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LETTERS TO THE EDITOR

COMMENTS ON "HARMONIC BALANCE AND CONTINUATION TECHNIQUES IN THE DYNAMIC ANALYSIS OF DUFFING'S EQUATION"

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In reference [1] Blair *et al*. applied a harmonic balance technique coupled with a continuation algorithm to study the dynamic response to changes in the amplitude of the applied harmonic force for Duffing's equation with a negative linear stiffness written as [2]

$$\ddot{x} + \gamma\dot{x} - (x/2)(1 - x^2) = F \sin \omega t. \tag{1}$$

The stability of the solutions was investigated by the Floquet theory. The harmonic balance technique is very efficient and it has the advantage of also discovering the unstable solutions. Among other interesting results, Blair *et al*. found new cascades of period doubling solutions ending in the limit in chaotic motion. With $\gamma = 0{\cdot}168$ and $\omega = 1$ an examination of the change in the Fourier coefficients of the solutions reveals the occurrence of several period doublings. Sequences of period doubling orbits represented in the phase plane have been illustrated in reference [1], e.g., for the cascade near $F = 0{\cdot}177$ in Figure 4(a), Figure 4(b) and Figure 4(c) representing $1T$, $2T$ and $4T$-solutions, respectively, and for the reverse cascade near $F = 0{\cdot}975$ in Figure 4(l), Figure 4(m) and Figure 4(n) illustrating $4T$, $2T$ and $1T$-solutions. The cascade near $F = 0{\cdot}975$ has not been found previously. In reference [1] it is cited that the solutions with the periods higher than $4T$ become computationally difficult to obtain since many Fourier coefficients have to be retained in the solution.

In this letter the bifurcation diagram is established for the two cascades of period doubling solutions mentioned above thus confirming the results of reference [1] related to this matter. In addition, by the use of a continuation technique based on the shooting method, it is illustrated that the solutions having the higher periods can also be readily obtained and that the distances between two consecutive transition values for F in the bifurcation tree satisfy Feigenbaum's relation [3] from Universality Theory.

With $x_1 = x$, $x_2 = \dot{x}$, equation (1) is rewritten as:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = (x_1/2)(1 - x_1^2) - \gamma x_2 + F \sin \omega t. \tag{2}$$

This system of differential equations is integrated by the use of the Runge–Kutta–Hŭta method of order six [4], which is a very accurate scheme. With

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the transient regime omitted, the Poincaré section point for x_1 at $t = 0$ is plotted with sampling period $T = 2\pi/\omega$ in terms of the parameter F. Figure 1 shows the bifurcation diagram with the first period doubling from the asymmetric $1T$-solution appearing near $F = 0{\cdot}177$. Four transitions are readily seen in this figure. In the limit, chaotic behavior is observed in the vicinity of $F = 0{\cdot}205$. The reverse cascade near $F = 0{\cdot}975$ is represented in Figure 2. These bifurcation diagrams suggest some regularity for the distances between the transition values for F.

A more precise computation of the transition values is investigated by the use of the continuation technique based on the shooting method [5–7]. Equation (2) is written in the form

$$\dot{x} = X(x, t), \tag{3}$$

with x two-dimensional and in which X is periodic with period $T = 2\pi/\omega$. One can look for a P-periodic solution of equation (3). In the period doubling cascade one alternatively chooses $P = 1T$, $P = 2T$, $P = 4T, \ldots$. One takes a starting point x_0 corresponding with $t = 0$. In the shooting method the correction vector $\Delta x_0 = x_{new} - x_0$, has to satisfy the system of linear equations

$$[I - A(P)]\Delta x_0 = e_0, \tag{4}$$

where e_0 is the error at the end of the numerical integration of equation (3) for $t = P$:

