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TRANSVERSE VIBRATIONS OF A CIRCULAR PLATE OF RECTANGULAR ORTHOTROPY WITH A CONCENTRIC CIRCULAR SUPPORT AND CARRYING A CONCENTRATED MASS

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1. INTRODUCTION

A recent publication [1] dealt with vibrating clamped and simply supported circular plates of rectangular orthotropy carrying a central, concentrated mass.

The present letter is an extension of the previously quoted paper. A concentric, circular support is provided in order to increase the stiffness of the system; see Figure 1. As the radius of the concentric support, R_o, approaches zero, the structural system degenerates into the central point support situation for which an exact solution is available in the case of isotropic plates. The fundamental eigenvalues for clamped and simply supported isotropic plates obtained in the present study are in good agreement with those determined exactly in terms of Bessel's functions [2]. On the other hand, an exact solution seems out of the question in the case of circular plates of rectangular orthotropy.

The optimized Rayleigh–Ritz method is used whereby three exponential optimization parameters are employed, extending the procedure developed in reference [3] where two optimization exponents were used.

2. APPROXIMATE DETERMINATION OF THE FUNDAMENTAL FREQUENCY COEFFICIENT

Using Lekhnitskii's well known notation [4], the maximum strain energy is expressed in the form

$$U_{max} = \frac{1}{2}\int\int\left[D_1\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + 2D_1\nu_2\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + D_2\left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 4D_k\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y, \qquad (1)$$

while the maximum kinetic energy is given by

$$T_{max} = \frac{1}{2}\left[\rho\omega^2 h\int\int W^2\,\mathrm{d}x\,\mathrm{d}y + M\omega^2 W|^2_{x=y=0}\right]. \qquad (2)$$

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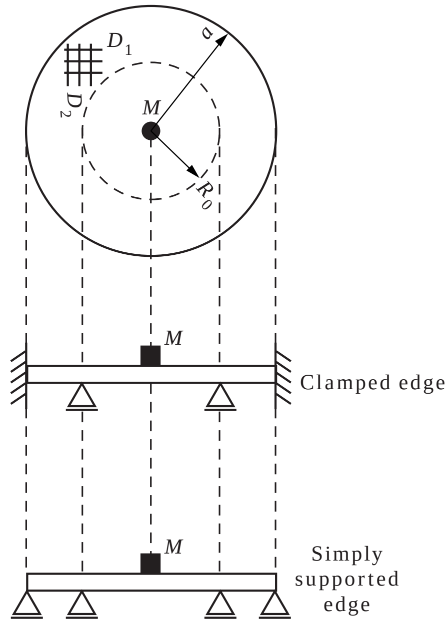

Figure 1. Structural system executing transverse vibrations, considered in the present study.

The energy functional is then defined as

$$J[W] = U_{max} - T_{max}. \quad (3)$$

The displacement amplitude is approximated by means of

$$W \cong W_a = C_1(\alpha_p r^p + \alpha_q r^q + \alpha_s r^s + 1) + C_2(\beta_p r^{p+1} + \beta_q r^{q+1} + \beta_s r^{s+1} + 1), \quad (4)$$

where p, q and s are Rayleigh's optimization parameters and α's and β's are determined substituting each co-ordinate function into

$$W(a) = \frac{\mathrm{d}W}{\mathrm{d}r}(a) = W(R_o) = 0. \quad (5a)$$

in the case of a clamped edge, Figure 1, and into

$$W(a) = \left(\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + \frac{\nu_2}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right)\bigg|_{r=a} = W(R_o) = 0 \quad (5b)$$

when the plate is simply supported. Clearly, the natural boundary conditions are not satisfied in the case of rectangular orthotropy but when $\nu_2 = \nu$ one has the isotropic situation and in this circumstance the polynomial co-ordinate functions expressed in equation (4) yield excellent accuracy [5].

