lambda_i)^{1/2}; \tag{32}$$

and the equation (28) now gives the unknown factor $\alpha$, and hence the $(y_n^{(i)})_1^n$. Now construct **C** from $(v_i,\ y_n^{(i)})_1^n$ using the Lanczos process; equation (25) gives **A**; equation (29) gives other pairs **A**$'$, **C**$'$. In particular, the stiffness transformation gives a pair (**K**, **M**) solving Problem 2.

## 4. AN EXAMPLE

Consider the K-matrix **K** and J-matrix **M**, of order $n$, given by

$$\mathbf{K} = \frac{1}{6}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix},$$

$$\mathbf{M} = \frac{1}{6}\begin{bmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{bmatrix}. \tag{33}$$

The eigenvalue equation (1) is equivalent to the recurrence relation

$$-(1+\lambda)u_{i-1}+(2-4\lambda)u_i-(1+\lambda)u_{i+1}=0, \tag{34}$$

with the end condition

$$u_0=0, \qquad (1-2\lambda)u_n=(1+\lambda)u_{n+1}. \tag{35}$$

This recurrence has the solution

$$u_i=\sin i\theta, \qquad \cos\theta=(1-2\lambda)/(1+\lambda), \tag{36}$$

where the second end condition (35) yields the eigenvalue equation

$$\cos n\theta=0. \tag{37}$$

This has the solution

$$\theta=\frac{(2i-1)\pi}{2n}, \qquad i=1,2,\ldots,n, \tag{38}$$

so that the eigenvalues of equation (34) are

$$\lambda_i=\frac{1-\cos\theta_i}{2+\cos\theta_i}, \qquad \theta_i=\frac{(2i-1)\pi}{2n}, \qquad i=1,2,\ldots,n. \tag{39}$$

The eigenvalues of the problem with $u_n=0$ are obtained by solving the recurrence (34) subject to the conditions

$$u_0=0=u_n. \tag{40}$$

The eigenvalue equation is

$$\sin n\theta=0, \tag{41}$$

which has solutions

$$\phi_i=\frac{i\pi}{n}, \qquad i=1,2,\ldots,n-1. \tag{42}$$

TABLE 1

*Two pairs of matrices with the specified spectra*

| | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $k_5$ | $k_6$ |
|---|---|---|---|---|---|---|
| $a=1$ | 1·0000 | 1·0000 | 1·0000 | 1·0000 | 1·0000 | 1·0000 |
| $a=2$ | 1·8150 | 0·1850 | 0·1166 | 0·1135 | 0·1134 | 0·1134 |
| | $m_{11}$ | $m_{22}$ | $m_{33}$ | $m_{44}$ | $m_{55}$ | $m_{66}$ |
| $a=1$ | 4·0000 | 4·0000 | 4·0000 | 4·0000 | 4·0000 | 2·0000 |
| $a=2$ | 2·2348 | 0·7107 | 0·6786 | 0·6805 | 0·6805 | 0·3403 |
| | $m_{12}$ | $m_{23}$ | $m_{34}$ | $m_{45}$ | $m_{56}$ | |
| $a=1$ | 1·0000 | 1·0000 | 1·0000 | 1·0000 | 1·0000 | |
| $a=2$ | 0·3700 | 0·2333 | 0·2269 | 0·2268 | 0·2268 | |

The eigenvalues are therefore

$$\mu_i = \frac{1-\cos\phi_i}{2+\cos\phi_i}, \qquad \phi_i = \frac{i\pi}{n}, \qquad i = 1, 2, \ldots, n-1. \tag{43}$$

Now apply the steps described in section 3, to find a pair $(\mathbf{K}, \mathbf{M})$ with the two spectra $(\lambda_i)_1^n$, $(\mu_i)_1^{n-1}$. Table 1 shows the results obtained for $n = 6$ and $a = 1, 2$.

Both these systems have their two spectra identical to those of the pair (33).

