Journal of Sound and Vibration (1999) **221**(2), 325–333
Article No. jsvi.1998.1968, available online at http://www.idealibrary.com on IDEAL

LETTERS TO THE EDITOR

A NOVEL NUMERICAL METHOD FOR EVALUATING THE NATURAL VIBRATION FREQUENCY OF A BENDING BAR CONSIDERING ROTARY INERTIA AND SHEAR EFFECT

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(*Received* 25 *February* 1997, *and in final form* 2 *September* 1998)

1. Introduction

Many numerical methods have been developed to investigate the natural vibration frequency of a bending bar [1–3]. Among them the Rayleigh–Ritz method is well known and significant. In this method, by minimizing the Rayleigh quotient with respect to relevant coefficients in the deflection function, the eigenvalue equation can be obtained [1–3]. On the other hand, one can use the differential equation to solve the problem, e.g., the shooting method [4, 5]. Though the previously proposed numerical methods are satisfactory for finding the fundamental vibration frequency, not enough attention has been paid to the problem of finding the vibration frequency in more complicated cases. Probably, the previously suggested methods have not exhausted the investigation in this field. In this paper, more complicated cases are considered, which include consideration of the varying cross section, the rotary inertia and the shear effect. In addition, a novel numerical method is developed in this paper. In the method, the problem for evaluating the natural vibration frequencies of a bar can be reduced to finding zeros of a target function. The details will be described in the following analysis.

2. Analysis

The governing equation for a bending bar considering the following factors: (a) the translation inertia, (b) the rotary inertia and (c) the shear effect of materials was proposed in [1, 3]. In this case, one has to introduce two functions $w(x, t)$ and $\psi(x, t)$, where $w(x, t)$ is the deflection of the bending bar and $\psi(x, t)$ is the rotation of a section, x is the position of a bar section, and t is the time variable (Figure 1). In the free vibration analysis, after letting

$$w(x, t) = W(x)\sin(\omega t), \qquad \psi(x, t) = \Psi(x)\sin(\omega t), \tag{1}$$

the governing equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\kappa G A(x)\left(\Psi - \frac{\mathrm{d}W}{\mathrm{d}x}\right)\right] - \omega^2 \rho A(x) W = 0, \qquad (0 \leqslant x \leqslant L) \tag{2}$$

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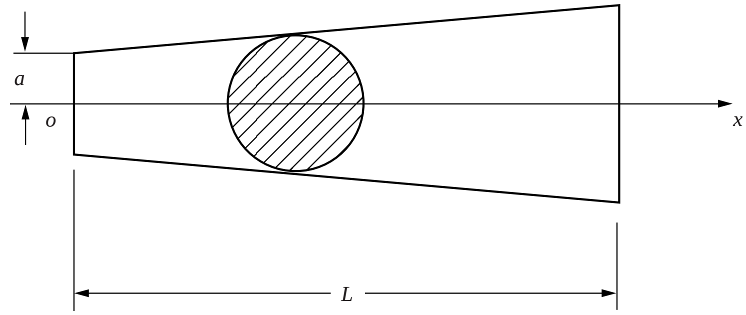

Figure 1. A truncated conical bar with two simply supported ends.

$$\frac{d}{dx}\left[EI(x)\frac{d\Psi}{dx}\right] - \kappa GA(x)\left(\Psi - \frac{dW}{dx}\right) + \beta\omega^2\rho I(x)\Psi = 0, \qquad (0 \leqslant x \leqslant L) \quad (3)$$

where ω denotes vibration frequency, $A(x)$ is the area of section, $I(x)$ is the moment inertia of section, G is the shear modulus of elasticity, $E = 2G(1 + \nu)$ is the Young's modulus of elasticity, ρ is the mass density of materials, and [3]

$$\kappa = (6 + 6\nu)/(7 + 6\nu). \quad (4)$$

The parameter β in (3) plays the following role: if the rotary inertia effect of the section is considered, we choose $\beta = 1$; otherwise we take $\beta = 0$. In this paper, the bar has a truncated conical configuration (Figure 1), and the two functions $I(x)$ and $A(x)$ will be

$$I(x) = I_0 g(x), \quad \text{with} \quad I_0 = I(0) = \frac{\pi a^4}{4}, \qquad g(x) = \left(1 + \frac{mx}{L}\right)^4 \quad (5)$$

$$A(x) = A_0 h(x), \quad \text{with} \quad A_0 = A(0) = \pi a^2, \qquad h(x) = \left(1 + \frac{mx}{L}\right)^2. \quad (6)$$

