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# STRAIN-BASED ROTATIONAL MODE MEASUREMENT IN A BEAM

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A new method to obtain beam rotational modes from dynamic strain measurements is proposed. Unlike existing measurement techniques that utilize transverse acceleration signals, rotational modes are spatially integrated from the measured strain modes. The use of the strain gage eliminates the mass loading effect, which may not be negligible in accelerometer applications. The spatial integration of the signal makes the rotational modes obtained less sensitive to local measurement errors, that may be serious in existing measurement techniques. Another important feature is that the present technique directly estimates the rotation of the normal of the middle surface, and not the rotation of the middle surface. Several experiments are performed to verify the validity of the present method, and the limitation and potential applications are discussed. This paper also presents the theoretical background that allows the strain mode measurement in the Timoshenko beam.


## 1. INTRODUCTION

Due to difficulties in measuring the time signal of rotational mobilities and thus rotational modes in structures, several rotational mobility measurement techniques have been suggested. Among others, the direct measurement techniques [1, 2] may be preferred because of the applicability in general situations. However, the mass loading effect of the special fixture needed in the measurement cannot be ignored and careful corrections must be made to obtain meaningful rotational mobilities.

Alternatively, techniques [3–5] that utilize translational mobilities and modes may be employed. The process to determine rotational mobilities from mobilities of the transverse displacement is equivalent to the numerical differentiation of a function. Therefore, local measurement errors may affect the accuracy of the rotational modes quite significantly.

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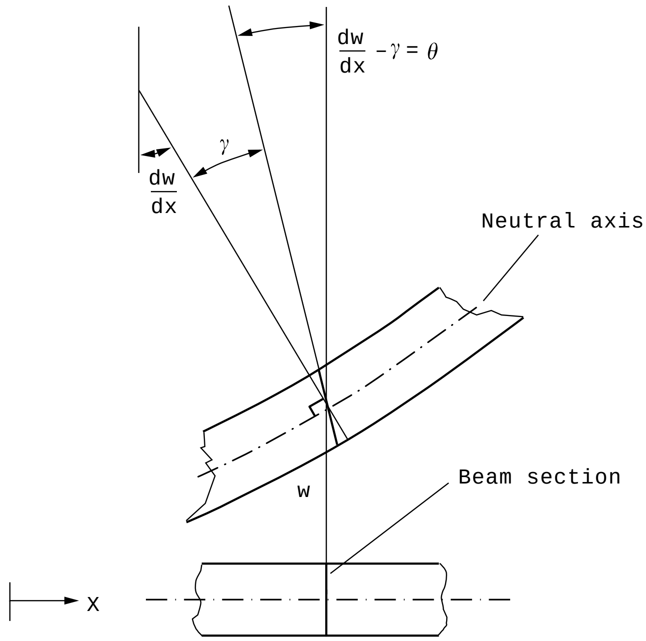


Figure 1. The deformation of the Timoshenko beam.

In this paper, a new method to obtain the rotational mobilities or mode shapes from the dynamic strain measurement is proposed.* Unlike existing techniques, strain-signals, more specifically, curvature signals are integrated in space in order to determine the rotational mobility. The major advantages of the present technique are: (1) the results have insignificant noise-like errors unlike those obtained from the differentiation of translational mobility; (2) the mass loading effect of the strain gage is almost negligible; and (3) the present technique estimates directly the rotation of the normal of the middle surface, and not the rotation of the middle surface. Since the normal rotation is directly measured, the present technique is not only useful in relatively thick beams but also desirable in component mode synthesizing coupled with finite element analysis.

## 2. THEORETICAL BACKGROUND

Although the present technique may be applied to any thin structure, the rotational degree measurement technique is examined in a beam. Consider an infinitesimal element of a beam during deformation as shown in Figure 1, where the rotation of the normal and the rotation of the middle plane are denoted by $\theta$ and $\mathrm{d}w/\mathrm{d}x$. In the Timoshenko beam theory where shear deformation is considered, the shear strain $\gamma$ is given by [7]

*The underlying idea of this method was presented by the present authors at the 15*th* *International Modal Analysis Conference* [6].