## 5. A WIDER FAMILY OF MATRICES

Instead of equation (25), assume that $\mathbf{A}$ and $\mathbf{C}$ are linked by

$$\mathbf{A} = \mathbf{I} - a\mathbf{C} + b\mathbf{E}_n, \tag{44}$$

where $\mathbf{E}_n = \mathbf{e}_n\mathbf{e}_n^T$ is the matrix with 1 in the lower right corner. ($\mathbf{A} = \mathbf{M}$ and $\mathbf{C} = \mathbf{K}$ given by equation (33) are linked by equation (44) with $a = 1$, $b = -1/2$.) Now equation (24) becomes

$$(\mathbf{C} - v\mathbf{I})\mathbf{x} = vb\mathbf{E}_n\mathbf{x}, \tag{45}$$

where again $v = \lambda/(1 + \lambda a)$. Now consider the eigenvalue equation for $\mathbf{C}$, namely

$$(\mathbf{C} - \gamma\mathbf{I})\mathbf{y} = \mathbf{0}. \tag{46}$$

Express $\mathbf{y}$ in terms of the $\mathbf{A}$-normalized eigenvectors $\mathbf{x}^{(i)}$ of equation (24):

$$\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}^{(i)}, \tag{47}$$

then equation (45) gives

$$(\mathbf{C} - \gamma\mathbf{I})\mathbf{y} = \sum_{i=1}^{n} \alpha_i\left(1 - \frac{\gamma}{v_i}\right)\mathbf{C}\mathbf{x}^{(i)} + \gamma b\mathbf{E}_n\mathbf{y} = \mathbf{0}. \tag{48}$$

Multiply by $\mathbf{x}^{(j)T}$ and use $\mathbf{x}^{(j)T}\mathbf{C}\mathbf{x}^{(i)} = \lambda_j\delta_{ij}$ to obtain

$$\alpha_j = \frac{-\gamma b x_n^{(j)} y_n}{\lambda_j(1 - \gamma/v_i)}. \tag{49}$$

If $\gamma = \sigma/(1 + \sigma a)$, this can be written

$$\alpha_j = \frac{-\sigma b x_n^{(j)} y_n}{\lambda_j - \sigma}. \tag{50}$$

Thus,

$$\mathbf{y} = -\sigma b y_n \sum_{i=1}^{n} \frac{x_n^{(i)}}{\lambda_j - \sigma}\,\mathbf{x}^{(i)}, \tag{51}$$

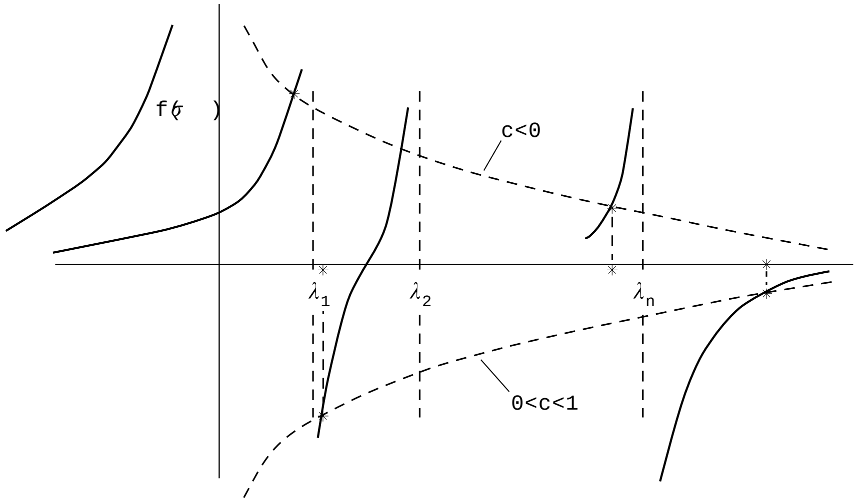


Figure 1. When $c < 1$ the curves intersect at $n$ positive values of $\sigma$: ——, $f(\sigma)$; – – –, $-1/(c\sigma)$.

so that the eigenvalue equation for $\mathbf{C}$ is

$$\sum_{i=1}^{n} \frac{[x_n^{(i)}]^2}{\lambda_i - \sigma} = -\frac{1}{b\sigma}. \tag{52}$$