In the present study, it is assumed that the two ends of the bar are simply supported. Therefore, the boundary value conditions will be

$$W|_{x=0} = 0, \qquad \left.\frac{d\Psi}{dx}\right|_{x=0} = 0 \quad (7a)$$

$$W|_{x=L} = 0, \qquad \left.\frac{d\Psi}{dx}\right|_{x=L} = 0. \quad (7b)$$

If the shear effect is not considered, we prefer to reduce the mentioned governing equations. In fact, eliminating the term $\kappa GA(x)(\Psi - \mathrm{d}W/\mathrm{d}x)$ in (3) by using (2), and substituting Ψ by $\mathrm{d}W/\mathrm{d}x$, yields

$$\frac{\mathrm{d}^2}{\mathrm{d}x}\left[EI(x)\frac{\mathrm{d}^2W}{\mathrm{d}x^2}\right] - \omega^2\rho A(x)W + \beta\omega^2\rho\frac{\mathrm{d}}{\mathrm{d}x}\left[I(x)\frac{\mathrm{d}W}{\mathrm{d}x}\right] = 0, \qquad (0 \leqslant x \leqslant L) \tag{8}$$

where the meaning of notations has been indicated above. The boundary conditions become

$$W|_{x=0} = 0, \qquad \left.\frac{\mathrm{d}^2W}{\mathrm{d}x^2}\right|_{x=0} = 0 \tag{9a}$$

$$W|_{x=L} = 0, \qquad \left.\frac{\mathrm{d}^2W}{\mathrm{d}x^2}\right|_{x=L} = 0. \tag{9b}$$

There are four types of frequency problems investigated below. To distinguish the characteristics of the four types, the governing equations and the boundary conditions for four types are listed in Table 1.

Previously, it has been pointed out that the eigenvalue problem of differential equation can be considered as a particular initial boundary value problem of the same equation [4, 5]. Following this idea, the target function method is suggested. In fact, the solution technique for four types of frequency equations is the same. The solution for types C and D will be introduced. In fact, for any given ω, we can solve the following initial boundary value problem

$$W|_{x=0} = 0, \qquad \left.\frac{\mathrm{d}\Psi}{\mathrm{d}x}\right|_{x=0} = 0, \qquad \left.\frac{\mathrm{d}W}{\mathrm{d}x}\right|_{x=0} = 1,$$

$$\Psi|_{x=0} = 0 \quad \text{(the fundamental problem P)} \tag{10}$$

TABLE 1

Classification of the studied frequency problems

	Consideration of rotary inertia	Consideration of shear effect	Governing equations	Boundary conditions
Type A	No	No	(8) $\beta = 0$	(9a), (9b)
Type B	Yes	No	(8) $\beta = 1$	(9a), (9b)
Type C	No	Yes	(2), (3) $\beta = 0$	(7a), (7b)
Type D	Yes	Yes	(2), (3) $\beta = 1$	(7a), (7b)

$$W|_{x=0}=0, \qquad \left.\frac{\mathrm{d}\Psi}{\mathrm{d}x}\right|_{x=0}=0, \qquad \left.\frac{\mathrm{d}W}{\mathrm{d}x}\right|_{x=0}=0,$$

$$\Psi|_{x=0}=1 \quad \text{(the fundamental problem Q).} \tag{11}$$

Note that both boundary conditions given by equations (10) and (11) contain the simply supported condition at the point $x=0$, which was shown by (7a). The relevant solution is called the fundamental solution P or Q, respectively. The solutions obtained are denoted by

$$W=p_1(x,\omega), \qquad \Psi=p_2(x,\omega),$$

$$(0 \leqslant x \leqslant L) \qquad \text{(for the fundamental problem P)} \tag{12}$$

$$W=q_1(x,\omega), \qquad \Psi=q_2(x,\omega),$$

$$(0 \leqslant x \leqslant L) \qquad \text{(for the fundamental problem Q).} \tag{13}$$

