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NUMERICAL EXPERIMENTS ON VIBRATING CIRCULAR PLATES OF RECTANGULAR ORTHOTROPY AND CARRYING A CENTRAL, CONCENTRATED MASS

P. A. A. LAURA AND E. ROMANELLI

Institute of Applied Mechanics (CONICET) and Department of Engineering, Universidad Nacional del Sur 8000 *Bahía Blanca, Argentina*

AND

V. SONZOGNI AND S. IDELSOHN

INTEC (UNL–CONICET), 3000 *Santa Fé, Argentina*

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1. INTRODUCTION

As stated in reference [1], "in the case of an isotropic circular plate of uniform thickness, many basic dynamic problems are solved in a classical fashion using Bessel functions". This is not the case when dealing with vibrating circular plates of rectangular orthotropy, the problem being of considerable technological importance in view of the continuously increasing use of composites and also of the orthotropic constitutive characteristics originated by metallurgical processes.

A recent study [1] dealt with clamped and simply supported circular plates carrying a central, concentrated mass using polynomial approximations and the straight Rayleigh–Ritz method.

The present publication deals with the solution of the title problem using the optimized Rayleigh–Ritz method. Polynomial co-ordinate functions which contain two independent optimization parameters are used following the approach developed by Grossi *et al*. [2]. The co-ordinate functions are expressed in terms of the radial variable neglecting the azimuthal dependence and the first two natural frequency coefficients corresponding to quasi-axisymmetric modes are obtained. In the case of isotropic plates the modes under investigation are axisymmetric and the results obtained are in excellent agreement with those available in the open literature [3].

An independent solution is also obtained using a very efficient and accurate finite element code [4]. Good agreement with the analytical predictions is shown to exist.

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2. APPROXIMATE ANALYTICAL SOLUTION

Using Lekhnitskii's well known notation [5] one expresses the maximum strain energy in the form

$$U_{max} = \frac{1}{2}\int\int\left[D_1\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + 2D_1\nu_2\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + D_2\left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 4D_k\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y, \qquad (1)$$

while the maximum kinetic energy is given by

$$T_{max} = \frac{1}{2}\left[\rho\omega^2 h\int\int W^2\,\mathrm{d}x\,\mathrm{d}y + M\omega^2 W|^2_{x=y=0}\right]. \qquad (2)$$

The energy functional is then defined as

$$J[W] = U_{max} - T_{max}. \qquad (3)$$

The displacement amplitude is approximated by means of

$$W \cong W_a = C_1[\alpha_p r^p + \alpha_q r^q + 1] + C_2[\beta_q r^{p+1} + \beta_q r^{q+1} + 1], \qquad (4)$$

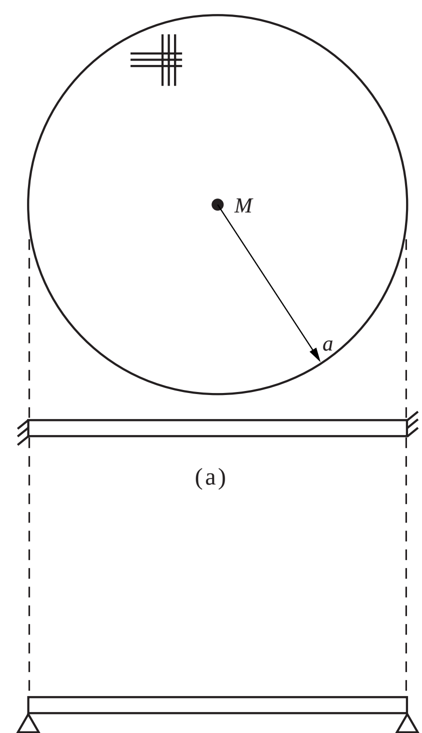

Figure 1. Vibrating system under study: (a) clamped edge, (b) simply supported edge.