If $\mathbf{y}$ is to be normalized so that $\mathbf{y}^T\mathbf{y} = 1$, then $\mathbf{y}^T\mathbf{C}\mathbf{y} = \gamma$ so that equation (51) gives

$$\sigma^2 b^2 y_n^2 \sum_{i=1}^{n} \frac{\lambda_i [x_n^{(i)}]^2}{(\lambda_i - \sigma)^2} = \gamma = \frac{\sigma}{1 + a\sigma}. \tag{53}$$

Having obtained these results, return to Problem 2 of section 3. The eigenvalues $(\lambda_i)_1^n$, $(\mu_i)^{n-1}$ give $(x_n^{(i)})_1^n$ apart from a common factor; thus equation (22) gives

$$[x_n^{(i)}]^2 = \alpha t_i, \qquad t_i > 0, \tag{54}$$

where, without loss of generality, one can take $\sum_{i=1}^{n} t_i = 1$. Equation (52) becomes

$$f(\sigma) \equiv \sum_{i=1}^{n} \frac{t_i}{\lambda_i - \sigma} = -\frac{1}{c\sigma}, \qquad c = b\alpha, \tag{55}$$

where for the $k$th value, $\sigma_k$, equation (53) becomes

$$[y_n^{(k)}]^2 = \frac{\alpha}{c^2} \frac{1}{p(\sigma_k)}, \tag{56}$$

where

$$p(\sigma) = \sigma(1 + a\sigma) \sum_{i=1}^{n} \frac{\lambda_i t_i}{(\lambda_i - \sigma)^2}. \tag{57}$$

The parameter $c$ must be chosen so that equation (55) has positive roots. Figure 1 shows $f(\sigma)$ and $-1/(c\sigma)$. When $c<0$ there are $n$ roots satisfying

$$0<\sigma_1<\lambda_1<\sigma_2<\cdots<\sigma_n<\lambda_n. \tag{58}$$

When $c>0$ there is always just one root in each of the $(n-1)$ intervals $(\lambda_i, \lambda_{i+1})$, $i=1,2,\ldots,n-1$. As $c$ increases through 1 the remaining root goes from $\sigma>1$, through infinity at $c=1$, to $\sigma<0$ at $c>1$. The condition for the roots to be positive is thus $c<1$.

Equation (56) shows how $\alpha$ must be chosen: to make

$$\sum_{k=1}^{n}[y_n^{(k)}]^2=1. \tag{59}$$

Thus,

$$\frac{1}{\alpha}=\frac{1}{c^2}\sum_{k=1}^{n}\frac{1}{p(\sigma_k)}. \tag{60}$$

This leads to the following procedure for solving Problem 2:

(1) Use $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$ in equation (22) to find $x_n^{(i)}$ $(\equiv u_n^{(i)})$.
(2) Find $(t_i)_1^n$ satisfying $\sum_{i=1}^n t_i=1$ from equation (54).
(3) Choose $c<1$,
(4) Find the roots $(\sigma_k)_1^n$ of equation (55),
(5) Find $\alpha$ from equation (60).
(6) Find $[y_n^{(k)}]^2$ from equation (56).
(7) Use the Lanczos process to find the NJ-matrix **C** from $\gamma_k=\sigma_k/(1+a\sigma_k)$ and $y_n^{(k)}$, $k=1,2,\ldots,n$.
(8) Find $b=c/\alpha$.
(9) Find **A** from equation (44). Note that since **C** is p-d, and equation (24) has positive eigenvalues $(\lambda_i)_1^n$, **A** will be p-d also.

One may find a **K**, **M** model by applying a stiffness transformation to **C**, **A**.

This procedure may be used with $a=1$ and $c=-1{\cdot}7322$ to reconstruct the pair (33) from the data (39) and (43).