Note that, for example, $p_1(x,\omega)$, $p_2(x,\omega)$, $(0 \leqslant a \leqslant L)$ are obtained in the form of a numerical solution, rather than in the form of an analytical solution. That is to say, from the governing equations (2) and (3) and the initial boundary condition (10), we can obtain the values of functions $p_1(x,\omega)$, $\mathrm{d}p_1(x,\omega)/\mathrm{d}x$, $p_2(x,\omega)$, $\mathrm{d}p_2(x,\omega)/\mathrm{d}x$ at the discrete points $x=0,\ L/N,\ 2L/N,\ 3L/N, \ldots, L$, where N is the division number used in integration of an ordinary differential equation. The numerical solution mentioned can be obtained easily by using the well known Runge–Kutta integration rule [6, p. 290], and the solution technique is cited in the Appendix.

Clearly, we can seek the general solution in the form

$$W(x,\omega)=c_1p_1(x,\omega)+c_2q_1(x,\omega) \tag{14}$$

$$\Psi(x,\omega)=c_1p_2(x,\omega)+c_2q_2(x,\omega). \tag{15}$$

Substituting (14) and (15) into (7b) yields

$$c_1p_1(L,\omega)+c_2q_1(L,\omega)=0, \qquad c_1p_2'(L,\omega)+c_2q_2'(L,\omega)=0. \tag{16}$$

In order that a non-trivial solution for c_1, c_2 exists, the relevant determinant should vanish. Therefore, from equation (16) we have the following equation

$$T(\omega)=0 \tag{17}$$

where

$$T(\omega)=p_1(L,\omega)q_2'(L,\omega)-q_1(L,\omega)p_2'(L,\omega). \tag{18}$$

This function is called the target function in this paper. Thus, the eigenvalues are equal to finding zeros of the target function. The zeros of the target function $T(\omega)$ can be easily obtained by using the half-division method in numerical computation.

3. NUMERICAL EXAMPLES

Numerical results are presented to verify the accuracy of the solution. In addition, the rotary inertia effect and the shear effect can also be found from the present examples. In computation, $N = 40$ divisions are used in the numerical integration of the ordinary differential equation, and $\nu = 0{\cdot}3$ is assumed.

3.1. *Numerical solution for the problem of type A* (*see Table* 1)

In the first case, both the rotary inertia effect and shear effect have not been considered. The calculated results for the natural frequency are expressed by

$$\omega = f(m)\left(\frac{EI_0}{\rho A_0}\right)^{1/2}\left(\frac{\pi}{L}\right)^2. \tag{19}$$

The results for the first six natural frequencies are listed in Table 2. From Table 2 we see that in the constant section case [$m = 0$ in (5), (6)], the deviation between the numerical computation and the analytical solution is negligible.

3.2. *Numerical solution for the problem of type B* (*see Table* 1)

In the second case, the rotary inertia effect of the section is considered and the shear effect has not been considered. The calculated results for the natural frequency are expressed by

$$\omega = B\left(m, \frac{a}{L}\right)\left(\frac{EI_0}{\rho A_0}\right)^{1/2}\left(\frac{\pi}{L}\right)^2. \tag{20}$$

The results for the first six natural frequencies are listed in Table 3 for two cases $a/L = 0{\cdot}05$ and $a/L = 0{\cdot}1$. In this case, the coefficients $B(m, a/L)$ depend not only on the factor (m) but also on the ratio (a/L). From the calculated results we see that in the case of $a/L = 0{\cdot}05$, the influence of the rotary inertia on the fundamental frequency is not significant. However, in an extreme case ($m = 4$, $a/L = 0{\cdot}1$) the 6th frequency is reduced from 95·698 to 35·265.

TABLE 2

The first six normalized natural frequency $f(m)$ for the problem of type A [*see Figure* 1, *Table* 1 *and equation* (19)]

	1st	2nd	3rd	4th	5th	6th
$m = 0$	1·000	4·000	9·000	16·003	25·009	36·027
$m = 0$*	1·000	4·000	9·000	16·000	25·000	36·000
$m = 1$	1·410	5·899	13·219	23·439	36·579	52·644
$m = 2$	1·719	7·686	17·123	30·247	47·094	67·694
$m = 3$	1·978	9·417	20·871	36·738	57·081	81·948
$m = 4$	2·206	11·115	24·522	43·031	66·733	95·698

* Exact.

