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ON OPTIMAL SELECTION OF INTERIOR POINTS FOR APPLYING DISCRETIZED BOUNDARY CONDITIONS IN DQ VIBRATION ANALYSIS OF BEAMS AND PLATES

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The direct implementation of derivative boundary conditions at edge points was demonstrated to be simple, flexible and computationally efficient in the vibration analysis of beams and plates by using the differential quadrature (DQ) method. In the approach, the discretized governing equations at certain interior points need to be replaced by the discretized boundary conditions. Naturally, one may have questions as to whether the replacement points can be selected arbitrarily, and how to choose the optimal replacement points. This paper tries to answer these questions, and its focus is to investigate systematically the effect of the replacement location on the accuracy of the numerical solution. From the error distribution analysis of derivative approximation, it is found that the optimal position for the replacement is the interior point just adjacent to the boundary. This conclusion is confirmed by the numerical experiments of free vibration analysis for beams and plates with various boundary conditions.

1. INTRODUCTION

Since the computation of weighting coefficients was improved by Quan and Chang [1], and generalized by Shu [2], Shu and Richards [3], the differential quadrature (DQ) method has been increasingly applied in engineering. A good review on the application of the DQ-type method in engineering was given by Bert and Malik [4]. For the application of the DQ-type method in structural and vibration analysis, pioneer work was done by Bert *et al.* [5]. The standard governing equations in structural mechanics usually involve fourth order derivatives. To ensure that the problem is well-posed, two boundary conditions at each boundary should be specified. Thus, the structural and vibration problems are actually high order boundary value problems with multiple boundary conditions. Numerically, some careful considerations are needed to implement properly the multiple boundary conditions at each edge.

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For the application of the DQ-type method in structural mechanics, there are various approaches to implement the multiple boundary conditions. One is the so-called δ-technique proposed by Bert *et al.* [5] and Jang *et al.* [6], in which two grid points, separated from each other by a small distance δ, are placed near each boundary edge. Then, the two boundary conditions at each boundary are applied at the boundary point itself and its adjacent δ-point. Usually, the derivative condition is applied at the δ-point. To obtain an accurate numerical result, the δ-value should be very small ($\delta \cong 10^{-5}$ in dimensionless value [4–7]). If the value of δ is small enough, the approach produces good results in some situations such as the clamped condition. However, the approach failed to work well in the other situations such as free edges. In addition, arbitrariness in the choice of the δ value may introduce unexpected oscillation behavior of the solutions. To overcome the above difficulties in the δ approach, Wang and Bert [8] presented an approach which firstly incorporates the boundary conditions into the DQ weighting coefficient matrices, and the modified weighting coefficients are then used to discretize the governing equations. The technique resulted in an obvious improvement in the DQ solution of beams and plates with free and simply-supported boundary conditions [8]. However, the technique is not applicable to problems with cross derivative and free corner conditions. The modification of the weighting coefficient matrices also causes some loss of flexibility and incurs greater additional computational efforts due to some matrix products, which require $O(N^4)$ scalar operations.

An intuitive methodology is to implement directly the double boundary conditions exactly at edge points [4, 9–11]. Recently, Shu and Du [12, 13] have given a systematic use of the methodology in the solution of vibration problems of beams and plates with various boundary conditions, including the first application in plates with free corners. The work shows that this approach is uniformly successful for all known boundary conditions and seems to have no limitation for its use so far. The accuracy, efficiency and stability of the approach are consistently superior to the traditional δ-technique in all cases examined. Wang and Bert's approach [8] worked slightly better than Shu and Du's approach in very limited situations [11]. However, the applicability of Wang and Bert's approach is subject to grave limitations in engineering problems due to its incapability in handling cross derivative boundary conditions. It seems that Shu and Du's approach has a potential to be further developed into an efficient and flexible numerical technique for solving high order boundary value problems in practical engineering. When the derivative condition is discretized by the DQ-type method, the resultant algebraic equation involves the boundary points as well as the interior points. The idea of Shu and Du's approach is to replace the discretized governing equation by the discretized boundary condition equation for some interior points. In the work of Shu and Du [12, 13], the interior point is chosen as the immediate neighboring point to the boundary. Here, one may raise the question as to whether this choice is optimal in terms of the accuracy and efficiency of the approach. This is the purpose of the present study.