## 6. VESILIC'S RESULT

Vesilic [16] proved the following result. Suppose that $2n$ eigenvalues are given in the left-hand half of the complex plane. They are made up of $2s$ negative real numbers

$$-\alpha_1, -\alpha_2, \ldots -\alpha_{2s}<0 \tag{61}$$

corresponding to "overdamped" modes, and $n-s$ complex conjugate pairs

$$-\gamma_1\pm i\beta_1, -\gamma_2\pm i\beta_2, \ldots, -\gamma_{n-s}\pm i\beta_{n-s}, \tag{62}$$

where $(\gamma_j, \beta_j)_1^{n-s} > 0$. Then there is a unique set of numbers $(\omega_j)_1^n$ satisfying

$$0 < \omega_1 < \omega_2 < \cdots < \omega_n \tag{63}$$

and a unique set of positive numbers $b_k$ making up the vector $\mathbf{b} = \{b_1, b_2, \ldots, b_n\}$ such that the matrix pencil

$$\mathbf{I}\lambda^2 + \mathbf{b}\mathbf{b}^T\lambda + \mathbf{\Omega}, \tag{64}$$

with $\mathbf{\Omega} = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_n)$, has the $2n$ specified eigenvalues.

Vesilic shows that if $\gamma_j = 0$ for some $j$, so that one mode is completely undamped, then $\omega_j = \beta_j$ and $b_j = 0$. The $\omega_k$ are given as the solutions of the equation

$$f(\omega_k) = \frac{(2k-1)\pi}{2}, \qquad k = 1, 2, \ldots, n, \tag{65}$$

where

$$f(x) = \sum_{r=1}^{2s} \arctan\left(\frac{x}{\alpha_r}\right) + \sum_{r=1}^{n-s}\left\{\arctan\left(\frac{x+\beta_r}{\gamma_r}\right) + \arctan\left(\frac{x-\beta_r}{\gamma_r}\right)\right\}. \tag{66}$$

Since $f(x)$ increases monotonically from 0 to $n\pi$ as $x$ increases from 0 to $\infty$, the solutions $\omega_k$ are distinct, so that equation (63) holds. The $b_k$ are given by

$$b_k^2 = \frac{\Pi_{r=1}^{2s}(\alpha_r^2 + \omega_k^2)\Pi_{r=1}^{n-s}\{(\omega_k + \beta_r)^2 + \gamma_r^2\}^{1/2}\{(\omega_k - \beta_r)^2 + \gamma_r^2\}^{1/2}}{\omega_k \Pi_{r=1}^{n'}|\omega_k^2 - \omega_r^2|}, \tag{67}$$

where $'$ denotes $r \neq k$.

## 7. RECONSTRUCTION OF A DAMPED FEM MODEL

First consider the problem of constructing matrices $\mathbf{A}$, $\mathbf{C}$ such that $\mathbf{A}$ is a J-matrix, $\mathbf{C}$ is an NJ-matrix, and

$$\mathbf{A}\lambda^2 + \gamma^2\mathbf{E}_n\lambda + \mathbf{C} \tag{68}$$

has specified eigenvalues given by equations (61) and (62).

Using Vesilic's process, one finds $\mathbf{b}$ and $\Omega$ such that

$$\mathbf{I}\lambda^2 + \mathbf{b}\mathbf{b}^T\lambda + \mathbf{\Omega}^2 \tag{69}$$

has the specified eigenvalues. Now seek a square matrix $\mathbf{X}$ such that

$$\mathbf{X}(\mathbf{I}\lambda^2 + \mathbf{b}\mathbf{b}^T\lambda + \mathbf{\Omega}^2)\mathbf{X}^T = \mathbf{A}\lambda^2 + \gamma^2\mathbf{e}_n\mathbf{e}_n^T\lambda + \mathbf{C}, \tag{70}$$

where

$$\mathbf{A} = \mathbf{I} - a\mathbf{C}. \tag{71}$$

Thus, $\mathbf{X}$ must satisfy

$$\mathbf{X}\mathbf{X}^T = \mathbf{A}, \qquad \mathbf{X}\mathbf{\Omega}^2\mathbf{X}^T = \mathbf{C}, \qquad \mathbf{X}\mathbf{b} = \gamma\mathbf{e}_n. \tag{72}$$