Table 3

The first six normalized natural frequency $B(m, a/L)$ for the problem of type B [see Figure 1, Table 1 and equation (20)]

	1st	2nd	3rd	4th	5th	6th
$a/L = 0{\cdot}05$ case						
$m = 0$	0·997	3·952	8·761	15·267	23·278	32·588
$m = 1$	1·398	5·742	12·482	21·272	31·675	43·282
$m = 2$	1·691	7·337	15·582	25·918	37·699	50·408
$m = 3$	1·922	8·779	18·211	29·598	42·165	55·381
$m = 4$	2·110	10·083	20·450	32·543	45·546	58·979
$a/L = 0{\cdot}10$ case						
$m = 0$	0·988	3·816	8·142	13·550	19·667	26·213
$m = 1$	1·366	5·339	10·846	17·207	23·954	30·845
$m = 2$	1·614	6·524	12·668	19·366	26·226	33·099
$m = 3$	1·779	7·451	13·944	20·755	27·601	34·410
$m = 4$	1·885	8·182	14·871	21·710	28·516	35·265

3.3. *Numerical solution for the problem of type C* (*see Table* 1)

In the third case, the rotary inertia effect of the section is not considered and the shear effect has been considered. The calculated results for the natural frequency are also expressed by

$$\omega = C\left(m, aL\right)\left(\frac{EI_0}{\rho A_0}\right)^{1/2}\left(\frac{\pi}{L}\right)^2. \qquad (21)$$

The results for the first six natural frequencies are listed in Table 4 for two cases: $a/L = 0{\cdot}05$ and $a/L = 0{\cdot}1$. From the calculated results we see that in the case of

Table 4

The first six normalized natural frequency $C(m, a/L)$ for the problem of type C [see Figure 1, Table 1 and equation (21)]

	1st	2nd	3rd	4th	5th	6th
$a/L = 0{\cdot}05$ case						
$m = 0$	0·991	3·863	8·364	14·092	20·751	28·031
$m = 1$	1·381	5·485	11·371	18·395	26·033	33·966
$m = 2$	1·657	6·835	13·578	21·149	29·047	37·045
$m = 3$	1·866	7·972	15·223	23·000	30·922	38·857
$m = 4$	2·030	8·930	16·465	24·293	32·170	40·029
$a/L = 0{\cdot}10$ case						
$m = 0$	0·966	3·523	7·004	10·893	14·919	18·968
$m = 1$	1·305	4·624	8·519	12·536	16·537	20·500
$m = 2$	1·503	5·353	9·326	13·293	17·218	21·107
$m = 3$	1·619	5·851	9·801	13·712	17·581	21·425
$m = 4$	1·680	6·199	10·104	13·973	17·805	21·620

TABLE 5

The first six normalized natural frequency $F(m, a/L)$ for the problem of type D [see Figure 1, Table 1 and equation (22)]

	1st	2nd	3rd	4th	5th	6th
$a/L = 0{\cdot}05$ case						
$m = 0$	0·988	3·822	8·179	13·685	20·010	26·903
$m = 1$	1·371	5·374	11·005	17·664	24·920	32·512
$m = 2$	1·631	6·631	13·035	20·241	27·833	35·527
$m = 3$	1·818	7·666	14·533	22·017	28·376	29·894
$m = 4$	1·950	8·523	15·707	22·955	23·990	31·134
$a/L = 0{\cdot}10$ case						
$m = 0$	0·955	3·421	6·722	10·418	14·291	18·237
$m = 1$	1·272	4·432	8·144	12·040	13·698	16·005
$m = 2$	1·429	5·091	8·880	9·892	12·883	16·200
$m = 3$	1·486	5·536	7·804	9·529	13·340	14·680
$m = 4$	1·475	5·805	6·736	9·842	13·481	13·924

$a/L = 0{\cdot}05$, the influence of the shear effect on the fundamental frequency is not significant. However, in an extreme case ($m = 4$, $a/L = 0{\cdot}1$), the 6th frequency is reduced from 95·698 to 21·620.