Substituting equation (4) into equations (1), (2) and (3) and applying the classical Rayleigh–Ritz method

$$\frac{\partial J[W]}{\partial C_i} = \frac{\partial U_{max}}{\partial C_i} - \frac{\partial T_{max}}{\partial C_i} = 0, \quad (6)$$

TABLE 1

Values of the frequency coefficient $\Omega_1 = \sqrt{\rho h/D}\omega_1 a^2$ for an isotropic circular plate with an intermediate circular support and a central, concentrated mass M: concentrated mass; M_p: plate mass; $\nu = 0{\cdot}30$

	M/M_p		R_o/a →0	0·2	0·4	0·6	0·8	→1
Clamped	0	(a)	22·963	30·297	39·838	23·956	14·618	11·504
		(b)	22·738	29·612	39·348	23·123	14·455	10·218
		(c)	4·6–2·7	3·6–3·4	4·3–4·0	5·5–5·1	13·3	21·400
			1·7	3·1	2·2	2·2	7·9–2·0	3·9–2·1
	0·25	(a)	22·963	26·355	15·966	10·869	8·209	7·044
		(b)	22·738	25·878	15·816	10·763	8·083	6·424
		(c)	4·6–2·7	3·5–2·2	3·3–3·2	5·3–4·9	19·0	20·900
			1·7	1·9	1·8	1·7	6·3–1·7	2·7–1·8
	0·50	(a)	22·963	22·792	11·677	8·082	6·273	5·474
		(b)	22·738	20·731	11·533	8·009	6·172	5·022
		(c)	4·6–2·7	2·5–2·2	3·4–3·1	6·4–4·0	18·1	21·200
			1·7	1·9	1·8	1·7	6·3–1·6	2·5–1·8
	0·75	(a)	22·963	20·120	9·646	6·717	5·268	4·628
		(b)	22·738	17·450	9·515	6·658	5·181	4·257
		(c)	4·6–2·7	2·4–2·1	3·4–3·1	6·7–3·8	19·5	21·200
			1·7	2·0	1·8	1·7	6·0–1·6	2·4–1·8
	1	(a)	22·963	18·136	8·403	5·870	4·629	4·082
		(b)	22·738	15·315	8·284	5·819	4·551	3·759
		(c)	4·6–2·7	2·4–2·1	3·4–3·0	6·9–3·7	20·2	21·400
			1·7	2·0	1·8	1·7	5·8–1·6	2·4–1·8
Simply supported	0	(a)	14·973	19·428	29·054	22·508	14·138	11·504
		(b)	14·816	19·134	28·769	21·906	14·047	10·218
		(c)	5·9–2·4	3·6–3·3	6·2–5·4	5·3–4·9	12·3	21·400
			1·8	2·9	2·1	2·2	7·6–2·2	3·9–2·1
	0·25	(a)	14·973	18·411	15·210	10·504	8·007	7·044
		(b)	14·816	18·386	15·030	10·430	7·926	6·424
		(c)	5·9–2·4	4·4–3·9	3·2–3·1	5·1–4·8	18·9	20·900
			1·8	1·8	1·8	1·7	6·0–1·7	2·7–1·8
	0·50	(a)	14·973	17·327	11·260	7·833	6·130	5·474
		(b)	14·816	17·103	11·084	7·780	6·063	5·022
		(c)	5·9–2·4	4·4–2·3	3·3–3·0	6·1–3·9	17·5	21·200
			1·8	1·8	1·8	1·7	6·0–1·6	2·5–1·8
	0·75	(a)	14·973	16·267	9·337	6·517	5·152	4·628
		(b)	14·816	15·518	9·176	6·473	5·093	4·257
		(c)	5·9–2·4	2·4–2·1	3·3–3·0	6·4–3·7	19·4	21·200
			1·8	2·0	1·8	1·7	5·7–1·6	2·4–1·8
	1	(a)	14·973	15·287	8·149	5·699	4·529	4·082
		(b)	14·816	14·086	8·001	5·659	4·475	3·759
		(c)	5·9–2·4	2·4–2·1	3·3–2·9	6·6–3·6	20·4	21·400
			1·8	2·0	1·8	1·7	5·5–1·6	2·4–1·8

(a) Eigenvalue determined taking $p = 4$, $q = 3$ and $s = 2$; (b) eigenvalue determined by optimization with respect to p, q and s; (c) optimum values of p, q and s.