Based on truncation error analysis for derivative approximation, this paper will provide a systematic investigation on the choice of interior points in which the discretized governing equation is replaced by the discretized boundary condition equation. The truncation error formulas newly developed by Chen [11] will be used to study the error distribution for the first, second, third, and fourth order derivatives. Since in Shu and Du's approach, the discretized governing equation is replaced by the discretized boundary condition equation at a selected interior point, it can be expected that the optimal position could be the one at which the truncation error of discretized governing equation reaches maximum. When an equation with maximum truncation error is replaced by another equation with less truncation error, the overall truncation error of the equation system will be reduced. Thus, the accuracy of numerical solution can be improved. The guidance of error distribution analysis for selecting the optimal replacement points will be validated by vibration analysis of beams and plates. Through the error distribution analysis and many numerical experiments by changing the replacement point, it was found that the optimal position is the interior point immediately neighboring to the boundary.

2. GENERALIZED DIFFERENTIAL QUADRATURE (GDQ)

The GDQ method will be used to discretize the derivatives in the governing equation and the boundary conditions. The GDQ approach was developed by Shu *et al.* [2, 3] to improve the DQ technique [14] for the computation of weighting coefficients. The nth order derivative of a function $f(x, t)$ with respect to x at a grid point x_i, can be approximated by the DQ approach as

$$f_x^{(n)}(x_i, t) = \sum_{k=1}^{N} c_{ik}^{(n)} \cdot f(x_k, t), \qquad n = 1, 2, \ldots, N-1, \quad \text{for} \quad i = 1, 2, \ldots, N, \quad (1)$$

where N is the number of grid points in the whole domain. $c_{ik}^{(n)}$ are the weighting coefficients to be determined by the GDQ method as:

weighting coefficients for the first order derivative

$$c_{ij}^{(1)} = \frac{A^{(1)}(x_i)}{(x_i - x_j) \cdot A^{(1)}(x_j)}, \qquad i, j = 1, 2, \ldots, N \quad \text{but} \quad j \neq i \quad (2)$$

$$c_{ii}^{(1)} = -\sum_{j=1, j\neq i}^{N} c_{ij}^{(1)}, \qquad i = 1, 2, \ldots, N, \quad (3)$$

where

$$A^{(1)}(x_i) = \sum_{k=1, k\neq i}^{N} (x_i - x_k);$$

weighting coefficients for the second and higher order derivatives

$$c_{ij}^{(n)} = n \cdot \left(c_{ii}^{(n-1)} \cdot c_{ij}^{(1)} - \frac{c_{ij}^{(n-1)}}{x_i - x_j} \right) \quad \text{for} \quad i,\ j = 1, 2, \ldots, N,$$

$$\text{but} \quad j \neq i, \quad n = 2, 3, \ldots, N-1, \tag{4}$$

$$c_{ii}^{(n)} = -\sum_{j=1, j\neq i}^{N} c_{ij}^{(n)}, \quad \text{for} \quad i = 1, 2, \ldots, N, \quad n = 2, 3, \ldots, N-1. \tag{5}$$

It is obvious from the above equations that the weighting coefficients of the second and higher order derivatives can be completely determined from those of the first order derivative.

3. TRUNCATION ERROR OF DERIVATIVE APPROXIMATION

Recently, Chen [11] has presented a number of new formulas for the analysis of truncation error distribution of derivatives in the DQ approximation. These formulas are based on the polynomial interpolation approach and the formulation of weighting coefficients given by Shu and Richards [3]. The formulas are different from those given by Bellman *et al*. [14] in that they can estimate the truncation error at every mesh point and expose the definite convergence speed more accurately.

If function $f(x)$ is smooth enough, it can be approximated by a Lagrangian polynomial,

$$f(x) = -\sum_{j=1}^{N} p_j(x) f_j(x)(p_j(x) f(x_j)) + R(x), \qquad j = 1, 2, \ldots, N, \tag{6}$$

where $p_j(x)$ is the Lagrangian interpolated polynomial, $R(x)$ is the truncation error, given by

$$R(x) = \frac{f^{(N)}(\xi) W(x)}{N!}, \tag{7}$$

where

$$W(x) = \prod_{i=1}^{N} (x - x_i).$$

Differentiating equation (6) with respect to x and then applying at each mesh point gives