$$e_0 = x(P) - x_0. \tag{5}$$

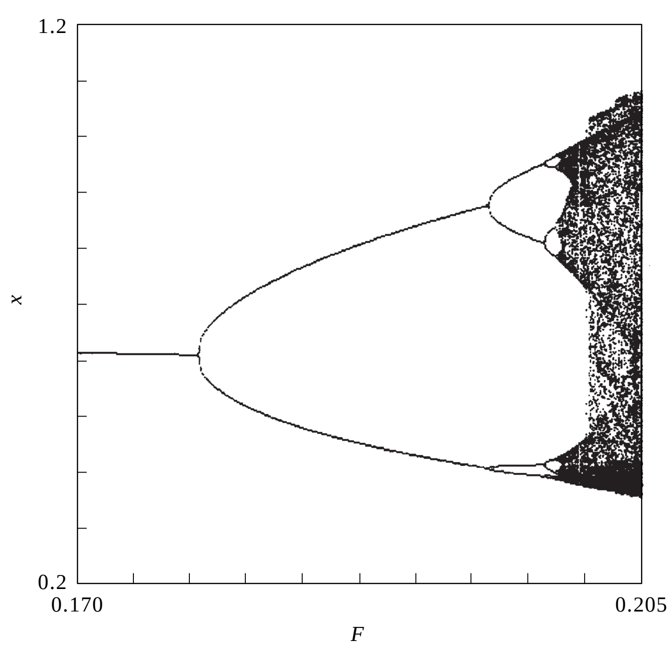

Figure 1. Period doubling bifurcations near $F = 0{\cdot}177$ with $\gamma = 0{\cdot}168$ and $\omega = 1$.

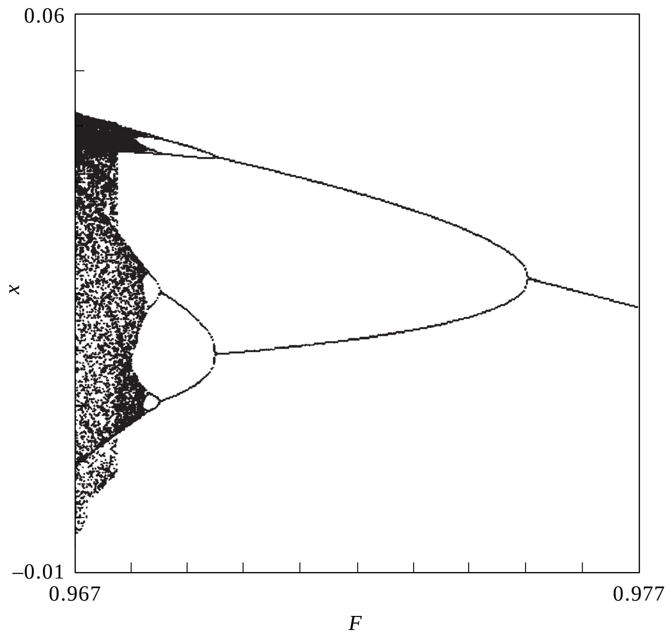

Figure 2. Period doubling bifurcations near $F = 0{\cdot}975$ with $\gamma = 0{\cdot}168$ and $\omega = 1$.

I is the identity matrix and $A(P)$ is the fundamental matrix of the system of the first variational equations derived from equation (3) with respect to the reference solution $x(t, x_0)$:

$$\dot{y} = X_x[x(t, x_0), t]y, \qquad (6)$$

with $A(0) = I$ and where X_x denotes the relevant partial derivative.

Equation (4) is used now in an iterative manner. In each iteration one has to solve the linear system for the corrections Δx_0, thus determining the ameliorated value x_{new}. This is continued until numerical convergence of the iterative method is reached. The suggested technique allows one to compute the stable as well as the unstable solutions. Stable periodic solutions correspond to eigenvalues of $A(P)$ which are lying inside the unit circle. At the transition from a stable iT-solution ($i = 1, 2, 4, \ldots$) to an unstable iT-solution one of the eigenvalues of $A(iT)$ leaves the unit circle along the real axis at the value -1. The passage through the value -1 is computed with high accuracy by applying polynomial interpolation and using a few additional calculations with small changes of F near the transition value. By repeated use of this procedure each transition in the sequence $1T \to 2T \to 4T \to 8T \to 16T \ldots$ is computed with high precision.

