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VIBRATION ANALYSIS OF RECTANGULAR AND SKEW PLATES BY THE RAYLEIGH–RITZ METHOD

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1. INTRODUCTION

There are two generally accepted computational techniques which frequently meet on the pages of the *Journal of Sound and Vibration* in vibration analysis of rectangular and skew plates. Through the last three decades many authors [1–3] have utilized the Rayleigh–Ritz method and spline approximation in computing transverse vibrations of plates because of their extensive need in a variety of applications in engineering design. In contrast to this practical importance of vibration analysis of plates in many areas of mechanical, aerospace, ocean, electronic and optical engineering, the methods used for solving the corresponding eigenvalue problems still do not take advantage of all possibilities offered by these popular techniques.

The aim of this study is to present an effective easy to build and easy to use technique for the computation of transverse vibrations of rectangular and skew plates by the Rayleigh–Ritz method using B-spline trial functions.

2. RAYLEIGH–RITZ METHOD AND B-SPLINES

The Rayleigh–Ritz method applied to the equation

$$\Delta^2\Phi = \lambda\Phi \qquad \text{in} \quad \Omega, \tag{1}$$

describing the free transverse vibrations of an isotropic uniform plate, results in the minimization of the following Rayleigh quotient

$$\frac{D}{2}\int_\Omega \left\{\Delta\Phi\Delta\Phi - 2(1-\nu)\left[\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2\Phi}{\partial y^2} - \left(\frac{\partial^2\Phi}{\partial x\,\partial y}\right)^2\right]\right\}\mathrm{d}\Omega \Big/ \int_\Omega \Phi\Phi\,\mathrm{d}\Omega \tag{2}$$

over the set of functions from the Sobolev space $W_2^2(\Omega)$ satisfying the corresponding boundary conditions [4]. Here $\lambda = \rho\omega^2/D$, where ρ is the mass density per unit area of the plate, ω is the circular frequency, D is the flexular rigidity and ν is Poisson's ratio. For simplicity only the boundary conditions most frequently used in practice for an edge parallel to the y-axis will be mentioned. Thus,

$$\Phi = \frac{\partial\Phi}{\partial x} = 0, \qquad \Phi = \frac{\partial^2\Phi}{\partial x^2} + \nu\frac{\partial^2\Phi}{\partial y^2} = 0, \tag{3, 4}$$

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and

$$\frac{\partial^2 \Phi}{\partial x^2} + v\,\frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial^3 \Phi}{\partial x^3} + (2 - v)\,\frac{\partial^3 \Phi}{\partial x\,\partial y^2} = 0 \tag{5}$$

represent the boundary conditions for a clamped, simply supported, and free edge, respectively. The corresponding boundary conditions for an edge parallel to the x-axis are obtained by interchanging x and y in equations (3), (4), and (5).

The success in the minimization of Rayleigh's quotient depends on a suitable choice of n trial functions ϕ_i satisfying the corresponding boundary conditions and effective solution of the generalized matrix eigenvalue problem

$$\mathbf{Ru} = \lambda \mathbf{Su} \tag{6}$$

resulting from the Ritz method applied to expression (2).

The suitable choice of trial functions must reflect the quantitative and qualitative properties of the exact eigenfunctions which exhibit singular behaviour at corner points of the boundary [5–8] and the higher eigenfunctions oscillations. To achieve good approximations of eigenfunctions with corner singularities, i.e., exhibiting non-polynomial behaviour in the neighbourhood of the corner points, it is necessary to use a local approximation scheme (finite elements, spline functions, multidomain approach) instead of the classical algebraic and goniometric polynomials. Moreover, in the case of the local trial functions the resulting matrices are sparse and well conditioned which results in more straightforward and reliable solution of equation (6). For these reasons the local approach represented by B-spline trial functions has been use in this study.