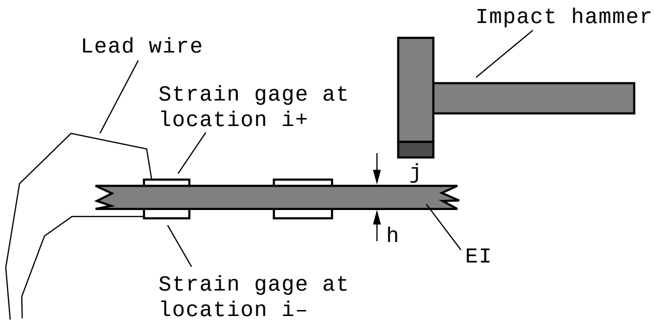


Figure 2. The schematic diagram of the experimental set-up.

$$\gamma = \frac{dw}{dx} - \theta, \tag{1}$$

and the curvature change $\kappa$ is expressed by

$$\kappa = \frac{d\theta}{dx}. \tag{2}$$

Symbols $i$ and $j$ in Figure 2 denote the locations of measurement and excitation. For bending deformation in a beam, the strain distribution through thickness can well be approximated by a linear function, as indicated in Figure 3. The time signal of the strains measured at location $i$ from the dynamic strain gages mounted on the top and bottom surfaces of a beam may be represented as

$$\begin{aligned} \varepsilon^{+} &= \varepsilon_{axial} + \varepsilon_{bending} \\ &= \frac{P}{EA} + \frac{1}{2}\frac{Mh}{EI}, \end{aligned} \tag{3}$$

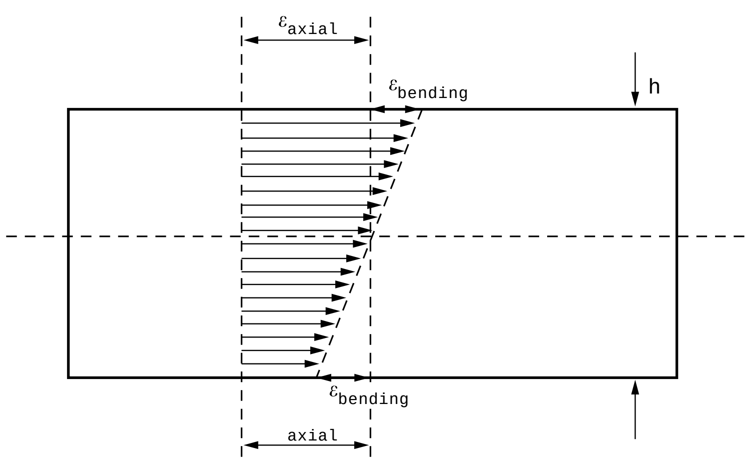


Figure 3. The strain distribution through the beam thickness which may be assumed to be linear.

$$\varepsilon^- = \varepsilon_{axial} - \varepsilon_{bending} = \frac{P}{EA} - \frac{1}{2}\frac{Mh}{EI}. \tag{4}$$

The superscripts + and − imply signals measured on the top and bottom surfaces.

It is rather straightforward to measure the curvature change $\kappa$ from the bending strain $\varepsilon_{bending}$ since

$$\varepsilon_{bending} \equiv \varepsilon^+ - \varepsilon^- = \frac{Mh}{EI} = h\kappa. \tag{5}$$

Equation (5) also shows that curvature change $\kappa$ is equal to bending strain, divided by beam thickness. In experiment, the quantity $\varepsilon_{bending}$ (which is proportional to $\kappa$), can be easily measured by employing the Wheatstone bridge circuit [8] from the strain signals $\varepsilon^+$ and $\varepsilon^-$ acquired from the dynamic strain gages.

For later development, it is useful to write the governing differential equations for the Timoshenko beam theory [7] as

$$EI\frac{\partial^2\theta}{\partial x^2} + kAG\left(\frac{\partial w}{\partial x} - \theta\right) = \rho I\frac{\partial^2\theta}{\partial t^2} - m(x, t), \tag{6}$$

$$kAG\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\theta}{\partial x}\right) = \rho A\frac{\partial^2 w}{\partial t^2} - f(x, t). \tag{7}$$

