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A THEORETICAL PREDICTION OF INCREASE OF NATURAL FREQUENCY TO FERROMAGNETIC PLATES UNDER IN-PLANE MAGNETIC FIELDS

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To predict the experimental phenomenon of the increase in natural frequency of a cantilever ferromagnetic beam–plate with low susceptibility under in-plane applied magnetic fields, a theoretical model for the magnetoelastic interaction is developed in this paper using the approach of the variational principle with energy functional of the system. It is found that the expression derived for the magnetic force exerted on the plate in this case is distinct from the existing models in the literature. Following this theoretical model, the experimental phenomenon of increase of natural frequency is successfully simulated in theory. After a revision related to the change of natural frequency is considered in the calculations of magnetic damping, the theoretical predictions of the magnetic damping ratio agree well with the corresponding experimental data.

1. INTRODUCTION

Recently, the ferromagnetic structures with low susceptibility, such as a ferritic steel: $8\%Cr$–$2\%W$-0·2%$V-0{\cdot}04\%$Ta-Fe (it is abbreviated by F-82H in reference [1]), for which $\chi \leqslant 15$, has been suggested as a candidate for the first wall in a fusion reactor [1]. Research on the behavior of interaction between magnetic field and mechanical deformation for the structures which are made of such materials has attracted the attention of researchers and engineers. An interesting experiment for the mechanical behavior of a cantilever ferromagnetic beam–plate under in-plane applied magnetic fields was conducted by Talagi *et al.* [2]. The experimental results show that the natural frequency of the plates increases with the applied magnetic fields.

Since Moon and Pao [3] conducted the experiments of magnetoelastic buckling to the cantilever ferromagnetic beam–plates in transverse magnetic fields, some

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theoretical models have been proposed using different approaches—including an intuitive method [3, 4], an axiomatic method [5–7], and a variational principle [8–10]—in order to simulate the experimental phenomenon of magnetoelastic instability. The main difference between the models is that the expressions of magnetic forces are different. It has been found that these theoretical models can behave like the experimental phenomenon of magnetoelastic buckling when a cantilever beam–plate is under a uniform transverse magnetic field except for some differences among the theoretical predictions [3, 4, 10–14]. Once they are chosen to simulate the experimental phenomenon of the increase in natural frequency [2], however, the numerical results will show that the natural frequency of the plates decreases rather than increases with the applied magnetic field. This contradiction between the theoretical predictions and the experimental results motivates us to find a new theoretical model for simulating the experimental phenomenon of the increase in natural frequency.

Based on the variational principle in which the functional is taken as the summation of the magnetic energy and the strain energy in the magnetoelastic system, here, a new theoretical model associated with a distinct expression of magnetic force to the ferromagnetic beam–plate with low susceptibility under in-plane magnetic fields is established such that the experimental phenomenon of the increase in natural frequency of the ferromagnetic structures can be simulated theoretically.

2. THEORETICAL GOVERNING EQUATIONS

For reason of simplicity, the effect of eddy currents on the natural frequency in the derivation of expression of magnetic force will be neglected for the problem dealt with in this paper. That is, a stationary magnetic field without electric field, charge distribution and conduction current in the magnetoelastic medium will be considered. According to the Maxwell equations of electromagnetism, one can introduce scalar potential function ϕ which satisfies

$$-\nabla\phi = \mathbf{H}. \tag{1}$$

For linearly constitutive relations between magnetic field vector $\mathbf{H}$ and magnetization $\mathbf{M}$ or magnetic induction $\mathbf{B}$, one has

$$\mathbf{M}^{+} = \chi\mathbf{H}^{+} \quad \text{in} \quad \Omega^{+} \tag{2}$$

or

$$\mathbf{B}^{+} = \mu_0\mu_r\mathbf{H}^{+} \quad \text{in} \quad \Omega^{+}, \tag{3a}$$

$$\mathbf{B}^{-} = \mu_0\mathbf{H}^{-} \quad \text{in} \quad \Omega^{-}. \tag{3b}$$

Here, Ω^{+} and Ω^{-} represent the inside region of the deformed ferromagnetic medium and the region out of the ferromagnetic medium; χ is susceptibility of the soft ferromagnetic medium; μ_0 and μ_r are the magnetic permeability of vacuum and the relative permeability of the ferromagnetic medium, and

$$\mu_r = \chi + 1. \tag{5}$$

After taking a closed surface S_0 which surrounds the ferromagnetic plate and is far away from it, the magnetic energy of the system can be written as [10]

$$\Pi_1\{\phi, \mathbf{U}\} = \frac{1}{2}\int_{\Omega^-(\mathbf{U})} \mu_0 (\nabla\phi^-)^2 \, \mathrm{d}v + \frac{1}{2}\int_{\Omega^+(\mathbf{U})} \mu_0\mu_r (\nabla\phi^+)^2 \, \mathrm{d}v + \int_{S_0} \mathbf{n}\cdot\mathbf{B}_0\phi^- \, \mathrm{d}s, \tag{6}$$

in which the last term on the right-hand side is the external work by applied magnetic field $\mathbf{B}_0$; $\mathbf{n}$ is a unit vector outward normal to S_0; and $\mathbf{U}$ denotes a displacement vector at a point of the medium.

From the theory of elasticity, one can write the mechanical strain energy in Ω^+ associated with the external work on S_t of the form

$$\Pi_2\{\phi, \mathbf{U}\} = \frac{1}{2}\int_{\Omega^+} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, \mathrm{d}v - \int_{S_t} \mathbf{F}^* \cdot \mathbf{U} \, \mathrm{d}s. \tag{7}$$

For linear elasticity of both material and geometry, one has

$$\boldsymbol{\sigma} = \mathbf{Y} : \boldsymbol{\varepsilon} \quad \text{in} \quad \Omega^+, \tag{8}$$

for constitutive equations of elasticity, and

$$\boldsymbol{\varepsilon} = (\nabla\mathbf{U} + (\nabla\mathbf{U})^T/2 \quad \text{in} \quad \Omega^+, \tag{9}$$

for geometrical relations. Here, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ represent stress and strain tensors respectively; $\mathbf{Y}$ is a tensor of order 4 of elastic constants of the material; $\mathbf{F}^*$ is a specific external force on S_t; and the superscript T represents the transpose of a tensor or matrix.