Denote by $B_i^\ell(x)$ the ith algebraic B-spline of order ℓ created over ℓ subintervals $\langle x_i, x_{i+1}\rangle$, $\langle x_{i+1}, x_{i+2}\rangle, \ldots \langle x_{i+\ell-1}, x_{i+\ell}\rangle$, where $\{x_i\}$ is a finite increasing sequence of mesh points. On each of these ℓ subintervals $B_i^\ell(x)$ is an algebraic polynomial of order $(\ell - 1)$ and outside of these subintervals $B_i^\ell(x) \equiv 0$. Such a piecewise polynomial function $B_i^\ell(x)$ is continuous on the whole real line together with all derivatives up to order $(\ell - 2)$ [9].

3. SUBSPACE ITERATION METHOD

The subspace iteration method (SIM) is probably the most popular method in solving large and sparse eigenproblems in structural mechanics. If the $n \times q$ matrix $\mathbf{X}_1$ contains initial approximations of the first q eigenvectors of (6), then the basic version of SIM for computing the p lowest eigenvalues $p < q$ and associated eigenvectors consists of the following four steps:

(1) Solve q systems of linear equations

$$\mathbf{R}\mathbf{X}^*_{k+1} = \mathbf{S}\mathbf{X}_k.$$

(2) Compute the q-dimensional projections of the matrices $\mathbf{R}$ and $\mathbf{S}$

$$\mathbf{R}^q_{k+1} = (\mathbf{X}^*_{k+1})^T\mathbf{R}\mathbf{X}^*_{k+1} \qquad \mathbf{S}^q_{k+1} = (\mathbf{X}^*_{k+1})^T\mathbf{S}\mathbf{X}^*_{k+1}.$$

(3) Solve the projected $q \times q$ eigensystem

$$\mathbf{R}^q_{k+1}\mathbf{Q}_{k+1} = \mathbf{\Lambda}_{k+1}\mathbf{S}^q_{k+1}\mathbf{Q}_{k+1}.$$

(4) Compute improved approximations of q eigenvectors

$$\mathbf{X}_{k+1} = \mathbf{X}^*_{k+1}\mathbf{Q}_{k+1},$$

and repeat the steps (1)–(4) until convergence of the first p eigenvalues. Many theoretical and practical details concerning SIM can be found in references [10–12].

Clearly, the bottleneck of this simple method is its first step—repeated solution of systems of linear equations. The most effective way of how to solve repeatedly the linear system of equations $\mathbf{Ru} = \mathbf{b}_i$ with different right-hand sides $\mathbf{b}_i$ is to compute the Cholesky factorization $\mathbf{R} = \mathbf{LL}^T$, where the Cholesky factor $\mathbf{L}$ is a lower triangular matrix and, consequently, one can obtain the desired $\mathbf{u}_i$ by solving the triangular systems $\mathbf{Lw}_i = \mathbf{b}_i$ and $\mathbf{L}^T\mathbf{u}_i = \mathbf{w}_i$ with the same $\mathbf{L}$ for all i [13].

4. NUMERICAL RESULTS

The intention in this secton is to illustrate the convergence of the computed eigenvalues of some model problems with respect to the continuity ($\ell = 4, 6, 8$) and dimension ($n = 576, 1296, 2304$) of the used B-spline approximation. The matrices $\mathbf{R}$ and $\mathbf{S}$ of the generalized eigenvalue problem (6) are created by numerical integration using 20-point Gaussian quadrature on each subinterval $\langle x_i, x_{i+1}\rangle$ of the corresponding one-dimensional mesh. Because these matrices are of band structure, the Cholesky factor $\mathbf{L}$ of the matrix $\mathbf{R}$ can be computed by the LINPACK [14] subroutine DPBCO and, consequently, the LINPACK subroutine DPBSL solves the systems of linear equations needed in the first step of SIM.

4.1. *Clamped square plate*

The clamped square plate is one of the standard eigenvalue problems of interest for mathematicians and mechanical engineers and it deserves some remarks. The interest stems mainly from two phenomena—the stress singularities in angular points [6, 7, 15, 16] and the existence of nodal lines for the first eigenfunction [17–19].