Table 1 gives the results in the cascade near $F = 0{\cdot}177$. Listed values are the transition value for F, one of the Poincaré section points in the phase plane (the values x_1 and x_2 at $t = 0$) and the numbers δ_i defined from the transition values F_i as

$$\delta_i = \Delta F_i / \Delta F_{2i}, \qquad (7)$$

TABLE 1

The transition values F_i, the initial conditions x_1, x_2 and the numbers δ_i in the cascade near $F = 0{\cdot}177$

Transition	F_i	x_1	x_2	δ_i
$1T\rightarrow 2T$	0·177441472	0·608041	−0·341204	
$2T\rightarrow 4T$	0·195484923	0·871826	−0·327403	
$4T\rightarrow 8T$	0·198951258	0·945554	−0·290570	5·2053
$8T\rightarrow 16T$	0·199691674	0·959765	−0·282158	4·6816
$16T\rightarrow 32T$	0·199851192	0·962830	−0·279694	4·6416
$32T\rightarrow 64T$	0·199885412	0·962942	−0·279478	4·6615

with $\Delta F_i = F_i - F_{2i}$ and $i = 1, 2, 4, 8, \ldots$. From the last column in Table 1 it is seen that the numbers δ_i numerically tend to Feigenbaum's number $\delta = 4{\cdot}6692\ldots$ from Universality Theory [3]. At the limit of the sequence of the transition values the bevavior of the system becomes chaotic. Table 2 illustrates that similar results and conclusions hold for the cascade of period doubling solutions near $F = 0{\cdot}975$, thus confirming Feigenbaum's relation.

The orbits for the period doubling cascade with its characteristics given in Table 1, encircle the point $x_1 = 1$, $x_2 = 0$ in the phase plane. Note that there exists an analogous cascade of period doubling solutions for which the orbits encircle the point $x_1 = -1$, $x_2 = 0$. This mirrored cascade is characterized by the same transition values for F as those given in Table 1. The initial conditions at $t = 0$ for, e.g., the first transition $1T\rightarrow 2T$ at $F = 0{\cdot}177441$ are $x_1 = -1{\cdot}001778$ and $x_2 = -0{\cdot}470825$. Similarly, one finds a mirrored bifurcation tree with respect to the one with its characteristics listed in Table 2. The first transition occurs at $F = 0{\cdot}975036$ with $x_1 = -0{\cdot}233146$ and $x_2 = -0{\cdot}769908$.

The solutions with the higher periods, which are difficult to find by the harmonic balance method, are readily obtained by the continuation technique combined with the shooting method. A typical orbit is illustrated in Figure 3 representing the $8T$-solution at the transition value $F = 0{\cdot}199692$ in the first cascade. The relevant $1T$, $2T$ and $4T$-solutions in this cascade have been illustrated in Figure 4 in reference [1].

TABLE 2

The transition values F_i, the initial conditions x_1, x_2 and the numbers δ_i in the cascade near $F = 0{\cdot}975$

Transition	F_i	x_1	x_2	δ_i
$1T\rightarrow 2T$	0·975036326	0·0267011	−0·747650	
$2T\rightarrow 4T$	0·969514162	0·0420834	−0·760360	
$4T\rightarrow 8T$	0·968511396	0·0444446	−0·757648	5·5069
$8T\rightarrow 16T$	0·968299488	0·0449224	−0·756598	4·7321
$16T\rightarrow 32T$	0·968254025	0·0450303	−0·756378	4·6610
$32T\rightarrow 64T$	0·968244279	0·0450472	−0·756393	4·6649

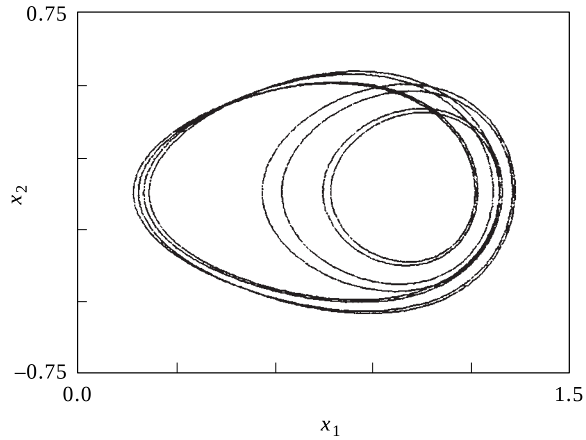

Figure 3. The 8T-orbit in the phase plane at the transition value F = 0·199692 with its characteristics given in Table 1.

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