Thus, the functional of total energy of the system can be obtained by adding Π_1 and Π_2. That is

$$\begin{aligned}\Pi\{\phi, \mathbf{U}\} &= \Pi_1\{\phi, \mathbf{U}\} + \Pi_2\{\phi, \mathbf{U}\} \\ &= \frac{1}{2}\int_{\Omega^-(\mathbf{U})} \mu_0 (\nabla\phi^-)^2 \, \mathrm{d}v + \frac{1}{2}\int_{\Omega^+(\mathbf{U})} \mu_0\mu_r (\nabla\phi^+)^2 \, \mathrm{d}v + \int_{S_0} \mathbf{n}\cdot\mathbf{B}_0\phi^- \, \mathrm{d}s \\ &\quad + \frac{1}{2}\int_{\Omega^+} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, \mathrm{d}v - \int_{S_t} \mathbf{F}^* \cdot \mathbf{U} \, \mathrm{d}s.\end{aligned} \tag{10}$$

It should be noted that the displacement $\mathbf{U}$ does not explicitly appear in ϕ; and the effect of displacement on the region Ω^+ in the integral of strain energy will be neglected for the case of small deformation. Let $\delta\mathbf{U}$ and $\delta\phi$ be the admissible variations of the displacement and the magnetic potential function of the system respectively. Then, one has

$$\delta\mathbf{U} = \mathbf{0} \quad \text{on} \quad S_u, \qquad \delta\phi^+ = \delta\phi^- \quad \text{on} \quad S. \tag{11, 12}$$

And the compatibility conditions for $\delta\mathbf{U}$ should be satisfied. Considering the variations $\delta\phi$ and $\delta\mathbf{U}$ independent from each other, according to the arithmetic of variations, one can write

$$\delta\Pi = \delta_\phi\Pi + \delta_{\mathbf{U}}\Pi, \tag{13}$$

in which $\delta_\phi \Pi$ and $\delta_\mathbf{U} \Pi$ denote the variations of Π caused by $\delta\phi$ and $\delta\mathbf{U}$ respectively. That is,

$$\delta_\phi \Pi = \delta_\phi \Pi_1 + \delta_\phi \Pi_2$$
$$= \int_{\Omega^-(\mathbf{U})} \mu_0 \nabla\phi^- \cdot \nabla(\delta\phi^-)\,\mathrm{d}v + \int_{\Omega^+(\mathbf{U})} \mu_0\mu_r \nabla\phi^+ \cdot \nabla(\delta\phi^+)\,\mathrm{d}v + \int_{S_0} \mathbf{n}\cdot\mathbf{B}_0 \delta\phi^-\,\mathrm{d}s,$$

$$\delta_\mathbf{U} \Pi = \delta_\mathbf{U} \Pi_1 + \delta_\mathbf{U} \Pi_2$$
$$= \frac{1}{2}\left[\int_{\Omega^+(\mathbf{U}+\delta\mathbf{U})} - \int_{\Omega^+(\mathbf{U})}\right] \mu_0\mu_r (\nabla\phi^+)^2\,\mathrm{d}v + \frac{1}{2}\left[\int_{\Omega^-(\mathbf{U}+\delta\mathbf{U})} - \int_{\Omega^-(\mathbf{U})}\right] \mu_0 (\nabla\phi^-)^2\,dv$$
$$+ \int_{\Omega^+} \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\,\mathrm{d}v - \int_{S_t} \mathbf{F}^* \cdot \delta\mathbf{U}\,\mathrm{d}s. \tag{15}$$

Since

$$\nabla\cdot(\nabla\phi\delta\phi) = \nabla^2\phi\delta\phi + \nabla\phi\cdot\nabla\delta\phi, \tag{16}$$

$$\int_V \nabla\cdot(\nabla\phi\delta\phi)\,\mathrm{d}v = \oint_{S^*} \mathbf{n}\cdot\nabla\phi\delta\phi\,\mathrm{d}s, \tag{17}$$

where S^* is the enclosed surface of region V, one can reduce equation (14) into

$$\delta_\phi \Pi = -\int_{\Omega^+(\mathbf{U})} \mu_0\mu_r (\nabla^2\phi^+)\delta\phi^+\,\mathrm{d}v - \int_{\Omega^-(\mathbf{U})} \mu_0 (\nabla^2\phi^-)\delta\phi^-\,\mathrm{d}v$$
$$+ \oint_S \mu_0 \left\{\mu_r \frac{\partial\phi^+}{\partial n} - \frac{\partial\phi^-}{\partial n}\right\}\delta\phi\,\mathrm{d}s + \int_{S_0} \left\{\mu_0 \frac{\partial\phi^-}{\partial n} + \mathbf{n}\cdot\mathbf{B}_0\right\}\delta\phi^-\,\mathrm{d}s. \tag{18}$$

For linear elasticity of the body, by variational arithmetic, one can find the relation of the form

$$\int_{\Omega^+} \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\,\mathrm{d}v - \int_{S_t} \mathbf{F}^* \cdot \delta\mathbf{U}\,\mathrm{d}s$$
$$= -\int_{\Omega^+} \nabla\cdot\boldsymbol{\sigma}\cdot\delta\mathbf{U}\,\mathrm{d}v + \int_{S_t} \{\mathbf{n}\cdot\boldsymbol{\sigma} - \mathbf{F}^*\}\cdot\delta\mathbf{U}\,\mathrm{d}s + \int_{S_U} \mathbf{n}\cdot\boldsymbol{\sigma}\cdot\delta\mathbf{U}\,\mathrm{d}s. \tag{19}$$