The applied moment and force are denoted by $m(x, t)$ and $f(x, t)$ in equations (6, 7). Denoting the translational and rotational normal modes by $\{W^{(r)}\}$ and $\{\Theta^{(r)}\}$, one can show the following orthogonality relation [9] exists

$$\int_0^L \rho A[W^{(r)}(x)W^{(s)}(x) + r_g^2\Theta^{(r)}(x)\Theta^{(s)}(x)]\,\mathrm{d}x = \delta_{rs} \quad (r, s = 1, 2, \ldots), \tag{8}$$

where $r_g$ the radius of gyration. This orthogonality relation applies to beams with any type of boundary conditions. If the applied force is assumed to be harmonic such that

$$f(x, t) = F(x)\,\mathrm{e}^{i\omega t}, \quad m(x, t) = 0, \tag{9, 10}$$

the steady-state forced-vibration response may be written as

$$w(x, t) = W(x)\,\mathrm{e}^{i\omega t} = \sum_{r=1}^{\infty} C_r(\omega)W^{(r)}(x)\,\mathrm{e}^{i\omega t}, \tag{11}$$

$$\theta(x, t) = \Theta(x)\, e^{i\omega t} = \sum_{r=1}^{\infty} C_r(\omega)\Theta^{(r)}(x)\, e^{i\omega t}, \tag{12}$$

$$\varepsilon(x, t) = S(x)\, e^{i\omega t} = \sum_{r=1}^{\infty} C_r(\omega) S^{(r)}(x)\, e^{i\omega t}. \tag{13}$$

The strain mode shape $S^{(r)}(x)$ is found to be

$$S^{(r)}(x) = h\frac{d\Theta^{(r)}(x)}{dx} \tag{14}$$

from equations (2, 5).

Once the strain mode shape is known, the rotation mode shape can be integrated as

$$\Theta^{(r)}(x) = \frac{1}{h}\int_0^x S^{(r)}(x)\, dx + \Theta_0^{(r)} \equiv \hat{\Theta}^{(r)}(x) + \Theta_0^{(r)}, \tag{15}$$

where $\Theta_0^{(r)}$ is the integration constant. The procedure to determine $\Theta_0^{(r)}$ will be discussed later.

In order to find $C_r$ for given $F(x)$, one may use the orthogonality condition (8). The result is

$$(-\omega^2 + \Omega_r^2)C_r = \int_0^L W^{(r)}(x)F(x)\, dx. \tag{16}$$

In the derivation of equation (16), the following relation is used:

$$EI\int_0^L \Theta^{(r)}\frac{d^2\Theta^{(r)}}{dx^2}\, dx + kAG\int_0^L \Theta^{(r)}\left(\frac{dW^{(r)}}{dx} - \Theta^{(r)}\right) dx + \int_0^L W^{(r)}\frac{d^2W^{(r)}}{dx^2}\, dx - \int_0^L W^{(r)}\frac{d\Theta^{(r)}}{dx}\, dx = -\Omega_r^2.$$

Now substituting $C_r$ in equation (16) into equations (12, 13) gives

$$\Theta(x, \omega) = \sum_{r=1}^{\infty}\frac{\Theta^{(r)}(x)}{\Omega_r^2 - \omega^2}\int_0^L W^{(r)}(x)F(x)\, dx, \tag{17}$$

$$S(x, \omega) = \sum_{r=1}^{\infty}\frac{S^{(r)}(x)}{\Omega_r^2 - \omega^2}\int_0^L W^{(r)}(x)F(x)\, dx. \tag{18}$$

If $F(x)$ is a point force applied at $x = x_j$ such that

$$F(x) = F_j \delta(x - x_j), \tag{19}$$

the frequency response of the rotation and strain at location $x = x_i$ are given by

$$\alpha_{ij} \equiv \frac{\Theta(x_i, \omega)}{F_j} = \sum_{r=1}^{\infty} \frac{\Theta_i^{(r)} W_j^{(r)}}{\Omega_r^2 - \omega^2}, \tag{20}$$

$$\beta_{ij} \equiv \frac{S(x_i, \omega)}{F_j} = \sum_{r=1}^{\infty} \frac{S_i^{(r)} W_j^{(r)}}{\Omega_r^2 - \omega^2}. \tag{21}$$

In equations (20, 21), the following definition is used

$$W_i^{(r)} = W^{(r)}(x_i); \quad \Theta_i^{(r)} = \Theta^{(r)}(x_i); \quad S_i^{(r)} = S^{(r)}(x_i). \tag{22}$$

From equation (21), it is clear that the relative ratios of $\beta_{ij}$ measured at various locations with a fixed excitation point give the strain mode shape. On the other hand, the relative ratios of $\beta_{ij}$ measured at one location with varying excitation locations give displacement mode shape.