Equations (70) and (71) give

$$\mathbf{X}\mathbf{X}^T = \mathbf{I} - a\mathbf{X}\mathbf{\Omega}^2\mathbf{X}^T \quad \text{i.e.,} \quad \mathbf{X}(\mathbf{I} + a\mathbf{\Omega}^2)\mathbf{X}^T = \mathbf{I}. \tag{73}$$

This equation states that the matrix

$$\mathbf{Y} = \mathbf{X}(\mathbf{I} + a\mathbf{\Omega}^2)^{1/2} \tag{74}$$

is orthogonal. Thus,

$$\mathbf{Y}\mathbf{Y}^T = \mathbf{I}, \qquad \mathbf{Y}\mathbf{\Omega}^2(\mathbf{I} + a\mathbf{\Omega}^2)^{-1}\mathbf{Y}^T = \mathbf{C}. \tag{75}$$

This shows that

$$\mathbf{C}\mathbf{Y} = \mathbf{Y}\mathbf{\Lambda}, \tag{76}$$

where

$$\mathbf{\Lambda} = \mathbf{\Omega}^2/(\mathbf{I} + a\mathbf{\Omega}^2). \tag{77}$$

In other words, the matrix $\mathbf{C}$ has eigenvalues

$$\lambda_i = \omega_i^2/(1 + a\omega_i^2), \tag{78}$$

and the corresponding eigenvectors are the columns of $\mathbf{Y}$, i.e.,

$$(\mathbf{C} - \lambda_i\mathbf{I})\mathbf{y}^{(i)} = 0. \tag{79}$$

The last equation (71), when combined with equation (73), gives

$$(1 + a\omega_i^2)^{-1/2}b_i = \gamma y_n^{(i)}. \tag{80}$$

Since the $y_n^{(i)}$ must satisfy

$$\sum_{i=1}^{n} [y_n^{(i)}]^2 = 1, \tag{81}$$

one must choose $\gamma$ so that

$$\sum_{i=1}^{n} (1 + a\omega_i^2)^{-1}b_i^2 = \gamma^2. \tag{82}$$

Now apply the Lanczos process to construct the NJ-matrix $\mathbf{C}$ from $(\lambda_i, y_n^{(i)})_1^n$, and then find $\mathbf{A}$ from equation (70). One can now apply the stiffness transformation

$$\mathbf{M} = \mathbf{DAD}, \qquad \mathbf{B} = \gamma^2\mathbf{D}\mathbf{E}_n\mathbf{D} = \gamma^2 d_n^2\mathbf{E}_n, \qquad \mathbf{K} = \mathbf{DCD} \tag{83}$$

TABLE 2

*The specified complex spectrum*

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\gamma_i$ | 0·0050 | 0·0048 | 0·0191 | 0·0185 | 0·0761 | 0·0165 |
| $\beta_i$ | 0·2230 | 0·4349 | 0·6292 | 0·7903 | 0·9023 | 0·9456 |

TABLE 3

*The constructed matrix pencil* $\mathbf{M}\lambda^2 + \mathbf{B}\lambda + \mathbf{K}$ *has the specified complex spectrum given in Table* 2

| | | | | | |
|---|---|---|---|---|---|
| $k_1$ | $k_2$ | $k_3$ | $k_4$ | $k_5$ | $k_6$ |
| 1·6402 | 0·3598 | 0·1157 | 0·0599 | 0·0423 | 0·0096 |
| $m_{11}$ | $m_{22}$ | $m_{33}$ | $m_{44}$ | $m_{55}$ | $m_{66}$ |
| 3·8473 | 1·2082 | 0·3581 | 0·2814 | 0·1498 | 0·0145 |
| $m_{12}$ | $m_{23}$ | $m_{34}$ | $m_{45}$ | $m_{56}$ | $m_{66}$ |
| 0·1799 | 0·0579 | 0·0299 | 0·0211 | 0·0048 | 0·0040 |

to find $\mathbf{M}$, $\mathbf{B}$, $\mathbf{K}$ such that

$$\mathbf{M}\lambda^2 + \mathbf{B}\lambda + \mathbf{K} \tag{84}$$

has the given eigenvalues.