3.4. *Numerical solution for the problem of type D* (*see Table* 1)

In the fourth case, both the rotary inertia effect of section and shear effect have been considered. The calculated results for the natural frequency are also expressed by

$$\omega = F\left(m, \frac{a}{L}\right)\left(\frac{EI_0}{\rho A_0}\right)^{1/2}\left(\frac{\pi}{L}\right)^2. \qquad (22)$$

The results for the first six natural frequencies are listed in Table 5 for two cases: $a/L = 0{\cdot}05$ and $a/L = 0{\cdot}1$. From the calculated results we see that in the case of $m = 0$ and $a/L = 0{\cdot}05$, the influence of the rotary inertia and shear effect on the fundamental frequency is not significant. However, in an extreme case ($m = 4$, $a/L = 0{\cdot}1$), the 6th frequency is reduced from 95·698 to 13·924. From the above mentioned results we see that both the rotary effect and the shear effect have contributed to a lowering of the relevant vibration frequency.

4. REMARKS

Previously, when the computer was not available, investigators paid attention to the solution which could be performed manually by using very elementary computation. In contrast, the present study is an attempt to use the computer intensively, and the goal is achieved by using the suggested method. In fact, the mentioned target function $T(\omega)$ can be obtained as a result of the numerical solution of the ordinary differential equation, which can easily be solved on the computer. Secondly, it is also easy to find the zeros of the target function, for instance, by using the well known half-division technique. In fact, the mentioned computation only uses a fraction of a second on the computer.

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APPENDIX

Numerical solution of the ordinary differential equations (2) *and* (3) *under the initial boundary value condition* (10)

Generally, an ordinary differential equation with higher order derivative can be reduced to a simultaneous equation with the first order derivative. To solve (2) and (3), we make a substation as follows

$$W_1(x) = W(x), \quad W_2(x) = \mathrm{d}W/\mathrm{d}x, \quad W_3(x) = \Psi(x), \quad W_4(x) = \mathrm{d}\Psi/\mathrm{d}x. \quad \text{(A1)}$$

In this case, (2) and (3) are reduced to a simultaneous equation

$$\frac{\mathrm{d}W_1}{\mathrm{d}x} = F_1(W_1, W_2, W_3, W_4, x, \omega)$$

$$\frac{\mathrm{d}W_2}{\mathrm{d}x} = F_2(W_1, W_2, W_3, W_4, x, \omega)$$

$$\frac{\mathrm{d}W_3}{\mathrm{d}x} = F_3(W_1, W_2, W_3, W_4, x, \omega)$$

$$\frac{\mathrm{d}W_4}{\mathrm{d}x} = F_4(W_1, W_2, W_3, W_4, x, \omega) \quad \text{(A2)}$$

where

$$F_1(W_1, W_2, W_3, W_4, x, \omega) = W_2$$

$$F_2(W_1, W_2, W_3, W_4, x, \omega) = \frac{1}{h(x)}\left(h(x)W_4 - h'(x)(W_2 - W_3) - \frac{\omega^2\rho}{E}\frac{2(1+v)}{\kappa}h(x)W_1\right)$$

$$F_3(W_1, W_2, W_3, W_4, x, \omega) = W_4$$

$$F_4(W_1, W_2, W_3, W_4, x, \omega) = \frac{1}{g(x)}\left(-g'(x)W_4 + \frac{\kappa}{2(1+\nu)}\frac{A_0}{I_0}h(x)(W_3 - W_2) - \beta\frac{\omega^2\rho}{E}g(x)W_3\right) \qquad \text{(A3)}$$

and where the functions $g(x)$, $h(x)$, and A_0, I_0 and have been indicated in (5) and (6), and the meaning of E, ν, κ, ρ, ω, β can be found from the text. From (10), the boundary condition becomes

$$W_1|_{x=0} = 0, \qquad W_4|_{x=0} = 0, \qquad W_2|_{x=0} = 1, \qquad W_3|_{x=0} = 0. \qquad \text{(A4)}$$

The simultaneous equation (A1) under the initial boundary value condition (A4) can be easily solved numerically by using the Runge–Kutta method [6, p. 290].