As is known, the singular part sing (r, θ) (in the polar co-ordinates r, θ) of the asymptotic expansion of the clamped square plate eigenfunctions near the corner points has the form

$$\text{sing}\,(r,\theta) = \sum_{i=1}^{\infty} a_i \,\text{Re}\,\{r^{z_i} f_i(\theta)\}, \qquad \theta \in \langle 0, \pi/2\rangle,\ r \in \langle 0, \varepsilon\rangle,$$

where $f_i(\theta)$ are symmetric functions with respect to the axis of the corner angles for i odd and antisymmetric functions for i even. By virtue of the shapes of eigenfunctions (the graphical results of reference [18]) and the values of z_i($z_1 \approx 3{\cdot}74 + i1{\cdot}12$ and $z_2 \approx 5{\cdot}81 + i1{\cdot}47$) some eigenfunctions are smoother than others. For example, among the first 46 eigenfunctions plotted in reference [18] the eigenfunctions Φ_5, Φ_{12}, Φ_{16}, Φ_{21}, Φ_{27}, Φ_{32}, Φ_{34}, Φ_{38} and Φ_{45} are antisymmetric functions with respect to the axes of the corner angles and, consequently, because

TABLE 1

Values of the Rayleigh–Ritz approximations ω_i^n for the first five frequencies ω_i of the clamped square plate of the length a using the B-spline trial functions (7) *of order* $\ell = 4, 6, 8$ *and dimension* $n = 576, 1296, 2304$ ($n_1 = 24, 36, 48$); *note that* $\lambda_i = \omega_i^2 a^4 \rho / D$ *and* $a = \rho = D = 1$

ℓ	n_1	ω_1^n	$\omega_{2,3}^n$	ω_4^n	ω_5^n
4	24	35·98519140	73·3938486	108·2165096	131·58094
4	36	35·985191162	73·3938460	108·21650298	131·580799
4	48	35·985191129	73·39384567	108·21650206	131·580781
6	24	35·9851911162	73·3938455291	108·216501796	131·58077261762
6	36	35·98519111523	73·3938455242	108·2165016923	131·580772614364
6	48	35·98519111515	73·39384552392	108·2165016911	131·580772614307
8	24	35·98519111547	73·39384552496	108·2165016955	131·5807726143028
8	36	35·985191115149	73·39384552394	108·2165016912	131·5807726143026
8	48	35·985191115125	73·393845523863	108·21650169092	131·5807726143022

these functions do not contain the most singular term $r^{z_1} f_1(\theta)$, which is a symmetric function with respect to the axis of the corner angle, they are smoother than others.

The approximations of the first five circular frequencies of clamped square plate of the length a using the trial functions

$$\phi_{i,j}(x, y) = x^2(a-x)^2 y^2 (a-y)^2 B_i^{\ell}(x) B_j^{\ell}(y), \qquad i, j = 1, 2, \ldots, n_1 \tag{7}$$

are given in Table 1.

4.2. *Clamped skew plate*

The next examples to be solved are clamped skew plates with sharp boundary corners of magnitude $\pi/4$ (i.e., reentrant corners $3\pi/4$) and $\pi/12$ (i.e., reentrant corners $11\pi/12$). As is known [7, 8], the greater reentrant corners cause more singular behaviour of the corresponding eigenfunctions.

The usual approach in the solution of problems defined on a rhombus uses the following transformation

$$x = r - s \cos\kappa / \sin\kappa \qquad y = s/\sin\kappa,$$

which maps the rhombus (in the r, s plane) of side length a and sharp interior angle κ onto the square $P = [0, a] \times [0, a]$. Consequently, instead of the Laplace operator Δ in equation (2) defined on a complicated skew shape one has to work with the slightly more complicated operator

$$\frac{1}{\sin^2\kappa}\left[\frac{\partial^2\Phi}{\partial x^2} - 2\cos\kappa \frac{\partial^2\Phi}{\partial x\,\partial y} + \frac{\partial^2\Phi}{\partial y^2}\right]$$

defined on the square P. If either $U = 0$ or $\partial U/\partial \nu = 0$ on the boundary of the rhombus, then $\Phi = 0$ or $\partial\Phi/\partial\nu = 0$ on the boundary of P. The computations using the trial functions (7) presented in Table 2 ($\kappa = \pi/4$) and Table 3 ($\kappa = \pi/12$) very clearly demonstrate essentially slower convergence of the skew plate approximate