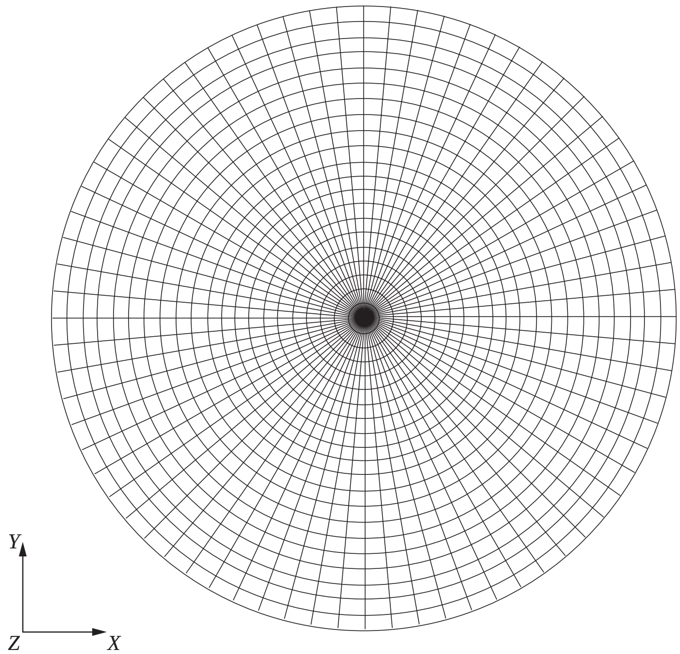

Figure 2. Finite element mesh.

where the α's and β's are determined substituting each co-ordinate function into

$$W(a) = \frac{\mathrm{d}W}{\mathrm{d}r}(a) = 0 \tag{5a}$$

in the case of a clamped edge, Figure 1, and into

$$W(a) = \left(\frac{\mathrm{d}^2W}{\mathrm{d}r^2} + \frac{\nu_2}{r}\frac{\mathrm{d}W}{\mathrm{d}r}\right)\Bigg|_{r=a} = 0 \tag{5b}$$

when the plate is simply supported. Clearly, the natural boundary condition is not satisfied in the case of rectangular orthotropy but when $\nu_2 = \nu$ one has the isotropic situation and in this circumstance the polynomial co-ordinate functions expressed in equation (4) yield excellent accuracy [6].

Substituting equation (4) in equations (1), (2) and (3) and applying the classical Rayleigh–Ritz method

$$\frac{\partial J[W]}{\partial C_i} = \frac{\partial U_{max}}{\partial C_i} - \frac{\partial T_{max}}{\partial C_i} = 0, \tag{6}$$

one obtains a linear system of homogeneous equations in C_1 and C_2. The non-triviality condition yields a bi-quadratic equation in Ω_{01} and Ω_{02} which are the frequency coefficients, $\Omega_{oi} = \sqrt{\rho h / D_a}\,\omega_{oi} a^2$, corresponding to quasi-axisymmetric modes†.

† The modes are axisymmetric only in the case of isotropic constitutive relations.

Since

$$\Omega_{oi} = \Omega_{oi}(p, q), \tag{7}$$

by minimizing equation (7) with respect to "p" and "q" one is able to optimize the frequency coefficients.

3. FINITE ELEMENT DETERMINATIONS

SAMCEF finite element code [4] has been used in the present investigation in order to obtain an independent solution. Hybrid plate elements of triangular and rectangular shape (Element Types 55 and 56 of the SAMCEF Library) have been used. They are displacement based elements which superimpose a Marguerre membrane and a hybrid plate, and allow one to model thin shells or plates. For

TABLE 1

The two lower natural frequency coefficients (axisymmetric modes) of an isotropic circular plate with a central, concentrated mass ($\nu = 0{\cdot}30$)

		Values of Ω_{01}			Values of Ω_{02}		
		Present study Analytical	Present study FE values	Exact [3]	Present study Analytical	Present study FE values	Exact [3]
Clamped	0	10·215	10·265	10·215	40·067	39·95	39·771
	0·05	9·010	–	9·012	33·080	–	–
	0·10	8·109	8·144	8·111	29·866	29·76	–
	0·20	6·869	6·896	7·00	27·070	26·95	–
	0·30	6·052	6·073	–	25·840	25·71	–
	0·40	5·466	5·484	–	25·155	25·02	–
	0·50	5·021	5·037	5·00	24·720	24·58	–
	0·60	4·669	4·683	–	24·419	24·28	–
	0·70	4·382	4·394	–	24·200	24·06	–
	0·80	4·141	4·153	–	24·032	23·89	–
	0·90	3·936	3·947	–	23·900	23·76	–
	1	3·759	–	3·75	23·794	–	–
Simply supported	0	4·935	4·965	4·935	29·807	29·95	29·72
	0·05	4·546	–	–	24·957	–	–
	0·10	4·231	4·255	–	22·329	22·38	–
	0·20	3·749	3·770	–	19·721	19·74	–
	0·30	3·398	3·416	–	18·447	18·45	–
	0·40	3·128	3·144	–	17·696	17·69	–
	0·50	2·912	2·927	–	17·202	17·20	–
	0·60	2·735	2·749	–	16·852	16·85	–
	0·70	2·587	2·600	–	16·592	16·58	–
	0·80	2·460	2·472	–	16·390	16·38	–
	0·90	2·351	2·362	–	16·230	16·22	–
	1	2·254	–	–	16·099	–	–

M: concentrated mass; M_p; plate mass.