Table 2

Values of the frequency coefficient $\Omega_1 = \sqrt{\rho h/D}\omega_1 a^2$ for an orthotropic circular plate with an intermediate circular support and a central, concentrated mass M: concentrated mass; M_p: plate mass; $D_2/D_1 = 1/2$; $D_k/D_1 = 1/2$; $\nu_2 = 0{\cdot}30$

			R_o/a					
	M/M_p		→0	0·2	0·4	0·6	0·8	→1
Clamped	0	(a)	21·633	28·542	37·530	22·569	13·771	10·837
		(b)	21·421	27·898	37·070	21·784	13·617	9·626
		(c)	4·6–2·7	3·8–3·5	4·4–3·9	5·5–5·1	14·0	21·400
			1·7	2·9	2·2	2·2	7·6–2·2	3·9–2·1
	0·25	(a)	21·633	24·828	15·041	10·239	7·734	6·636
		(b)	21·421	24·379	14·900	10·139	7·615	6·052
		(c)	4·6–2·7	3·5–2·2	3·3–3·2	5·3–4·9	19·0	20·900
			1·7	1·9	1·8	1·7	6·3–1·7	2·7–1·8
	0·50	(a)	21·633	21·472	11·001	7·614	5·910	5·257
		(b)	21·421	19·530	10·865	7·545	5·814	4·731
		(c)	4·6–2·7	2·5–2·2	3·4–3·1	6·4–4·0	18·1	21·200
			1·7	1·9	1·8	1·7	6·3–1·6	2·5–1·8
	0·75	(a)	21·633	18·955	9·087	6·328	4·963	4·360
		(b)	21·421	16·440	8·964	6·272	4·881	4·010
		(c)	4·6–2·7	2·3–2·1	3·4–3·1	6·7–3·8	19·5	21·200
			1·7	2·0	1·8	1·7	6·0–1·6	2·4–1·8
	1	(a)	21·633	17·086	7·916	5·530	4·361	3·845
		(b)	21·421	14·428	7·804	5·481	4·288	3·541
		(c)	4·6–2·7	2·3–2·1	3·1–3·0	6·9–3·9	20·2	21·400
			1·7	2·0	1·8	1·7	5·8–1·6	2·4–1·8
Simply supported	0	(a)	13·960	18·115	27·176	21·169	13·314	10·837
		(b)	13·806	17·858	26·935	20·621	13·231	9·625
		(c)	4·9–2·4	3·9–3·3	6·4–5·3	5·4–4·8	12·5	21·400
			1·8	1·7	2·1	2·2	7·5–2·0	3·9–2·1
	0·25	(a)	13·960	17·190	14·313	9·889	7·542	6·636
		(b)	13·806	17·171	14·151	9·821	7·466	6·052
		(c)	4·9–2·4	4·4–3·9	3·2–3·1	3·4–3·0	18·7	20·900
			1·8	1·8	1·8	1·8	6·1–1·7	2·7–1·8
	0·50	(a)	13·960	16·203	10·600	7·375	5·774	5·157
		(b)	13·806	16·002	10·433	7·326	5·711	4·731
		(c)	4·9–2·4	4·3–2·3	3·3–3·0	6·0–3·9	17·5	21·200
			1·8	1·8	1·8	1·7	6·0–1·6	2·5–1·8
	0·75	(a)	13·960	15·231	8·790	6·136	4·853	4·360
		(b)	13·806	14·552	8·637	6·095	4·797	4·010
		(c)	4·9–2·4	2·4–2·1	3·3–2·3	6·3–3·7	17·4	21·200
			1·8	2·0	1·8	1·7	5·5–1·6	2·4–1·8
	1	(a)	13·960	14·330	7·672	5·366	4·266	3·845
		(b)	13·806	13·229	7·532	5·329	4·216	3·541
		(c)	4·9–2·4	2·3–2·1	3·3–2·9	6·5–3·6	20·4	21·400
			1·8	2·0	1·8	1·7	5·5–1·6	2·4–1·8