One may find a larger family of matrices by combining this procedure with that given in section 5.

## 8. AN EXAMPLE OF A DAMPED FEM SYSTEM

The specified complex spectrum shown in Table 2 and Table 3 shows the constructed matrix pencil $\mathbf{M}\lambda^2 + \mathbf{B}\lambda + \mathbf{K}$.

## 9. CONCLUSIONS

Earlier it was stated that we are concerned essentially with mathematical inverse problems. However, it is considered how the mathematical problem as stated relates to the engineering problem of identifying a system corresponding to given spectral data. There are two principal considerations: it is not possible in practice to obtain the data needed for the solution of the problem as stated; the solution of the stated problem is given in terms of the stiffness and inertia matrices of the system, not as the physical parameters such as densities and radii of cross-section of, say, an actual non-uniform rod. These matters are treated one by one.

First, the analysis shows that, from given spectral data, it is possible to construct two matrices which have the generic forms corresponding to the stiffness and mass matrices of an on-line system of elements attached to each other by one co-ordinate at each end. This is an important new result; in previous analyses, references [8, 9] for example, it had to be assumed that the mass matrix was diagonal, i.e., the mass was lumped. The procedure, like that of references [8, 9] requires a complete set of spectral data. One way this data could be assembled in practice is as follows. The underlying system which is being modelled can be thought of as a straight rod, with varying cross-section, undergoing longitudinal vibration. It is well known that a FEM model of such a system will correctly predict only low natural frequencies. This means that experiment can provide only the lower portion of the required spectral data. But it is also well known that for a rod with only small variations in cross-section, the higher natural frequencies are relatively insensitive to the cross-sectional variation. This means that, in principle, one may augment that experimentally determined spectral data, with the calculated higher spectral data corresponding to the FEM model of a *uniform* rod of the relevant length.

Now consider the second problem: the system identification. In any system identification problem of inverse eigenvalue type there are three kinds of quantities: system parameters, i.e., lengths, densities, radii of cross-section etc.; stiffness and mass matrices; and spectral data. Essentially there are two ways of connecting these three sets of quantities. The first, which has been used extensively, and which is fully documented in reference [2], for instance, is as follows. Start with the system parameters, express the stiffness and mass matrices in terms of them, and then through some optimization process, determine the values which the parameters should take in order to yield the specified spectral data. Procedures for carrying out these opeations are now well advanced. In this paper we proceed quite differently: instead of starting with the system parameters and constructing the stiffness and mass matrices, we start from the spectral data and find stiffness and mass matrices which correspond to them. Since it is known that there are an infinity of pairs of matrices which have the correct generic form, and which correspond to the data, we limit our choice by seeking pairs in which the mass matrix is expressed in terms of the stiffness matrix (by equation (25)). We choose to write the mass matrix in this way because the generic form of the stiffness matrix is so simple; see equation (2). The ''solution'' of the problem as stated in this paper is a stiffness matrix **K** and a mass matrix **M** which both have the correct generic form for systems of the type considered. There remains the problem of reconstructing the system parameters from these matrices. This problem has already been addressed, in reference [15], and proceeds in two stages: the construction of element stiffness and mass matrices; and the identification of the system parameters from the element matrices.

It has already been noted that there is an infinite family of possible systems, of the required form, corresponding to the given data. Various members of the family can be obtained by varying the parameters $a$ (in equation (25)) and $b$ (in equation (44)). In the analysis two possible sets of data are considered: one spectrum of natural frequencies, or two spectra. In both cases there are families of solutions. When the data consists of just one spectrum, then the analysis of reference [15] can be used to construct an infinite family of isospectral pairs of matrices (**K**, **M**), all of the correct generic form, from one such pair. Note that there is an open theoretical problem: can one start from one member and construct the *complete* infinite isospectral family, as can be done for lumped-mass models [10]? I conjecture that the answer is NO.

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