TABLE 2

The two lower natural frequency coefficients (quasi-axisymmetric modes) of an orthotropic circular plate with a central concentrated mass ($D_2/D_1 = 0{\cdot}5$; $D_k/D_1 = 0{\cdot}5$; $\nu_2 = 0{\cdot}30$)

		Values of Ω_{01} Present study			Values of Ω_{02} Present study		
	M/M_p	Analytical	FE values	Reference [1]	Analytical	FE values	
Clamped	0	9·624	–	9·619	37·746	–	
	0·05	8·488	–	8·492	31·164	–	
	0·10	7·639	7·609	7·661	28·136	26·10	
	0·20	6·471	6·435	6·517	25·502	24·08	
	0·30	5·701	5·663	–	24·343	23·11	
	0·40	5·150	5·111	–	23·697	22·56	
	0·50	4·730	4·693	4·797	23·288	22·20	
	0·60	4·399	4·362	–	23·005	21·96	
	0·70	4·128	4·092	–	22·798	21·77	
	0·80	3·901	3·867	–	22·640	21·64	
	0·90	3·708	3·675	–	22·516	21·53	
	1	3·541	–	–	22·416	–	
Simply supported	0	4·481	4·475	4·482	27·940	28·99[a]	21·33[b]
	0·05	4·132	4·125	4·138	23·389	20·35	
	0·10	3·848	3·840	3·859	20·918	19·13	
	0·20	3·414	3·405	3·433	18·459	17·35	
	0·30	3·097	3·087	–	17·254	16·35	
	0·40	2·852	2·842	–	16·542	15·72	
	0·50	2·657	2·647	2·686	16·073	15·31	
	0·60	2·497	2·487	–	15·741	15·02	
	0·70	2·362	2·352	–	15·494	14·80	
	0·80	2·247	2·237	–	15·303	14·62	
	0·90	2·147	2·137	–	15·150	14·49	
	1	2·059	2·050	–	15·026	14·37	

[a] Corresponding to mode 6.
[b] Corresponding to mode 4.
Note: Apparently mode 6 possesses a higher degree of radial symmetry than mode 4.

the plate behavior they follow Kirchhoff's theory, and possess displacement connectors at the element vertices and equilibrium connectors at midsides.

The mesh for the circular plate is shown in Figure 2. It has 1513 elements and 7275 degrees of freedoms.

4. NUMERICAL RESULTS

Frequency coefficients have been determined for: (1) isotropic plates ($\nu = 0{\cdot}30$), Table 1; (2) orthotropic plates: $D_2/D_1 = 0{\cdot}5$, $D_k/D_1 = 0{\cdot}5$, $\nu_2 = 0{\cdot}30$ (Table 2); $D_2/D_1 = 1$, $D_k/D_1 = 0{\cdot}5$, $\nu_2 = 0{\cdot}30$ (Table 3); $D_2/D_1 = 0{\cdot}5$, $D_k/D_1 = 1/3$, $\nu_2 = 1/3$ (Table 4).

When implementing the optimized Rayleigh–Ritz method it was found that the optimized values of the frequency coefficients was attained, in general, for values of "p" between 2 and 5 and values of "q" between 1·8 and 2·5, the minimization procedure being carried out numerically.

Table 1 depicts values of Ω_{01} and Ω_{02} for an orthotropic circular plate with a central, concentrated mass. Poisson's ratio has been taken to be equal to 0·3 in all calculations. Present results, obtained for clamped and simply supported edges, are in good agreement with those available in the literature [3].

A similar situation takes place in the case of orthotropic plates (Tables 2, 3 and 4). Present analytical results are in good agreement with the finite element predictions and are lower than those obtained. Apparently, values of Ω_{02} have not been considered in previous investigations. From the analysis of the tables it is concluded that the agreement between finite element results (presumably very accurate) and the analytical predictions is, in general, always better in the case of

Table 3

The two lower natural frequency coefficients (quasi-axisymmetric modes) of an orthotropic circular plate with a central concentrated mass ($D_2/D_1 = 1{\cdot}0$; $D_k/D_1 = 0{\cdot}5$; $\nu_2 = 0{\cdot}30$)