(a) Eigenvalue determined taking $p = 4$, $q = 3$ and $s = 2$; (b) eigenvalue determined by optimization with respect to p, q and s; (c) optimum values of p, q and s.

TABLE 3

Values of the frequency coefficient $\Omega_1 = \sqrt{\rho h/D}\omega_1 a^2$ for an orthotropic circular plate with an intermediate circular support and a central, concentrated mass M: concentrated mass; M_p: plate mass; $D_2/D_1 = 1$; $D_k/D_1 = 1/2$; $\nu_2 = 0{\cdot}30$

			R_o/a					
	M/M_p		→0	0·2	0·4	0·6	0·8	→1
Clamped	0	(a)	23·809	31·413	41·305	24·838	15·156	11·927
		(b)	23·575	30·703	40·798	23·975	14·986	10·594
		(c)	4·6–2·6	3·8–3·5	4·3–3·9	5·5–5·1	14·0	21·400
			1·7	2·9	2·2	2·2	7·6–2·0	3·9–2·1
	0·25	(a)	23·809	27·325	16·554	11·269	8·511	7·303
		(b)	23·575	26·831	16·399	11·159	8·380	6·660
		(c)	4·6–2·6	3·5–2·2	3·3–3·2	5·3–4·9	19·1	20·900
			1·7	1·9	1·8	1·7	6·2–1·7	2·7–1·8
	0·50	(a)	23·809	23·631	12·107	8·380	6·504	5·676
		(b)	23·575	21·494	11·957	8·304	6·399	5·207
		(c)	4·6–2·6	2·5–2·2	3·4–3·1	6·4–4·0	18·1	21·200
			1·7	1·9	1·8	1·7	6·3–1·6	2·5–1·8
	0·75	(a)	23·809	20·861	10·001	6·964	5·463	4·799
		(b)	23·575	18·093	9·866	6·903	5·372	4·413
		(c)	4·6–2·6	2·4–2·1	3·4–3·1	6·7–3·8	19·5	21·200
			1·7	2·0	1·8	1·7	6·0–1·6	2·4–1·8
	1	(a)	23·809	18·804	8·712	6·066	4·800	4·232
		(b)	23·575	15·879	8·589	6·033	4·719	3·897
		(c)	4·6–2·6	2·3–2·1	3·4–3·0	6·9–3·7	20·2	21·400
			1·7	2·0	1·8	1·7	5·8–1·6	2·4–1·8
Simply supported	0	(a)	15·400	19·984	29·957	23·307	14·655	11·927
		(b)	15·232	19·695	29·635	22·699	14·562	10·594
		(c)	4·9–2·8	3·6–3·3	6·1–5·5	5·3–4·9	12·5	21·400
			1·7	2·9	2·1	2·3	7·5–2·0	3·9–2·1
	0·25	(a)	15·400	18·957	15·757	10·886	8·301	7·303
		(b)	15·232	18·935	15·578	10·810	8·217	6·660
		(c)	4·9–2·8	4·4–3·9	3·2–3·1	5·1–4·7	18·60	20·900
			1·7	1·8	1·8	1·7	6·1–1·7	2·7–1·8
	0·50	(a)	15·400	17·863	11·668	8·118	6·355	5·676
		(b)	15·232	17·639	11·485	8·064	6·286	5·207
		(c)	4·9–2·8	4·4–2·3	3·3–3·0	6·0–3·9	17·5	21·200
			1·7	1·8	1·8	1·7	6·0–1·6	2·5–1·8
	0·75	(a)	15·400	16·786	9·676	6·755	5·341	4·799
		(b)	15·232	16·033	9·507	6·709	5·280	4·413
		(c)	4·9–2·8	2·4–2·1	3·3–2·9	6·3–3·7	19·4	21·200
			1·7	2·0	1·8	1·7	5·7–1·6	2·4–1·8
	1	(a)	15·400	15·789	8·445	5·906	4·695	4·232
		(b)	15·232	14·570	8·304	5·866	4·640	3·897
		(c)	4·9–2·8	2·3–2·1	4·1–2·8	6·5–3·6	20·4	21·400
			1·7	2·0	1·7	1·7	5·5–1·6	2·4–1·8