The foregoing analysis gives the theoretical background on how to find the strain modes in the Timoshenko beam, and shows that the rotational mode shape can be found from the strain mode shape. One of the important issues in determining the rotation mode shape from equation (15) is how to determine the integration constant $\Theta_0^{(r)}$. This will be discussed below in detail.

In the first case, consider a beam fixed at any point, say fixed at $x = 0$. In this case the rotation as well as the deflection at $x = 0$ must vanish, i.e., $w(x = 0, t) = \theta(x = 0, t) = 0$. Recall that $\theta(x, t)$ may be written as

$$\theta(x, t) = \sum_{r=1}^{\infty} C_r \Theta^{(r)}(x)\, e^{i\Omega_r t}. \tag{23}$$

In order to satisfy the boundary condition at $x = 0$ , it is obvious that every rotation mode shape must satisfy the condition

$$\Theta^{(r)}(0) = 0. \tag{24}$$

The integration constant $\Theta_0^{(r)}$ in equation (15) is simply selected to satisfy the condition expressed by equation (24).

To determine the integration constant $\Theta_0^{(r)}$ in equation (14) for other boundary conditions, it is proposed to use the fact that any elastic mode is orthogonal to the rigid-body rotational mode. In the Timoshenko beam, the rigid-body rotation mode (in addition to the rigid-body translation, which is not needed here) is given as

$$\begin{matrix} \Theta^0(x) = A \\ W^0(x) = Ax \end{matrix} \quad (A = \text{constant}). \tag{25}$$

Applying the orthogonality of the elastic mode and the rigid-body rotation gives the following result:

$$\Theta_0^{(r)} = -\frac{1}{r_g^2}\int_0^L \rho x W^{(r)}(x)\,\mathrm{d}x - \int_0^L \rho\hat{\Theta}^{(r)}(x)\,\mathrm{d}x. \tag{26}$$

Obviously, the determination of $\Theta_0^{(r)}$ requires the transverse displacement mode shapes, which can be determined by the use of either accelerometers or strain gages.

For a beam hinged at both ends, the procedure to determine $\Theta_0^{(r)}$ is similar to that for the free–free beam. However, the rigid-body mode in this case is the thickness shear mode [10] such that $\Theta^{(0)}(x) = A$ and $W^{(0)}(x) = 0$. Subsequently the integration constant $\Theta_0^{(r)}$ is determined as

$$\Theta_0^{(r)} = -\int_0^L \hat{\Theta}^{(r)}(x)\,\mathrm{d}x. \tag{27}$$

In the situation where the strain gage installation on both the top and bottom surfaces of a beam is not feasible, a flex gage [11] may be cemented only on one side of a beam to measure the bending strain, as depicted in Figure 4. From the strain measurement $\varepsilon_1$ and $\varepsilon_2$ at locations 1 and 2, the bending strain $\varepsilon_{bending}$ can be determined. Thus, one may use

$$\varepsilon_{bending} \equiv \frac{h}{h'}(\varepsilon_1 - \varepsilon_2) = h\kappa.$$

## 3. VERIFICATION OF THE PROPOSED TECHNIQUE

### 3.1. CASE I: FREE–FREE BEAM

To verify the present measuring technique of the rotational mode in a free–free beam, an aluminum beam with square cross-section (1 cm by 1 cm) is used. The beam length is 100 cm. Before presenting the results for the rotational mode

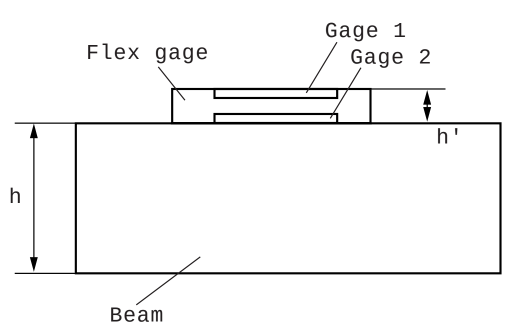


Figure 4. The one-sided strain gage for the bending strain measurement.