$$f^{(1)}(x_i) = \sum_{j=1}^{N} p_j^{(1)}(x_i) f(x_j) + R^{(1)}(x_i) = \sum_{j=1}^{N} c_{ij}^{(1)} f(x_j) + R^{(1)}(x_i), \tag{8}$$

where $c_{ij}^{(1)}$ are the GDQ weighting coefficients of the first order derivative, x_i is the co-ordinate of grid points. $R^{(1)}(x_i)$ is the truncation error of the first order derivative approximation by GDQ approach. Thus, one has

$$R^{(1)}(x_i) = \frac{f^{(n)}(\xi)W^{(1)}(x_i)}{N!}, \qquad i = 1, 2, \ldots, N. \tag{9}$$

Let $K_1 = \max\{|f^{(N)}(\xi)|\}$, one has

$$|R^{(1)}(x_i)| \leqslant K_1 \frac{|W^{(1)}(x_i)|}{N!} = K_1 e^{(1)}(x_i), \qquad i = 1, 2, \ldots, N. \tag{10}$$

In fact,

$$|W^{(1)}(x_i)| = \left| \prod_{k \neq i}^{N} (x_i - x_k) \right| = p(x_i). \tag{11}$$

Therefore, one obtains

$$e^{(1)}(x_i) = \frac{p(x_i)}{N!}. \tag{12}$$

For the second order derivative, one has

$$R^{(2)}(x_i) = \frac{2\xi_x f^{(N+1)}(\xi)W^{(1)}(x_i)}{N!} + \frac{f^{(N)}(\xi)W^{(2)}(x_i)}{N!}, \qquad i = 1, 2, \ldots, N. \tag{13}$$

Following the formulation of weighting coefficients given by Shu [3], one has

$$c_{ii}^{(m-1)} = \frac{W^{(m)}(x_i)}{mW^{(1)}(x_i)}. \tag{14}$$

where $c_{ii}^{(m-1)}$ are the diagonal entries of the DQ weighting coefficient matrix for the $(m-1)$th order derivative. $W^{(m)}$ denotes the mth order derivative of function $W(x)$. Therefore

$$W^{(m)}(x_i) = mc_{ii}^{(m-1)} W^{(1)}(x_i). \tag{15}$$

Substituting formula (15) into equaion (13), one has

$$|R^{(2)}(x_i)| \leqslant 2K_2(1 + |c_{ii}^{(1)}|) \frac{p(x_i)}{N!} = K_2 e^{(2)}(x_i), \tag{16}$$

where $K_2 = \max\{|f^{(N)}(\xi)|, |\xi_x f^{(N+1)}(\xi)|)\}$, and

$$e^{(2)}(x_i) = {}^2(1 + |c_{ii}^{(1)}|) \frac{p(x_i)}{N!}.$$

Similarly, one can obtain

$$|R^{(3)}(x_i)| \leqslant 3K_3(2 + 2|c_{ii}^{(1)}| + |c_{ii}^{(2)}|) \frac{p(x_i)}{N!} = K_3 e^{(3)}(x_i) \tag{17}$$

$$|R^{(4)}(x_i)| \leqslant 4K_4(5 + 6|c_{ii}^{(1)}| + 3|c_{ii}^{(2)}| + |c_{ii}^{(3)}|)\frac{p(x_i)}{N!} = K_4 e^{(4)}(x_i) \tag{18}$$

for the third and fourth order derivatives, respectively, where K_3 and K_4 are the maximum values of composite derivatives of ξ and $f(x)$ up to the $(N+3)$ order, and

$$e^{(3)}(x_i) = 3(2 + 2|c_{ii}^{(1)}| + |c_{ii}^{(2)}|\ \frac{p(x_i)}{N!},$$

$$e^{(4)}(x_i) = 4(5 + 6|c_{ii}^{(1)}| + 3|c_{ii}^{(2)}| + |c_{ii}^{(3)}|\ \frac{p(x_i)}{N!}.$$

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4. GOVERNING EQUATIONS AND NUMERICAL IMPLEMENTATION

The governing equations of transverse vibration of beams and plates and their numerical implementation by the GDQ approach will be shown in this section. In the present study, all independent variables are normalized to the interval [0, 1]. For beams, N is the number of grid points, and A_{ij}, B_{ij}, C_{ij} and D_{ij} represent the GDQ weighting coefficients of the first, second, third and fourth order derivatives. For plates, N and M are the number of grid points in the x and y directions, and A_{ij}, B_{ij}, C_{ij} and D_{ij} with superscript x and y denote the weighting coefficient matrices of the first, second, third and fourth order derivatives along the x and y directions. For simplicity, the simply-supported, clamped and free edge conditions are denoted by SS, C and F respectively.