Because the displacement vector $\mathbf{U}$ does not explicitly appear in the expression of ϕ in equation (15), one can write

$$\frac{1}{2}\left[\int_{\Omega^+(\mathbf{U}+\delta\mathbf{U})} - \int_{\Omega^+(\mathbf{U})}\right] \mu_0\mu_r (\nabla\phi^+)^2\,\mathrm{d}v + \frac{1}{2}\left[\int_{\Omega^-(\mathbf{U}+\delta\mathbf{U})} - \int_{\Omega^-(\mathbf{U})}\right] \mu_0 (\nabla\phi^-)^2\,\mathrm{d}v$$
$$= \frac{1}{2}\int_{\Omega^+(\mathbf{U}+\delta\mathbf{U})\cap\Omega^-(\mathbf{U})} \{\mu_0\mu_r(\nabla\phi^+)^2 - \mu_0(\nabla\phi^-)^2\}\,\mathrm{d}v$$
$$+ \frac{1}{2}\int_{\Omega^-(\mathbf{U}+\delta\mathbf{U})\cap\Omega^+(\mathbf{U})} \{\mu_0(\nabla\phi^-)^2 - \mu_0\mu_r(\nabla\phi^+)^2\}\,\mathrm{d}v \tag{20}$$

In fact, the regions of $\Omega^+(\mathbf{U}+\delta\mathbf{U})\cap\Omega^-(\mathbf{U})$ and $\Omega^-(\mathbf{U}+\delta\mathbf{U})\cap\Omega^+(\mathbf{U})$ are the parts of variation of volume of the region Ω^+ related to $\delta\mathbf{U}$ on the surface S. When $\delta\mathbf{U}$ is very small, one can write the variation of volume as follows:

$$\{\Omega^+(\mathbf{U}+\delta\mathbf{U}) \cap \Omega^-(\mathbf{U})\} \cup \{\Omega^-(\mathbf{U}+\delta\mathbf{U}) \cap \Omega^+(\mathbf{U})\} = S \times (\mathbf{n}\cdot\delta\mathbf{U}), \qquad (21)$$

where $\delta\mathbf{U}$ is taken on the surface S. The sign of $\mathbf{n}\cdot\delta\mathbf{U}$ is used to identify that the change of volume at a point on the surface (on the right-hand side of equation (21)) corresponds to one of the two sets of variations of volume on the left-hand side of the equation. After that, the integration of equation (20) can be expressed by

$$\frac{1}{2}\left[\int_{\Omega^+(\mathbf{U}+\delta\mathbf{U})} - \int_{\Omega^+(\mathbf{U})}\right]\mu_0\mu_r(\nabla\phi^+)^2\,\mathrm{d}v + \frac{1}{2}\left[\int_{\Omega^-(\mathbf{U}+\delta\mathbf{U})} - \int_{\Omega^-(\mathbf{U})}\right]\mu_0(\nabla\phi^-)^2\,\mathrm{d}v$$
$$= \frac{1}{2}\oint_S \{\mu_0\mu_r(\nabla\phi^+)^2 - \mu_0(\nabla\phi^-)^2\}\mathbf{n}\cdot\delta\mathbf{U}\,\mathrm{d}s \qquad (22)$$

Before reducing this integration further, let us check the physical meaning of the terms on the right-hand side of equation (22). Denote

$$T = \tfrac{1}{2}\{\mu_0\mu_r(\nabla\phi^+)^2 - \mu_0(\nabla\phi^-)^2\} \quad \text{on} \quad S. \qquad (23)$$

Then, one can find that T is the change of density of magnetic energy between the two sides on surface S. Thus, the integration of equation (22) represents the flux of magnetic energy flowing out of the region Ω_+ when a set of admissible variation of displacement, $\delta\mathbf{U}$, is given. According to the connected conditions of the magnetic field on the surface of a ferromagnetic medium, one can write

$$H_\tau^+ = H_\tau^- \quad \text{and} \quad B_n^+ = B_n^- \quad \text{on} \quad S. \qquad (24)$$

Here, the subscripts "n" and "τ" are used to represent the normal and tangential components of the quantities. From equations (4) and (24), one can get

$$H_n^- = \mu_r H_n^+, \quad \text{and} \quad B_\tau^- = \frac{1}{\mu_r}B_\tau^+ \quad \text{on} \quad S. \qquad (25)$$

Considering the definition of the magnetic potential function in equation (1), and substituting the formulas of equation (25) into equation (23), one has

$$T = -\frac{\mu_0\mu_r\chi}{2}(H_n^+)^2 + \frac{\mu_0\chi}{2}(H_r^+)^2 \quad \text{on} \quad S. \qquad (26)$$

For a thin ferromagnetic beam–plate with low susceptibility under an in-plane applied magnetic field, i.e., $\mathbf{B}_0 = B_0\boldsymbol{\tau}$, one introduces two assumptions of the form

$$\nabla\cdot\delta\mathbf{U} \approx 0 \quad \text{in} \quad \Omega^+; \qquad (27)$$

$$\frac{\mu_0\chi}{2}(H^+)^2 \gg \frac{\mu_0\chi(\mu_r+1)}{2}(H_n^+)^2 \quad \text{on} \quad S. \qquad (28)$$

In the next section, some numerical valuations will be given to show the assumptions are reasonable and acceptable for this case of magnetoelastic interaction. Under the assumptions of equations (27) and (28), the integration on the right-hand side of equation (22) can be reduced into

$$\frac{1}{2}\int_S \{\mu_0\mu_r(\nabla\phi^+)^2 - \mu_0(\nabla\phi^-)^2\}\mathbf{n}\cdot\delta\mathbf{U}\,\mathrm{d}s \approx \frac{\mu_0\chi}{2}\int_{\Omega^+}\{\nabla(H^+)^2\}\cdot\delta\mathbf{U}\,\mathrm{d}v. \tag{29}$$