		Values of Ω_{01} Present study			Values of Ω_{02} Present study	
	M/M_p	Analytical	FE values	Reference [1]	Analytical	FE values
Clamped	0	10·521	–	10·592	41·542	–
	0·05	9·341	–	9·358	34·298	–
	0·10	8·407	8·394	8·446	30·966	30·69
	0·20	7·122	7·107	7·189	28·067	27·79
	0·30	6·275	6·260	–	26·791	26·52
	0·40	5·668	5·652	–	26·081	25·81
	0·50	5·206	5·192	5·296	25·630	25·36
	0·60	4·841	4·827	–	25·319	25·05
	0·70	4·543	4·529	–	25·091	24·82
	0·80	4·293	4·280	–	24·917	24·65
	0·90	4·081	4·068	–	24·781	24·51
	1	3·897	–	–	24·670	–
Simply supported	0	4·974	4·974	4·977	30·787	30·76
	0·05	4·586	4·585	4·593	25·773	25·71
	0·10	4·270	4·268	4·296	23·052	22·97
	0·20	3·788	3·785	3·833	20·345	20·25
	0·30	3·435	3·431	–	19·020	18·92
	0·40	3·163	3·160	–	18·238	18·14
	0·50	2·946	2·943	3·017	17·723	17·62
	0·60	2·768	2·765	–	17·358	17·26
	0·70	2·619	2·615	–	17·086	16·99
	0·80	2·491	2·488	–	16·876	16·78
	0·90	2·380	2·377	–	16·709	16·61
	1	2·283	2·280	–	16·572	16·47

TABLE 4

The two lower natural frequency coefficients (quasi-axisymmetric modes) of an orthotropic circular plate with a central concentrated mass ($D_2/D_1 = 0{\cdot}5$; $D_k/D_1 = 1/3$; $\nu_2 = 1/3$)

		Values of Ω_{01} Present study		Values of Ω_{02} Present study	
	M/M_p	Analytical	FE values	Analytical	FE values
Clamped	0	9·208	–	36·117	–
	0·05	8·121	–	29·818	–
	0·10	7·309	7·275	26·923	24·92
	0·20	6·192	6·152	24·401	22·98
	0·30	5·455	5·414	23·295	22·06
	0·40	4·927	4·886	22·679	21·53
	0·50	4·526	4·486	22·284	21·19
	0·60	4·209	4·170	22·013	20·95
	0·70	3·949	3·912	21·815	20·78
	0·80	3·733	3·696	21·664	20·65
	0·90	3·548	3·513	21·545	20·54
	1	3·388	–	21·449	–
Simply supported	0	4·492	–	26·904	–
	0·05	4·138	–	22·527	–
	0·10	3·849	3·841	20·159	18·49
	0·20	3·410	3·400	17·808	16·74
	0·30	3·090	3·078	16·661	15·78
	0·40	2·844	2·832	15·986	15·19
	0·50	2·647	2·635	15·540	14·79
	0·60	2·487	2·475	15·225	14·51
	0·70	2·352	2·339	14·991	14·30
	0·80	2·236	2·225	14·810	14·14
	0·90	2·136	2·125	14·666	14·01
	1	2·049	2·037	14·549	13·90

M: concentrated mass; M_p: plate mass.
$\Omega_{01} = \sqrt{\rho h/D_1}\,\omega_{01} a^2$; $\Omega_{02} = \sqrt{\rho h/D_1}\,\omega_{02} a^2$.

a simply supported edge in spite of the fact that the polynomial co-ordinate functions do not satisfy the natural boundary condition. On the other hand, the agreement is considerably better in the case of the fundamental frequency coefficient. When determining the second eigenvalue the analytical approach yields, in some instances, rough estimates.

Figures 3 through 12 depict modes 1 through 10 obtained by means of SAMCEF for a clamped plate of orthotropic characteristics: $D_2/D_1 = 0{\cdot}5$; $D_k/D_1 = 1/3$ and $\nu_2 = 1/3$. The parameter M/M_p is equal to 0·10. Mode 4, shown in Figure 6, corresponds to the second quasi-axisymmetric frequency coefficient, $\Omega_{02} = 24{\cdot}92$.

Figure 3 indicates the rather high degree of radial symmetry, for the mechanical parameters that come into play for this particular configuration.