TABLE 2

Values of the Rayleigh–Ritz approximations ω_i^n for the first four frequencies ω_i of the clamped skew plate ($\kappa = \pi/4$) of the length a using the B-spline trial functions (7) of order $\ell = 4, 6, 8$ and dimension $n = 576, 1296, 2304$ ($n_1 = 24, 36, 48$); note that $\lambda_i = \omega_i^2 a^4 \rho / D$ and $a = \rho = D = 1$

ℓ	n_1	ω_1^n	ω_2^n	ω_3^n	ω_4^n
4	24	65·6432	106·494975	148·3131	157·2371
4	36	65·642874	106·4949166	148·3121	157·23467
4	48	65·642817	106·4949087	148·31192	157·23423
6	24	65·64284	106·494905563	148·31191	157·23435
6	36	65·6427966	106·49490555747	148·311865	157·234074
6	48	65·6427916	106·49490555717	148·311859	157·234036
8	24	65·64281	106·4949055580	148·311876	157·23414
8	36	65·6427917	106·49490555720	148·3118596	157·234037
8	48	65·642790233	106·494905557077	148·311857844	157·23402535

frequencies in comparison with the square plate case, although the skew problems have been solved as the corresponding transformed square problems.

5. CONCLUDING REMARKS

The computer program producing the presented results uses the simplest basic version of SIM without any improvements considered in references [10, 20–22]. In spite of this simplicity the number of SIM iterations was always less than 15 using the initial eigenvectors created as the Cartesian product of the eigenvector approximations of the beam equations with the end conditions corresponding to the boundary conditions of the solved plate problem. The matrices **R** and **S** are stored in CSR (Compressed Sparse Row) format [23] in which only the non-zero

TABLE 3

Values of the Rayleigh–Ritz approximations ω_i^n for the first four frequencies ω_i of the clamped skew plate ($\kappa = \pi/12$) of the length a using the B-spline trial functions (7) of order $\ell = 4, 6, 8$ and dimension $n = 576, 1296, 2304$ ($n_1 = 24, 36, 48$); note that $\lambda_i = \omega_i^2 a^4 \rho / D$ and $a = \rho = D = 1$

ℓ	n_1	ω_1^n	ω_2^n	ω_3^n	ω_4^n
4	24	408·48	522·86	627·59	742·72
4	36	407·71	520·98	621·18	726·73
4	48	407·53	520·72	620·20	724·23
6	24	407·56	520·63	619·92	723·41
6	36	407·445	520·6156	619·827	723·2846
6	48	407·417	520·6143	619·815	723·2820
8	24	407·488	520·6197	619·848	723·293
8	36	407·420	520·61447	619·8162	723·2821
8	48	407·4046	520·614112	619·810104	723·28163

elements are considered, while the Cholesky factor of **R** uses band storage format [14]. In practice one needs 1·6, 4·6, and 9·9 MB of the main memory to keep the corresponding three matrices for $n = 576$, 1296, and 2304 ($\ell = 8$ for each of n), respectively. This amount of main memory is certainly no problem for the majority of computer environments used in engineering design and analysis.

Comparisons between the results for square shape and skew shape indicate that owing to small regularity of the approximated eigenfunctions, accuracy of the eigenvalue approximations is essentially smaller for the skew shape. This may cause some misleading conclusions in solving such problems using global trial functions as algebraic and goniometric polynomials which are more sensitive to the regularity of approximated functions than locally supported trial functions. Therefore, the stagnation of convergence in solving some problems defined on sharp skew shapes (having big reentrant corners) need not signify that the results are of the desired accuracy. In such cases *a posteriori* error estimations [24] of the eigenvalue approximations may be helpful.