According to equations (13), (15), (18), (19), (22) and (29), one gets

$$\begin{aligned}\delta\Pi\{\phi,\mathbf{U}\} = &-\int_{\Omega^+(\mathbf{U})}\mu_0\mu_r(\nabla^2\phi^+)\delta\phi^+\,\mathrm{d}v - \int_{\Omega^-(\mathbf{U})}\mu_0(\nabla^2\phi^-)\delta\phi^-\,\mathrm{d}v \\ &+\oint_S \mu_0\left\{\mu_r\frac{\partial\phi^+}{\partial n} - \frac{\partial\phi^-}{\partial n}\right\}\delta\phi\,\mathrm{d}s + \int_{S_0}\mathbf{n}\cdot\{\mu_0\nabla\phi^- + \mathbf{B}_0\}\delta\phi^-\,\mathrm{d}s \\ &-\int_{\Omega^+}\left\{\nabla\cdot\boldsymbol{\sigma} - \frac{\mu_0\chi}{2}\nabla(H^+)^2\right\}\cdot\delta\mathbf{U}\,\mathrm{d}v + \int_{S_t}\{\mathbf{n}\cdot\boldsymbol{\sigma} - \mathbf{F}^*\}\cdot\delta\mathbf{U}\,\mathrm{d}s \\ &+\int_{S_U}\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\delta\mathbf{U}\,\mathrm{d}s.\end{aligned} \tag{30}$$

From equation (30) and the arbitrariness and the independence of $\delta\phi$ and $\delta\mathbf{U}$, and letting $\delta\Pi\{\phi, \mathbf{U}\} = 0$, one can get all governing equations and boundary conditions of the magnetoelastic interaction as follows.

Governing equations for magnetic fields:

$$\nabla^2\phi^+ = 0 \quad \text{in} \quad \Omega^+, \qquad \nabla^2\phi^- = 0 \quad \text{in} \quad \Omega^-, \tag{31, 32}$$

with the connected conditions

$$\phi^+ = \phi^- \quad \text{on} \quad S, \qquad \mu_r\frac{\partial\phi^+}{\partial n} = \frac{\partial\phi^-}{\partial n} \quad \text{on} \quad S, \tag{33, 34}$$

and the boundary conditions on S_0

$$-\nabla\phi^- = \frac{1}{\mu_0}\mathbf{B}_0 \quad \text{at} \quad \infty \quad \text{or} \quad \text{on} \quad S_0. \tag{35}$$

Governing equations for deformation of plates:

$$\nabla\cdot\boldsymbol{\sigma} + \mathbf{f}^{em} = \mathbf{0} \quad \text{in} \quad \Omega^+, \tag{36}$$

with the boundary equations

$$\mathbf{U} = \mathbf{U}^* \quad \text{on} \quad S_u, \qquad \mathbf{n}\cdot\boldsymbol{\sigma} = \mathbf{F}^* \quad \text{on} \quad S_t. \tag{37, 38}$$

Here, the magnetic force $\mathbf{f}^{em}$ is obtained by

$$\mathbf{f}^{em} = -\frac{\mu_0\chi}{2}\nabla(\mathbf{H}^+)^2 \quad \text{in} \quad \Omega^+. \tag{39}$$

It is obvious that this expression of magnetic force is totally different from those formulas of magnetic forces which are expressed in the existing theoretical models for the magnetoelastic interaction of a ferromagnetic medium in an applied magnetic field. For example, when a soft ferromagnetic plate is under a transverse magnetic field, from the intuitive method of micro-current model, the magnetic body force is expressed by [4]

$$\mathbf{f}^{em} = \frac{\mu_0 \mu_r \chi}{2} \nabla (\mathbf{H}^+)^2 \quad \text{in} \quad \Omega^+. \tag{40}$$

When one uses the theoretical model associated with the magnetic force of equation (40) for the case of the plates considered in the experiments [2], one will find that its predictions are opposite to the experimental phenomenon, i.e., the decrease of natural frequency is predicted by it. Since there is a key difference of a negative symbol "−" between equations (39) and (40), it is possible for us to give a theoretical prediction of the increase phenomenon of natural frequency based on the model associated with the expression of magnetic force of equation (39). The quantitative results for this will be given in the following sections. For the same reason, it should be noted that the theoretical model of equations (31)–(39) cannot give a reasonable prediction of the experimental phenomenon of magnetoelastic buckling of a ferromagnetic plate in transverse magnetic fields since the experimental phenomenon of magnetoelastic buckling has been predicted by the theoretical model of equations (31)–(38) associated with the magnetic force of equation (40) [4, 10]. According to reference [10], it can be found that the theoretical model associated with the magnetic force of equation (40) is suitable for the special cases of applied magnetic fields in a transverse rather than in-plane direction. Since the theoretical model proposed in this section is based on the assumptions of equations (27) and, mainly, (28), the content of its application will be discussed in the next section.

3. APPLICABILITY OF THEORETICAL MODEL OF EQUATIONS (31)–(39)

From section 2, it was found that the derivations of equation (30) are dependent upon the two assumptions of equations (27) and (28) introduced. When the plates are thin, the variation of volume strain, $\nabla \cdot \delta \mathbf{U}$, in the region Ω^+ is so small that one can neglect its effect. Hence, the first assumption is reasonable and acceptable for thin plates. Next, it is shown how good is the second assumption of equation (28). Let

$$T_n = -\frac{\mu_0 \chi \mu_r}{2} (H_n^+)^2, \qquad T_\tau = \frac{\mu_0 \chi}{2} (H_\tau^+)^2. \tag{41, 42}$$

Then one has

$$T = T_n + T_\tau \quad \text{on} \quad S. \tag{43}$$

This formula shows that the change of density of magnetic energy on the surface S is generated from two parts. One is caused by the discontinuity of the normal component of the magnetic field (equation (41)) while another is generated by the jump of the tangential component of the magnetic field (equation (42)) on the surface S. In order to compare the values of the two terms in the inequality of equation (28) for the second assumption, denote

$$T_n^* = \frac{\mu_0 \chi (\mu_r + 1)}{2} (H_n^+)^2, \qquad T_\tau^* = \frac{\mu_0 \chi}{2} (H^+)^2.$$