(a) Eigenvalue determined taking $p = 4$, $q = 3$ and $s = 2$; (b) eigenvalue determined by optimization with respect to p, q and s; (c) optimum values of p, q and s.

TABLE 4

Values of the frequency coefficient $\Omega_1 = \sqrt{\rho h/D}\omega_1 a^2$ *for an orthotropic circular plate with an intermediate circular support and a central, concentrated mass* M*: concentrated mass;* M_p*: plate mass;* $D_2/D_1 = 1/2$*;* $D_k/D_1 = 1/2$*;* $\nu_2 = 1/3$

	M/M_p		R_o/a →0	0·2	0·4	0·6	0·8	→1
Clamped	0	(a)	20·699	27·309	35·909	21·594	13·176	10·369
		(b)	20·496	26·694	35·469	20·843	13·029	9·210
		(c)	4·6–2·7	4·0–3·4	4·3–3·9	5·6–5·0	14·0	21·400
			1·7	2·8	2·2	2·2	7·6–2·0	3·9–2·1
	0·25	(a)	20·699	23·756	14·392	9·797	7·400	6·349
		(b)	20·496	23·326	14·257	9·701	7·286	5·790
		(c)	4·6–2·7	3·5–2·2	3·5–3·1	5·3–4·9	19·0	20·900
			1·7	1·9	1·8	1·7	6·3–1·7	2·7–1·8
	0·50	(a)	20·699	20·545	10·526	7·285	5·655	4·934
		(b)	20·496	18·687	10·397	7·219	5·563	4·527
		(c)	4·6–2·7	2·5–2·2	3·4–3·1	6·4–4·0	18·1	21·200
			1·7	1·9	1·8	1·7	6·3–1·6	2·5–1·8
	0·75	(a)	20·699	18·136	8·694	6·055	4·749	4·172
		(b)	20·496	15·730	8·577	6·001	4·670	3·837
		(c)	4·6–2·7	2·4–2·1	3·4–3·1	6·7–3·8	19·5	21·200
			1·7	2·0	1·8	1·7	6·0–1·6	2·4–1·8
	1	(a)	20·699	16·348	7·574	5·291	4·173	3·679
		(b)	20·496	13·805	7·467	5·245	4·103	3·388
		(c)	4·6–2·7	2·3–2·1	3·4–3·0	6·9–3·7	20·2	21·400
			1·7	2·0	1·8	1·7	5·8–1·6	2·4–1·8
Simply supported	0	(a)	13·537	17·559	26·235	20·294	12·744	10·369
		(b)	13·395	17·294	25·978	19·751	12·663	9·210
		(c)	4·9–2·4	4·0–3·3	6·1–5·5	5·4–4·8	12·3	21·400
			1·8	2·7	2·1	2·2	7·6–2·0	2·1
	0·25	(a)	13·537	16·635	13·715	9·470	7·218	6·349
		(b)	13·395	16·613	13·562	9·403	7·145	5·790
		(c)	4·9–2·4	4·4–3·9	3·4–3·0	5·1–4·8	18·7	20·900
			1·8	1·8	1·8	1·7	6·1–1·7	3·9–2·1
	0·50	(a)	13·537	15·652	10·152	7·062	5·526	4·934
		(b)	13·395	15·446	9·994	7·013	5·465	4·527
		(c)	4·9–2·4	4·3–2·3	3·3–3·0	6·1–3·9	17·5	21·200
			1·8	1·8	1·8	1·7	6·0–1·6	2·5–1·8
	0·75	(a)	13·537	14·690	8·418	5·875	4·644	4·172
		(b)	13·395	14·006	8·273	5·835	4·591	3·837
		(c)	4·9–2·4	2·4–2·1	3·5–2·8	6·4–3·7	19·4	21·200
			1·8	2·0	1·7	1·7	5·7–1·6	2·4–1·8
	1	(a)	13·537	13·802	7·347	5·137	4·083	3·679
		(b)	13·395	12·709	7·214	5·102	4·034	3·388
		(c)	4·9–2·4	2·3–2·1	3·5–2·8	6·6–3·6	20·4	21·400
			1·8	2·0	1·7	1·7	5·5–1·6	2·4–1·8