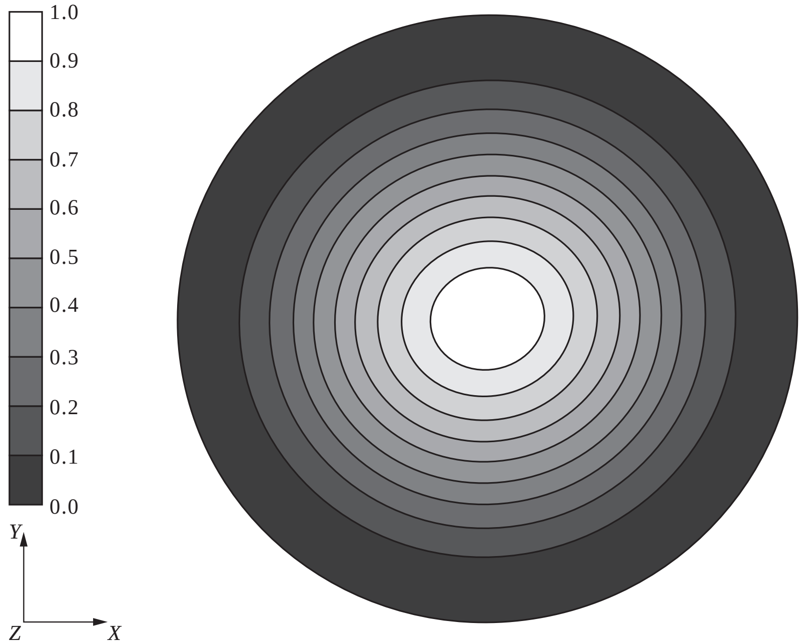

Figure 3. Mode No. 1 (SAMCEF [4]); $\Omega_1 = 7{\cdot}275$. Note: the shaded regions correspond to non-dimensional deformation steps of 0·1, from 0 for the darkest region to 1 for the lightest region.

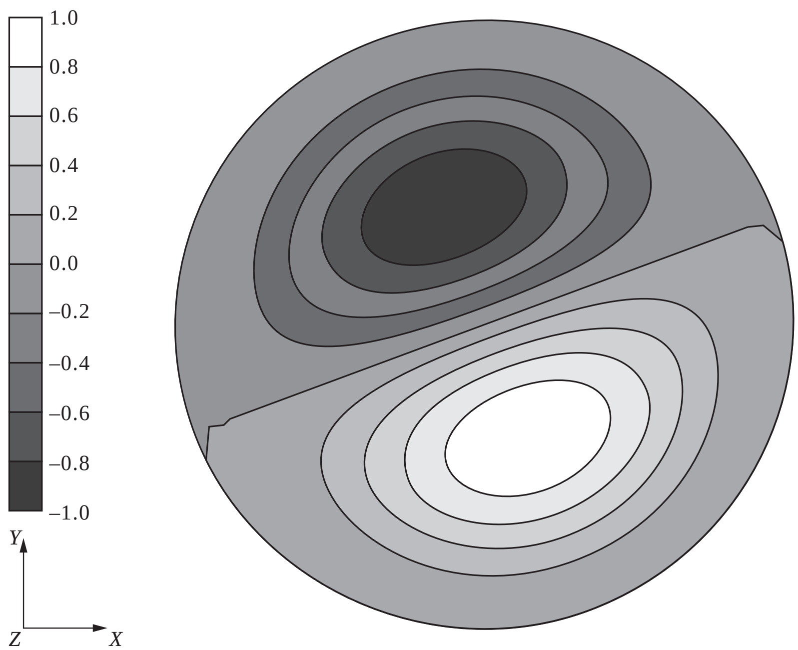

Figure 4. Mode No. 2 (SAMCEF [4]); $\Omega_2 = 17{\cdot}56$. Note: the shaded regions correspond to non-dimensional deformation steps of 0·2, from -1 for the darkest region to $+1$ for the lightest region.

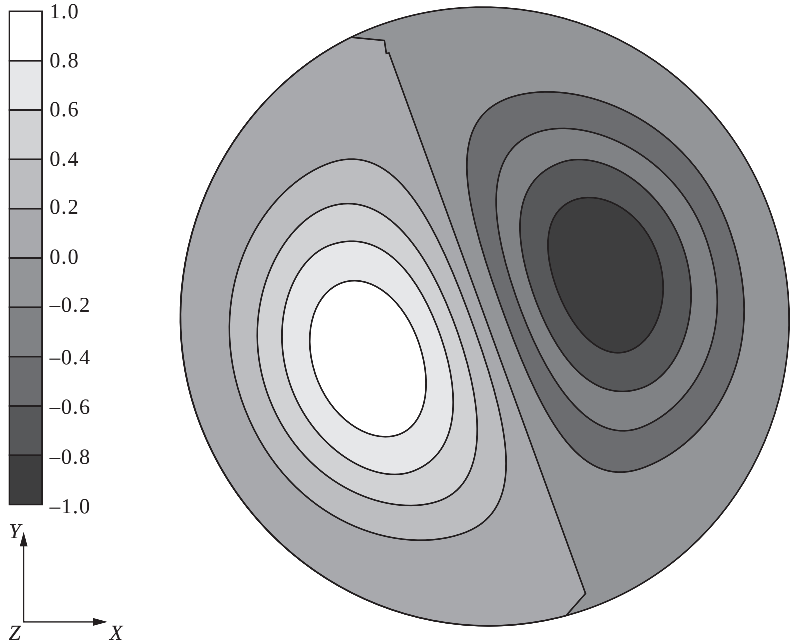

Figure 5. Mode No. 3 (SAMCEF [4]); $\Omega_3 = 20{\cdot}50$. Note: as for Figure 4.