If the value of T_n^* is much smaller than the value of T_τ^* on the main surfaces of S of a ferromagnetic plate except for its end points, one may say that the second assumption of equation (28) is reasonable and acceptable. The numerical test is conducted for the cantilevered ferromagnetic beam–plates under in-plane magnetic fields (see Figure 1). When the plates are placed in a uniformly distributed in-plane applied magnetic field $\mathbf{B}_0 = B_0\mathbf{i}$, and the susceptibility χ is small, the distributions of T_n^* and T_τ^* on the top and the bottom surfaces of the plate are plotted in Figure 2 for the ratio of length to thickness, $L/h = 100$, and $\chi = 14$ which are taken from the parameters of the experimental apparatus (see Table 2). In Figure 2, the solid and the dot dash line represent respectively the distributions of T_τ^* and T_n^* on the top and the bottom surfaces of the beam–plate varying with the x co-ordinate along the axial direction, which clearly shows that the magnitude of T_n^* is much smaller than the value of T_τ^* except for the small regions near the end points of the plate. As the susceptibility increases, the numerical results show that the value of T_n^* increases while T_τ^* decreases for a fixed ratio L/h. When χ is greater than a certain value, or a critical value, the condition of the second assumption of equation (28) will be violated. Thus, the expression of magnetic force of equation (39) cannot be used for the cases in which the susceptibility χ of the plate is greater than the critical value. The numerical output shows that the critical value of the parameter χ is dependent upon the ratio of length to thickness of the plate. The larger the ratio, the larger the critical value. Since it is dependent on the comparison of the values of T_n^* and T_τ^*, in practice, it is not easy to give the critical value of χ exactly. A rough confirmation for this critical value from the numerical tests is conducted here. The critical values for the three cases in which the plates have the ratio of 100, 200, and 333·3 are listed in Table 1, which gives the applicability region of the theoretical model of equations (31)–(39) for the plates considered in this paper.

4. NATURAL FREQUENCY

Here, the small free vibration of the cantilevered ferromagnetic beam–plate with low susceptibility and under in-plane applied magnetic fields will be dealt with. An approximate method such as the Galerkin method will be chosen to predict quantitatively the natural frequency while the magnetic fields are analyzed by the finite element method.

Since the experimental results in reference [2] for the electrically conducting plates such as copper plates vibrating in in-plane magnetic fields show almost no effect of eddy current in the vibrating plates on their vibrating frequency, here, the effect of the eddy current will be neglected. Taking the co-ordinate system xoz in which the x-axis is placed along the axis of the beam–plate in the mid-plane and the z-axis along the transverse direction of the plate (see Figure 1), for unit width of plate, the dynamic governing equations of free vibration in the transverse direction for the cantilevered beam–plate subjected to both transverse and axial magnetic forces can be expressed by

$$D\frac{\partial^4 w}{\partial x^4} - N_x\frac{\partial^2 w}{\partial x^2} - \frac{\partial N_x}{\partial x}\frac{\partial w}{\partial x} = -\rho h\frac{\partial^2 w}{\partial t^2} + q_z^{em}(x) \qquad 0 < x < L, \tag{44}$$

TABLE 1

The applicability region of susceptibilty of the theoretic model of equations (31)–(39)†: $\chi \leqslant \chi_{cr}$

L/h	χ_{cr}
100	500
200	2000
333·3	10 000

† Note: the soft ferromagnetic plates should be placed in in-plane magnetic fields only considered in this paper

with the boundary conditions

$$x = 0: \quad w = 0; \quad \frac{\partial w}{\partial x} = 0; \tag{45a, b}$$

$$x = L: \quad \frac{\partial^2 w}{\partial x^2} = 0; \quad \frac{\partial^3 w}{\partial x^3} = 0. \tag{46a, b}$$

in which the transverse magnetic force exerted on the plate is, from the theory of plates, expressed by

$$q_z^{em}(x) = \int_{-h/2}^{h/2} f_z^{em}(x, z)\, \mathrm{d}z. \tag{47}$$

According to the equilibrium equation in the x-direction, one can get the extension force

$$N_x(x) = -\int_x^L \int_{-h/2}^{h/2} f_x^{em}(x, z)\, \mathrm{d}z\, \mathrm{d}x. \tag{48}$$

Substitution of equation (39) into equations (47) and (48) leads to

$$q_z^{em}(x) = -\frac{\mu_0 \chi}{2} \{[H^+(x, h/2)]^2 - [H^+(x, -h/2)]^2\} \tag{49}$$

and

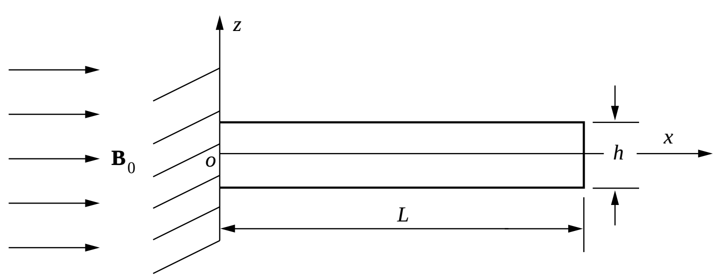

Figure 1. Schematic drawing of a cantilevered ferromagnetic beam–plate under in-plane

$$N_x(x) = \frac{\mu_0 \chi}{2} \int_{-h/2}^{h/2} \{[H^+(x,z)]^2 - [H^+(L,z)]^2\} \, \mathrm{d}z$$
$$\approx \frac{\mu_0 \chi h}{2} \{[H^+(x,0)]^2 - [H^+(L,0)]^2\}. \tag{50}$$

Here, D is the flexural rigidity of the plate, h the thickness of the plate, L the length of the plate, w deflection in the transverse direction, N_x the internal extension force in the axial direction, ρ density of mass of the plate, t a time variable, and f_x^{em} and f_z^{em} represent the components of magnetic force $\mathbf{f}^{em}$ in the x- and z-directions respectively.