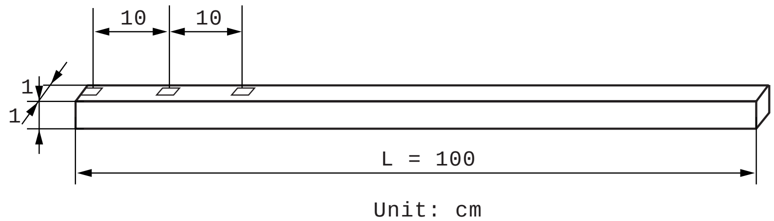


Figure 5. The test specimen made of aluminum. The metal foil strain gages with 350-$\Omega$ resistance are installed on the top and bottom surfaces. Eleven pairs of strain gages are used, which are 10 cm apart.

shapes obtained from strain signals, one must first determine the natural frequencies.

The metal foil strain gages with 350-$\Omega$ resistance are installed on the top and bottom surfaces. Eleven pairs of strain gages are used, which are 10 cm apart. The effect of the gage length on measurement error is discussed in reference [8]. The signal wavelength that can be accurately obtained also depends on the number of installed gages. Since the rotational normal modes are integrated from the strain normal modes, the strain measurement points are equivalent to the integration points for a function. The corresponding error analysis may be found in reference [12].

A typical strain frequency response function obtained for the test beam is shown in Figure 6 and the quality of the present measurement is demonstrated by the good coherence (denoted by C12 in Figure 6). A PCB impact hammer is used for beam excitation.

The measured dynamic strains at various locations are analyzed by the polyreference technique of T–DAS [13] and the results for the natural frequencies are listed in Table 1. These results are compared with those obtained from the transverse acceleration measurement. Table 1 indicates the well-known

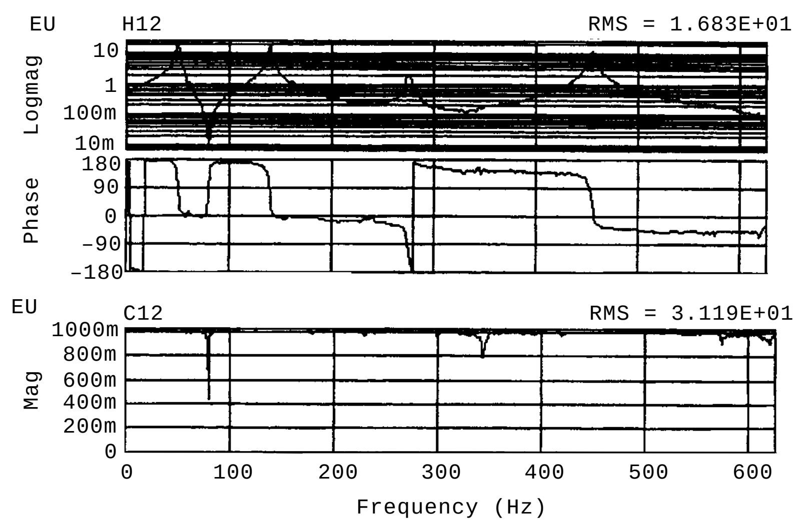


Figure 6. A typical strain frequency response function obtained for the test beam.

TABLE 1

*The natural frequencies measured by means of accelerometers and strain gages in the test specimen shown in Figure* 5

| Model No. | Transverse acceleration measurement (accelerometers) | Strain measurement (present) |
|---|---|---|
| 1 | 48.3 Hz | 50.86 Hz |
| 2 | 137.3 Hz | 139.6 Hz |
| 3 | 272.1 Hz | 275.9 Hz |
| 4 | 451.4 Hz | 455.2 Hz |

mass-loading effect of the accelerometer (weighing 10.5 g); the natural frequencies obtained by accelerometers are somewhat lower than those by the strain gages.

To obtain the rotational mode shapes from the measured strain, equation (15) is utilized. The orthogonality condition given in equation (8) is used to determine the integration constant, as explained in the previous section. The results for the first four rotational mode shapes are plotted in Figure 7. We measured the strain signal up to the frequency range of 625 Hz. In Figure 7, the solid lines are used to designate the analytic modes based on the Timoshenko beam theory [9]. It is apparent that the present results agree well with the analytic results.