While the trial functions for solving the clamped plate problem can be built very simply, in the remaining cases it is necessary to build them as a linear combination of the neighbouring B-splines. For example, let us have to build trial functions for a plate free at the edge $x = 0$ and clamped at the edge $x = a$. The simple functions $\psi_i^\ell(x) = B_i^\ell(x)(a - x)^2$ satisfy clamped edge condition at $x = a$, while free edge conditions at $x = 0$ can be constructed as the free end conditions of a beam by linear combination of the neighbouring $\psi_i^\ell(x)$. If one selects quartic B-splines ($\ell = 5$), the corresponding mesh points $\{x_i\}$ in the surrounding of $x = 0$ are distributed as

$$x_1 < x_2 < x_3 < x_4 < x_5 \equiv 0 < x_6 < x_7 < \dots,$$

and $\psi_i^\ell(x)$ which have non-zero values at $x = 0$, are the ones for $i = 1, 2, 3, 4$. In this case, we can take

$$\phi_1(x) = \psi_1^5(x) + \alpha_1\psi_2^5(x) + \beta_1\psi_3^5(x), \qquad \phi_2(x) = \psi_2^5(x) + \alpha_2\psi_3^5(x) + \beta_2\psi_4^5(x),$$
$$\phi_3(x) = \psi_5^5(x), \qquad \phi_4(x) = \psi_6^5(x), \dots,$$

where the coefficients α_i and β_i are determined from the system of two linear equations $\phi_i''(0) = \phi_i'''(0) = 0$. This approach produces $(\ell - 3)$ end trial functions for every ℓ. Moreover, the trial functions for the Rayleigh–Ritz method must satisfy exactly only the geometric boundary conditions and the remaining ones may be ignored. This means that trial functions for a simply supported plate must satisfy only the condition $u = 0$ on $\partial\Omega$. The error estimations of the first six frequencies of a simply supported plate using the trial functions

$$\phi_{i,j}(x, y) = x(a - x)y(a - y)B_i^\ell(x)B_j^\ell(y), \qquad i, j = 1, 2, \dots, n_1 \tag{8}$$

satisfying only $u = 0$ on $\partial\Omega$ are reported in Table 4. These results indicate that no more than the last three figures are destroyed by round-off error.

Although the simplest variant of the subspace iteration method has performed very well in all the presented computations, one can meet with requirements to use a more efficient and robust method for solving generalized large sparse matrix

TABLE 4

Error estimations for the Rayleigh–Ritz approximations ω_i^n of the first five frequencies ω_i of the simply supported square plate using the B-spline trial functions (8) *of order* $\ell = 4, 6, 8$ *and dimension* $n = 576, 1296, 2304$ ($n_1 = 24, 36, 48$)

ℓ	n_1	$\|\omega_1^n - \omega_1\|$	$\|\omega_{2,3}^n - \omega_{2,3}\|$	$\|\omega_4^n - \omega_4\|$	$\|\omega_5^n - \omega_5\|$
4	24	0·77D-8	0·61D-5	0·12D-4	0·60D-3
4	36	0·13D-8	0·99D-6	0·20D-5	0·97D-4
4	48	0·37D-9	0·29D-6	0·57D-6	0·28D-4
6	24	0·57D-13	0·24D-10	0·49D-10	0·14D-7
6	36	0·68D-12	0·11D-11	0·17D-11	0·26D-9
6	48	0·34D-12	0·17D-12	0·40D-12	0·19D-10
8	24	0·23D-12	0·14D-12	0·17D-12	0·15D-11
8	36	0·17D-12	0·16D-12	0·18D-12	0·14D-13
8	48	0·80D-12	0·21D-12	0·11D-12	0·14D-13

eigenproblems. In this case the Lanczos method [25, 26], the Rayleigh quotient iteration method [27], and the implicitly restarted Arnoldi method [28] are very promising alternatives. There are three reliable FORTRAN packages freely available on the INTERNET

LANZ at http://www.netlib.org/lanz/

BLZPACK at http://www.nersc.gov/˜osni/

ARPACK at http://www.caam.rice.edu/software/ARPACK/.

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