In the above equations (44)–(50), the magnetic force $\mathbf{f}^{em}$ will be determined by the distribution of the magnetic fields for which the governing equations are equations (32)–(35). From the symmetric property of distribution of the magnetic fields, there is no difficulty for one to find that $f_z^{em}(x) \equiv 0$ when the ferromagnetic plate is placed in a uniformly distributed in-plane (along the x-direction) applied magnetic field, and when $w \equiv 0$ [4]. On the another hand, from equation (44), the natural frequency of the ferromagnetic plate in the magnetic field will be influenced by two parts in which one arises from the axial extension force, $N_x(x)$, while another source is the equivalent transverse magnetic force, $q_z^{em}(x)$, which is non-linearly dependent on the deflection. The numerical results show that there is little effect of deflection on the axial internal force $N_x(x)$ when the deflection is small. Thus, internal force $N_x(x)$ will be taken unchanged with time while the plate is vibrating with small deflection. Let $\bar{w}(x)$ be a normalized eigen-function of the dynamic problem and

$$w(x,t) = \alpha \bar{w}(x) \mathrm{e}^{i\omega t}, \tag{51}$$

in which α is a small positive number and ω refers to the natural frequency of the loaded plate. When the deflection is small, one can expand the magnetic field vector $\mathbf{H}^+(x, \bar{z})$ under the deformed state on the undeformed state through the transformation between the co-ordinate, z, for the undeformed state and the co-ordinate, $\bar{z}$, for the deformed plate. That is

$$\bar{z} = z + \alpha \bar{w} \mathrm{e}^{i\omega t} \tag{52}$$

and

$$\mathbf{H}^+(x,\bar{z}) = \mathbf{H}^+(x, z + \alpha \bar{w} \mathrm{e}^{i\omega t})$$
$$= \mathbf{H}_0^+(x,z) + \alpha \bar{w}(x) \frac{\partial \mathbf{H}_0^+(x,z)}{\partial z} \mathrm{e}^{i\omega t} + \cdots, \tag{53}$$

where $\mathbf{H}_0^+(x, z)$ represents the magnetic field vector corresponding to the case of the undeformed plate. Considering the symmetric property of $\mathbf{H}_0^+(x, z)$ [10], and substituting equation (53) into equation (49), then neglecting the terms of α higher than order 1, one obtains the linear expression of the equivalent transverse magnetic force

$$q_z^{em}(x,t) = -2\mu_0\chi\alpha\mathbf{H}_0^+(x,h/2)\cdot\frac{\partial\mathbf{H}_0^+(x,h/2)}{\partial z}\bar{w}(x)e^{i\omega t} = \alpha\bar{q}_z^{em}(x)e^{i\omega t}, \qquad (54)$$

in which

$$\bar{q}_z^{em}(x) = -2\mu_0\chi\alpha\mathbf{H}_0^+(x,h/2)\cdot\frac{\partial\mathbf{H}_0^+(x,h/2)}{\partial z}\bar{w}(x). \qquad (55)$$

Substituting equations (51) and (54) into equations (44)–(46), one gets the eigenvalue equations as follows

$$D\frac{d^4\bar{w}}{dx^4} - N_x\frac{d^2\bar{w}}{dx^2} - \frac{dN_x}{dx}\frac{d\bar{w}}{dx} = \rho h\omega^2\bar{w}(x) + \bar{q}_z^{em}(x) \qquad 0 < x < L, \qquad (56)$$

with the boundary conditions

$$x = 0:\ \bar{w} = 0; \quad \frac{d\bar{w}}{dx} = 0; \qquad (57a, b)$$

$$x = L:\ \frac{d^2\bar{w}}{dx^2} = 0; \quad \frac{d^3\bar{w}}{dx^3} = 0. \qquad (58a, b)$$

Now, choose the eigen-function of the cantilevered beam–plate under no applied loads as an admissible function of the Galerkin method to calculate approximately the natural frequency of the ferromagnetic structure. From the textbook (see reference [15], pp. 161–166), one has

$$\begin{aligned}\bar{w}(x) = \bar{w}^*(x) = [(&\sin\beta L - \sinh\beta L)(\sin\beta x - \sinh\beta x)\\ &+ (\cos\beta L + \cosh\beta L)(\cos\beta x - \cosh\beta x)]/(\sin\beta L - \sinh\beta L), \qquad (59)\end{aligned}$$

which satisfies all boundary conditions listed in equations (57) and (58). By applying the Galerkin method to equation (56), one can obtain

$$\omega^2 = \omega_1^2 + \frac{1}{\Delta}\int_0^L N_x(x)\left[\frac{d\bar{w}^*}{dx}\right]^2 dx - \frac{1}{\Delta}\int_0^L \bar{q}_z^{em}(x)\bar{w}^*(x)\, dx, \qquad (60)$$

in which

$$\Delta = \rho h\int_0^L [\bar{w}^*(x)]^2\, dx, \qquad \omega_1^2 = \frac{1}{\Delta}\int_0^L D\frac{d^4\bar{w}^*(x)}{dx^4}\bar{w}^*(x)\, dx, \qquad (61, 62)$$

where ω_1 is an exact natural frequency of the cantilevered beam–plate under no applied loads. From equation (59), one can see that the last two terms on the right-hand side of the equation are the effect of the axial extension force and the equivalent transverse magnetic force on the natural frequency of the plate, respectively.