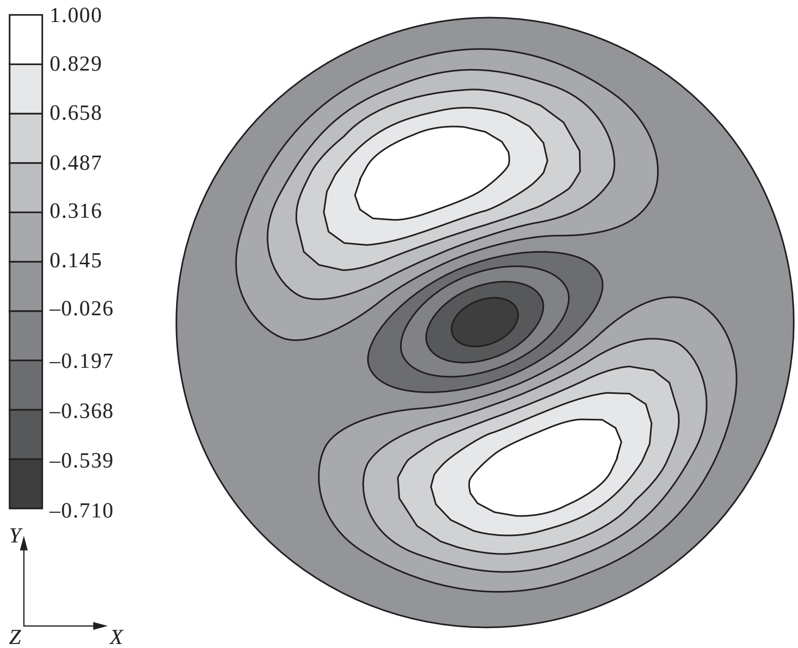

Figure 6. Mode No. 4 (SAMCEF [4]); $\Omega_4 = 24{\cdot}92$. Note: the shaded regions correspond to non-dimensional deformation steps of 0·171 from $-0{\cdot}71$ for the darkest region to $+1$ for the lightest region.

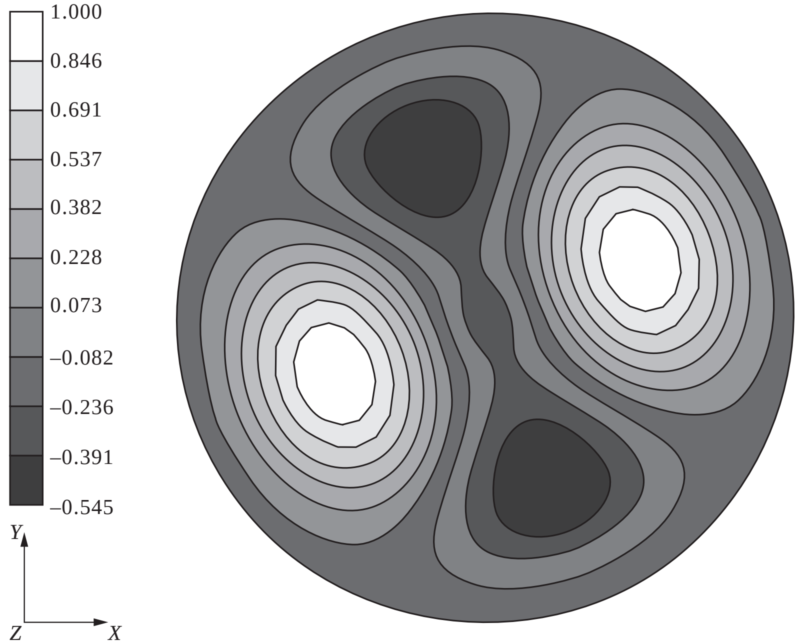

Figure 7. Mode No. 5 (SAMCEF [4]); $\Omega_5 = 31{\cdot}70$. Note: the shaded regions correspond to non-dimensional deformation steps of 0·154 from $-0{\cdot}545$ for the darkest region to $+1$ for the lightest region.