### 3.2. CASE II: CLAMPED–FREE BEAM

The process of determining an integration constant is rather straightforward in the case of a beam with a clamped end. We consider a clamped–free beam, which is 91 cm long. The integration constant is determined easily from the condition stated in equation (24). The present rotational mode shapes are plotted in Figure 8 and agree well with the analytic mode shapes.

## 4. DISCUSSION

The limitation of this approach is that the present technique may not be applicable to a wide class of practical problems at the present stage. However, the application of the present idea in measuring the tip rotation of some actuators is possible. For instance, the continuous piezoceramic coating on the upper and lower surface of a beam-type actuator, instead of discrete strain gage installation, may be utilized in measuring the actuator tip rotation. This paper also presents the theoretical background that allows the strain mode measurement in the Timoshenko beam.

## 5. CONCLUSIONS

A new experimental technique to measure the rotational modes of beams from the dynamic strain measurement has been proposed. The key idea is to perform the spatial integration of the curvature signals which are measured from paired strain gages through the Wheatstone bridge. The procedure to determine spatial

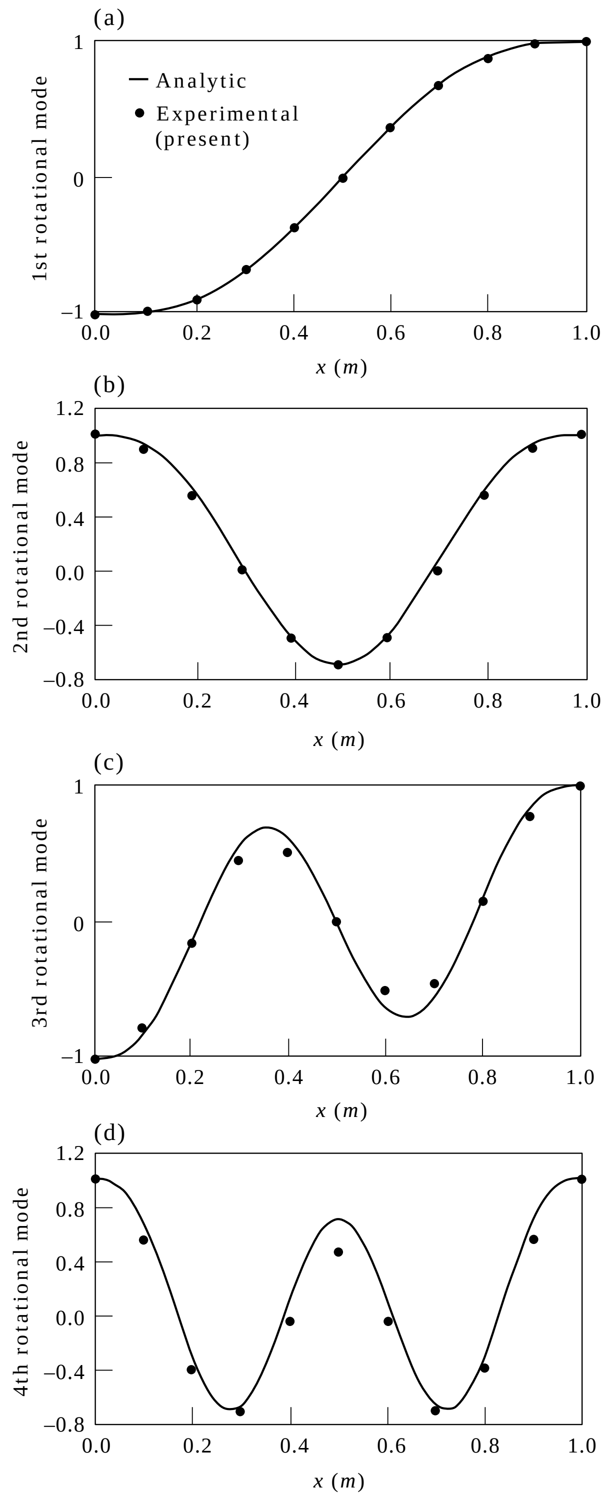


Figure 7. (a) The first, (b) the second, (c) the third, and (d) the fourth rotational mode of the free–free beam shown in Fig. 5.