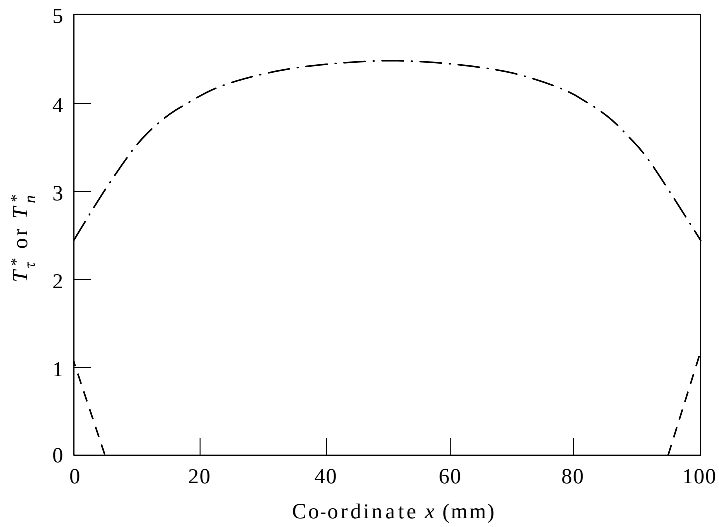

Figure 2. The distributions of T^*_τ (–·–·) and T^*_n (—) on the top and bottom surfaces of the beam–plate under an in-plane magnetic field ($B_0 = 1T$; $L/h = 100$; $\chi = 14{\cdot}0$).

5. RESULTS AND DISCUSSIONS

According to the analysis in previous sections, a numerical program to predict theoretically the natural frequency is carried out by computer for the free vibration of the first order of the cantilevered beam–plate in an in-plane applied magnetic field which is uniformly distributed. For the first mode of eigenfunction of vibration of a cantilever plate under no applied load, the value of βL in equation (59) is [15]

$$\beta L = 1{\cdot}875. \tag{63}$$

In order to make a comparison with the experimental data, the parameters of the apparatus are taken as those given in reference [2], which are listed in Table 2

The distribution of the magnetic field is analyzed by the finite element method [4, 10]. As pointed out in section 4, the effect of the magnetic field on the ferromagnetic plate with low susceptibility arises from two parts, that is, the axial extension force and the equivalent transverse magnetic force (equation (60)). Let

$$a_1 = \frac{1}{\varDelta}\int_0^L N_x(x)\left[\frac{\mathrm{d}\bar{w}^*(x)}{\mathrm{d}x}\right]^2 \mathrm{d}x, \qquad a_2 = -\frac{1}{\varDelta}\int_0^L \bar{q}_z^{em}(x)\bar{w}^*(x)\,\mathrm{d}x. \tag{64, 65}$$

Then, a_1 and a_2 display these two parts of the effect on the natural frequency. From the output values of a_1 and a_2 one finds that both a_1 and a_2 are greater than zero, and the value of a_2 is much greater than that of a_1. This shows that the increase of natural frequency is mainly generated from the equivalent transverse magnetic force $q_z^{em}(x)$, and a little from the axial force $N_x(x)$. The varying curves of natural frequency with the in-plane magnetic field $\mathbf{B}_0 = B_0\mathbf{i}$ are

TABLE 2

The parameters of apparatus employed in the experiments of cantilevered beam–plates made of the ferromagnetic material F-82H under in-plane magnetic fields (taken from reference [2])†

Density of mass, ρ	$7{\cdot}8 \times 10^3$ (kg/m^3)
Young modulus, E	$2{\cdot}0 \times 10^{11}$ (Pa)
Poisson's ratio, ν	0·30
Thickness of plate, h	0·29 and 0·50 (mm)
Length of plate, L	100 (mm)
Magnetic permeability, μ	$1-2 \times 10^{-5}$
Conductivity, σ	$2{\cdot}3 \times 10^6$ (S/m)

† The parameters of the experiments are supplied by Dr Takagi who re-checked them.

plotted in Figure 3(a) for $L/h = 100/0{\cdot}29$ and Figure 3(b) for $L/h = 100/0{\cdot}5$, and $\chi = 7 \sim 15$ which corresponds to the content of the parameter used in experiments [2]. From them, one can find that the change of natural frequency is effected by susceptibility χ. The larger the susceptibility of the ferromagnetic plate, the larger the change. The comparison of the theoretical predictions with the experimental data for the increase of natural frequency is shown in Figure 3. It is shown that the theoretical model proposed in this paper is successful in predicting the experimental phenomenon of the increase in natural frequency [2]. However, there is some quantitative difference between the theoretical predictions and the experimental data in Figure 3. It is possible that the non-linear magnetic field and non-linear interaction between magnetic fields and deformation would be responsible for the difference, which are not considered in this quantitative analysis.

When the small electricity conductivity σ is considered in the vibrating plates, according to the balance of the dissipated energy from the eddy current and the. equivalent magnetic damping, reference [2] gave a method to predict the magnetic damping coefficient c_j, that is

$$c_j = \frac{1}{\pi\sigma\omega_j}\int \mathrm{d}V \int_0^{2\pi/\omega_j} J^2 \,\mathrm{d}t. \tag{66}$$

Here, J is the density of the eddy current in the vibrating plates. Since the changes of vibrating frequency of the plates are not considered in their predictions, their predictions are all higher than the experimental data for the magnetic damping ratio ξ_j (see Figure 4) . Here, a revision of the predictions, after the change of natural frequency is considered in the method is given. According to Faraday's law of Maxwell equations, and Ohm's law, it is reasonable to assume that the density of the eddy current in the structures is proportional to the vibrating frequency or natural frequency of the structure. Under this assumption, from equation (66), one can find that the equivalent magnetic viscous damping coefficient c_j would be independent of the natural frequency ω_j. Let ξ_j^* be the theoretical prediction, given in reference [2], of the