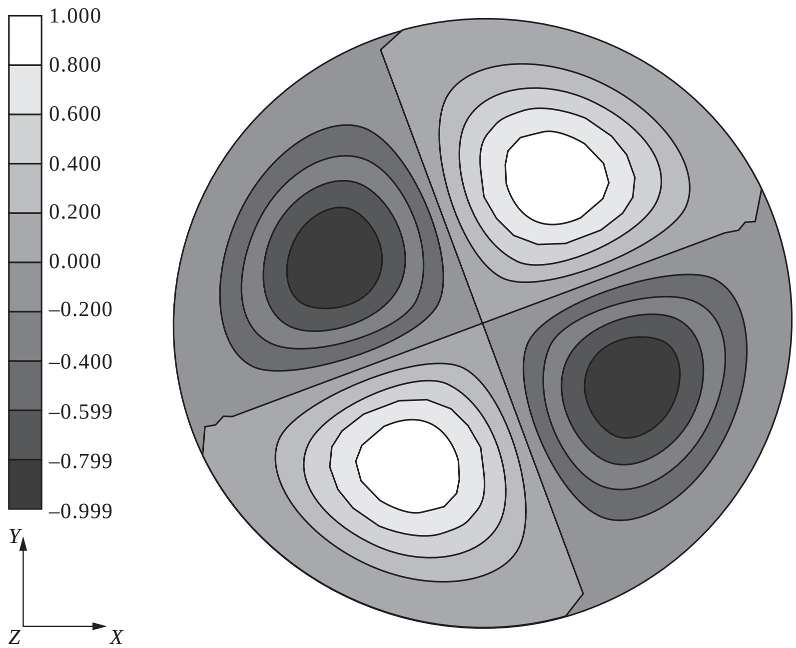

Figure 8. Mode No. 6 (SAMCEF [4]); $\Omega_6 = 31{\cdot}80$. Note: the shaded regions correspond to non-dimensional deformation steps of 0·20 from $-0{\cdot}999$ for the darkest region to $+1$ for the lightest region.

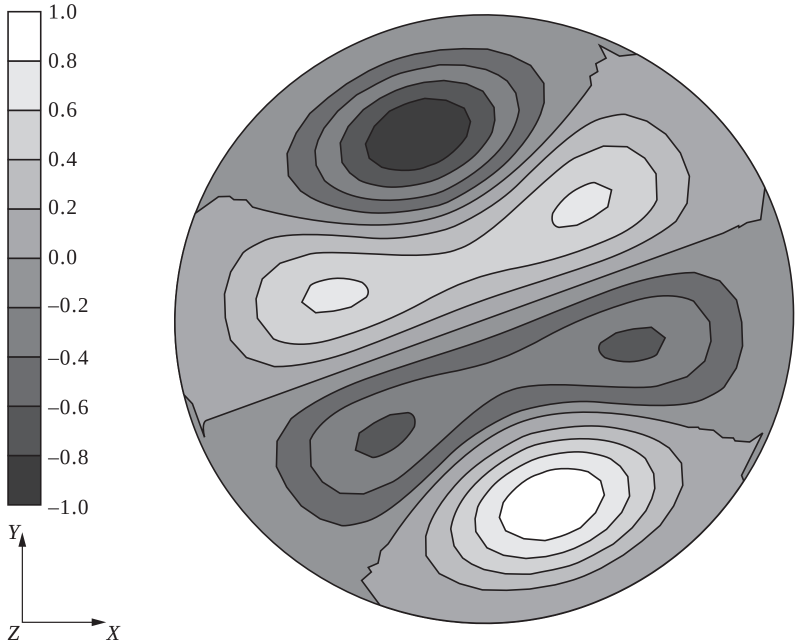

Figure 9. Mode No. 7 (SAMCEF [4]); $\Omega_7 = 42{\cdot}34$. Note: as for Figure 4.

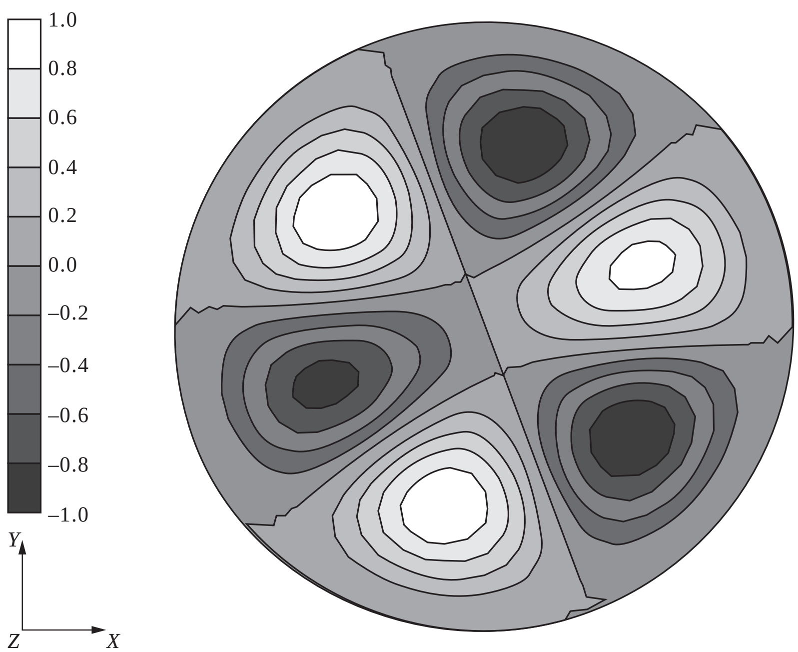

Figure 10. Mode No. 8 (SAMCEF [4]); $\Omega_8 = 44{\cdot}98$. Note: as for Figure 4.