(a) Eigenvalue determined taking $p = 4$, $q = 3$ and $s = 2$; (b) eigenvalue determined by optimization with respect to p, q and s; (c) optimum values of p, q and s.

one obtains a linear system of homogeneous equation in C_1 and C_2. The non-triviality conditions yield a determinantal equation whose lowest root constitutes the fundamental frequency coefficient $\Omega_1 = \sqrt{\rho h / D_1}\omega_1 a^2$. Obviously, the fundamental mode shape must be defined as a quasi-axisymmetric shape in view of the orthotropy of the material (the fundamental mode will be completely independent of the azimuthal variable only in the case of an isotropic plate). On the other hand, the existence of the circular inner support contributes to a higher degree of radial symmetry than in the case of reference [1].

Since, in view of equation (4),

$$\Omega_1 = \Omega_1(p, q, s), \tag{7}$$

by minimizing equation (7) with respect to p, q and s one is able to optimize the frequency coefficient.

3. NUMERICAL RESULTS

Frequency coefficients have been determined for isotropic plates ($\nu = 0{\cdot}30$), Table 1, and for orthotropic plates: (1) $D_2/D_1 = 1/2$, $D_k/D_1 = 1/2$, ($\nu_2 = 0{\cdot}30$), Table 2; (2) $D_2/D_1 = 1$, $D_k/D_1 = 1/2$, ($\nu_2 = 0{\cdot}30$), Table 3; (3) $D_2/D_1 = 1/2$, $D_k/D_1 = 1/3$, ($\nu_2 = 1/3$), Table 4. The values of the exponential parameters which minimize the fundamental eigenvalue are also indicated in the tables. The procedure is essentially a non-linear optimization process but it has been performed by a simple numerical searching process.

As $R_o \rightarrow a$, one immediately notices the fact that the optimized eigenvalue corresponds to very large values of the parameter p and when this situation occurs, the plate behaves as clamped.

In the case of a clamped, isotropic plate with a central point support one has $\Omega_1 = 22{\cdot}7$ and when the plate is simply supported $\Omega_1 = 14{\cdot}8$, where Poisson's ratio has been taken equal to 0·30 [2]. The present study yields 22·738 and 14·816, respectively, for $R_o/a \rightarrow 0$.

In each table the eigenvalues has been listed as a function of R_o/a and M/M_p, where M_p is the plate mass.

Also shown in each case, and in the first line, is the eigenvalue obtained using the classical Rayleigh–Ritz method with fixed values of the exponents appearing in the co-ordinate functions. One can notice the fact that in some instances the eigenvalues are lowered considerably by means of the optimization process.

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