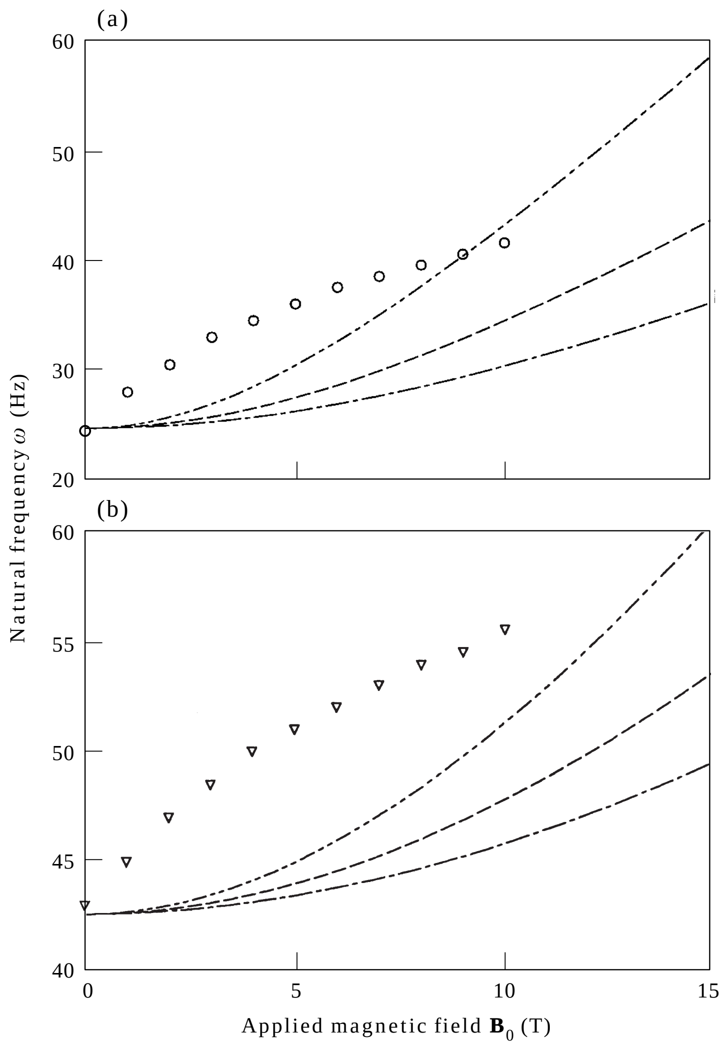

Figure 3. Comparison of theoretical predictions with experimental measurement of natural frequency of the increase of natural frequency for the F-82H ferromagnetic cantilevered beam–plates under in-plane magnetic field ($L = 100$ mm). (a) $h = 0{\cdot}29$ mm; (b) $h = 0{\cdot}50$ mm. - - - -, Theory: $\chi = 15{\cdot}0$; — — —, theory: $\chi = 10{\cdot}0$; —·—, theory: $\chi = 7{\cdot}0$; ○, experimental data (a); ∇, experimental data (b).

magnetic damping ratio which has the relation of

$$\xi_j^* = \frac{c_j}{2\omega_j^*}, \tag{67}$$

in which ω_j^* is a natural frequency of the plate structure when the applied magnetic field is equal to zero. Since the natural frequency is changed with the applied magnetic field, the real natural frequency is ω rather than ω_j^*. Hence, ω_j^* should be replaced by ω in equation (67). Denote

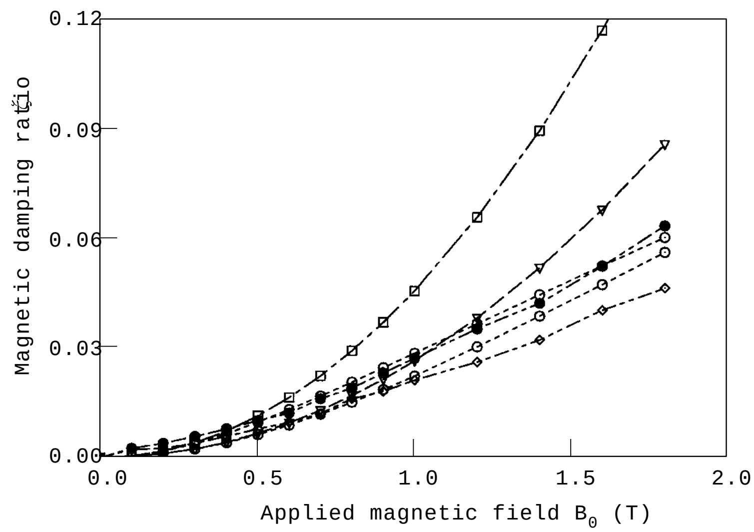

Figure 4. Comparison of theoretical prediction and experimental data of magnetic damping ratio versus applied magnetic fields for the ferromagnetic beam–plates vibrating in in-plane magnetic fields ($\chi = 14{\cdot}0$). ◇, Experiment, $h = 0{\cdot}5$; ▽, theory [2]; ○, theory (this paper); ●, experiment, $h = 0{\cdot}29$; □, theory [2].

$$\alpha = \frac{\omega_j^*}{\omega} < 1. \tag{68}$$

Then, the revised prediction of the magnetic damping ratio ξ is given by

$$\xi = \alpha \xi_j^*, \tag{69}$$

which makes it possible that the revised predictions of equation (69) will be close to the experimental measurement. Figure 4 exhibits the comparison of the, theoretical predictions and the experimental measurement of the magnetic damping ratio ξ for the ferromagnetic plates of $L/h = 100/0{\cdot}29$ and $L/h = 100/0{\cdot}5$ and the material F-82H of $\chi = 14$. It is clearly found that the theoretical predictions of the magnetic damping ratio, after it is revised by considering the increase in natural frequency, is close to the experimental data for the first case when $B_0 < 1{\cdot}60T$, and for the second case when $B_0 < 1{\cdot}20T$.

6. CONCLUSIONS

A theoretical model associated with a new expression of magnetic force for characterizing the magnetoelastic interaction of a ferromagnetic plate with low susceptibility in an in-plane applied magnetic field is obtained by means of the approach of the variational principle. It is numerically shown that this theoretical model can describe the experimental phenomenon of the increase in natural frequency of the cantilevered ferromagnetic beam–plates under in-plane magnetic fields. After the change in natural frequency is considered, the predictions of magnetic damping ratio is in good agreement with the experimental data.

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