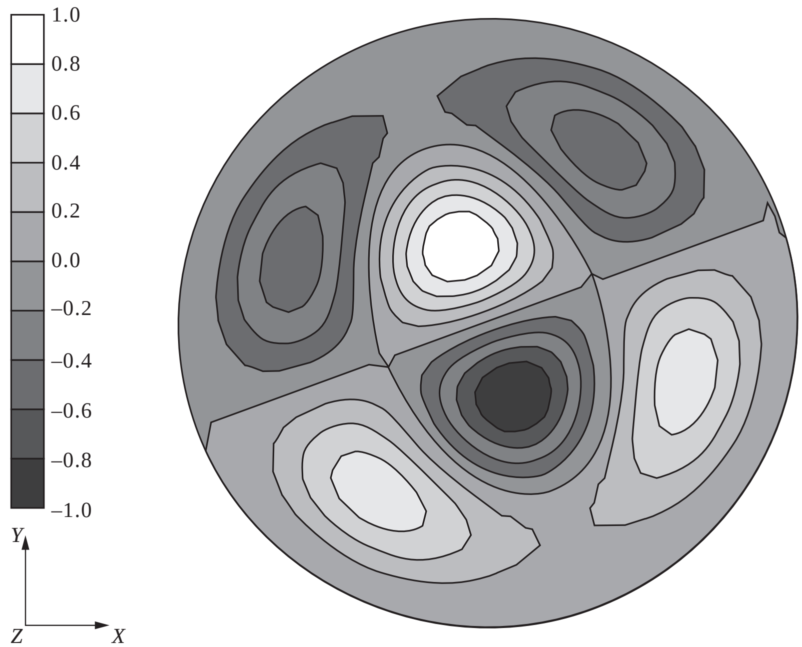

Figure 11. Mode No. 9 (SAMCEF [4]); $\Omega_9 = 52{\cdot}95$. Note: as for Figure 4.

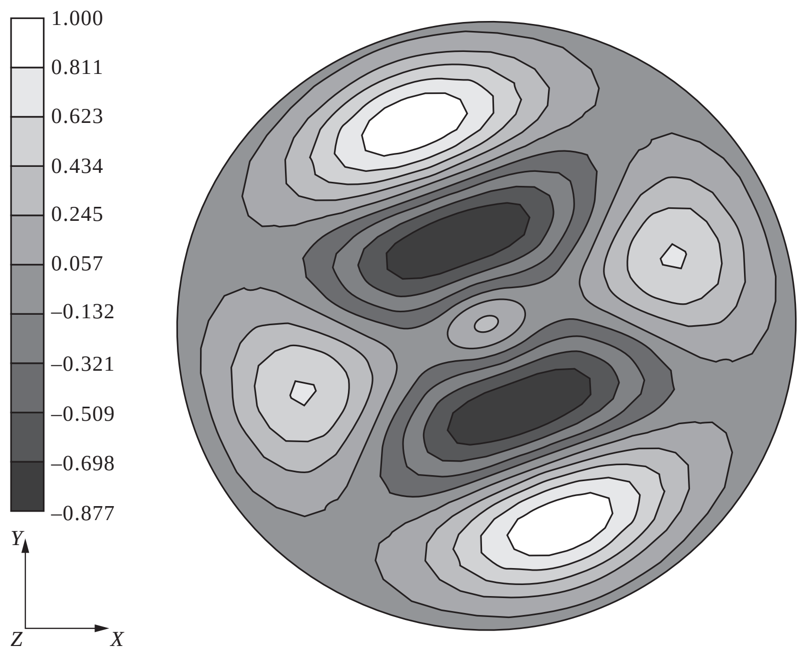

Figure 12. Mode No. 10 (SAMCEF [4]); $\Omega_{10} = 55{\cdot}89$. Note: the shaded regions correspond to non-dimensional deformation steps of 0·189 from $-0{\cdot}887$ for the darkest region to $+1$ for the lightest region.

The results shown in Table 1 indicate the high accuracy when the structural system is isotropic.

The results are upper bounds with respect to the exact eigenvalues and, in many instances, they are lower than the frequency coefficients obtained by means of the finite element method.

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