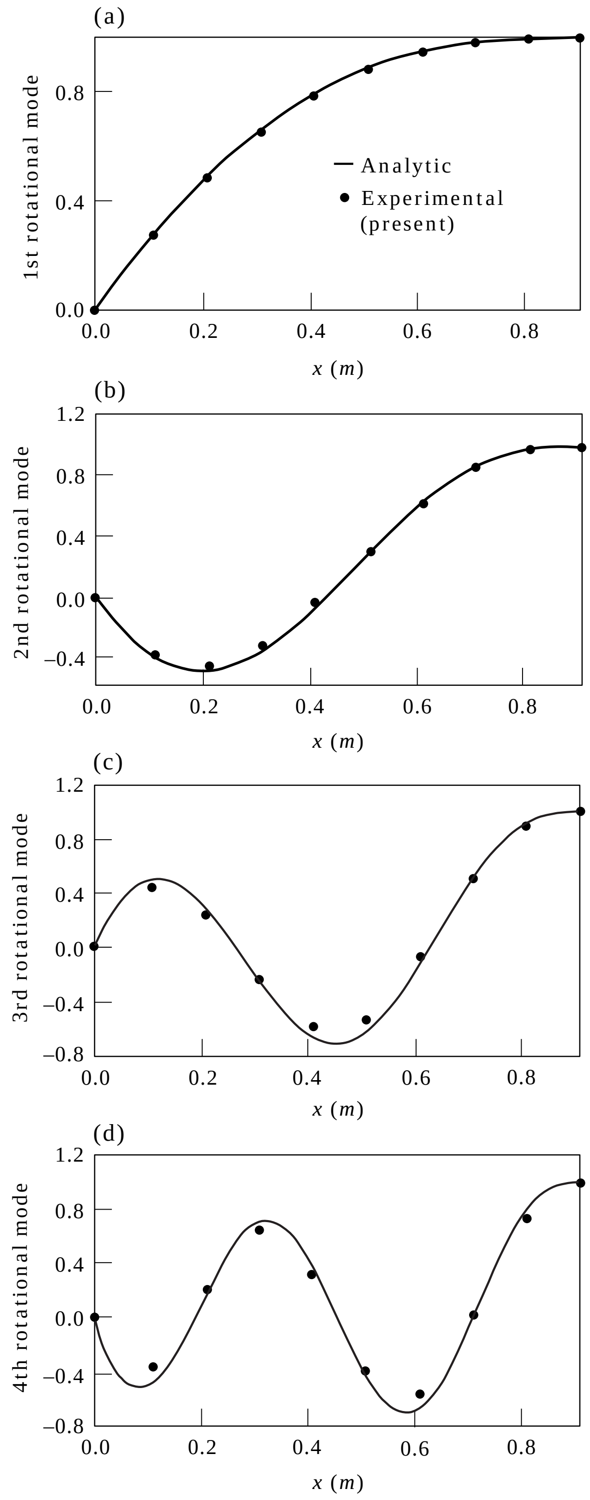


Figure 8. (a) The first, (b) the second, (c) the third; and (d) the fourth rotation mode shape for the clamped-free beam.

integration constants is developed, and a theoretical background to this approach is presented. Several experiments on a test beam confirmed the validity of the present method. Applications of the present technique to tapered beams and plates/shells appear to be interesting future research topics.

## ACKNOWLEDGMENT

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## APPENDIX: NOMENCLATURE

$w$ transverse displacement of the middle plane
$\theta$ rotation of the normal
$M$ bending moment
$P$ axial force
$EI$ bending rigidity of a beam
$\varepsilon$ strain
$A$ cross-sectional area of a beam

| | |
|---|---|
| $S^{(r)}$ | the $r$th bending strain mode |
| $W^{(r)}$ | the $r$th transverse displacement mode |
| $\Theta^{(r)}$ | the $r$th rotational mode |
| $h$ | beam thickness |
| $h'$ | flex gage thickness |
| $r_g$ | radius of gyration |
| $\Omega_r$ | the $r$th natural frequency |
| $\omega$ | excitation frequency |
| $\overline{(\ )}$ | quantity expressed in the frequency domain |