4.1. TRANSVERSE VIBRATION OF BEAMS

The non-dimensional governing equation for the free vibration of a uniform beam is given by

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = \bar{\omega}^2 w, \tag{19}$$

where $\bar{\omega}^2 = \rho A_0 L^4 \omega^2 / EI$ is the dimensionless frequency, ω is the natural frequency of free vibration. A_0, L and ρ are the constant cross-sectional area, the length of the beam, and the density, respectively, E is the elastic modulus and I is the constant area moment of inertia about the neutral axis. Since equation (19) is the fourth order, two boundary conditions are needed at each end. In the present work, three types of boundary conditions are considered:

Simply supported end (SS):

$$W = 0 \quad \text{and} \quad \frac{\partial^2 W}{\partial x^2} = 0 \quad \text{at} \quad x = 0, 1. \tag{20}$$

Clamped end (C):

$$W = 0 \quad \text{and} \quad \frac{\partial W}{\partial x} = 0 \quad \text{at} \quad x = 0, 1. \tag{21}$$

Free end (F):

$$\frac{\partial^2 W}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial^3 W}{\partial x^3} = 0 \quad \text{at} \quad x = 0, 1. \tag{22}$$

Using then the GDQ method, equation (19) can be discretized as

$$\sum_{j=1}^{N} D_{ij} w_j = \bar{\omega}^2 w_i. \tag{23}$$

It is noted that equation (23) should be applied at interior points. Similarly, the derivatives in equations (20)–(22) can also be discretized by the GDQ method. The discretized form of equation (20) can be written as

$$w_1 = 0, \sum_{k=1}^{N} B_{1,k} \cdot w_k = 0, \qquad \text{at}\, x = 0, \tag{24a}$$

$$w_N = 0, \sum_{k=1}^{N} B_{N,k} \cdot w_k = 0, \qquad \text{at}\, x = 1, \tag{24b}$$

and the discretized form of equation (2l) is

$$w_1 = 0, \sum_{k=1}^{N} A_{1,k} \cdot w_k = 0, \qquad \text{at}\, x = 0, \tag{25a}$$

$$w_N = 0, \sum_{k=1}^{N} A_{N,k} \cdot w_k = 0, \qquad \text{at}\, x = 1. \tag{25b}$$

In a similar way, equation (22) is discretized as

$$\sum_{k=1}^{N} B_{1,k} \cdot w_k = 0, \sum_{k=1}^{N} C_{1,k} \cdot w_k = 0 \qquad x = 0 \tag{26a}$$

$$\sum_{k=1}^{N} B_{N,k} \cdot w_k = 0, \sum_{k=1}^{N} C_{N,k} \cdot w_k = 0 \qquad x = 1 \tag{26b}$$

It should be indicated that all the boundary conditions are discretized exactly on the boundary. The detailed description of implementing boundary conditions can be found in reference [12]. It can be seen from equations (24)–(26) that at each end, there are two boundary conditions. So, for two ends, there are four boundary conditions. To close the system, equation (23) can only be applied at (N–4) interior points since the number of unknowns (function values) is just N.

In other words, at two interior points, equation (23) should be replaced by the boundary condition equation which is usually given from the derivative condition. This is possible since the discretized derivative condition involves the functional values at the boundary points as well as the interior points. Mathematically, the discretized derivative condition can be considered as the governing equation for any point involved in the equation. In the work of Shu and Du [12], the derivative condition equations were taken as the equations for two interior points which are immediately adjacent to the left and right boundaries. An interesting question is whether this choice is optimal and what is the effect on accuracy when the derivative condition equations are applied at other interior points. To investigate these problems, this study considers the discretized derivative condition at the left boundary as the equation for an interior point $i = DL$, and the discretized derivative condition at the right boundary as the equation for an interior point $i = DR$. DL and DR are changeable, $2 \leqslant DL \leqslant N/2$, $N/2 \leqslant DR \leqslant N-1$. When $DL = 2$ and $DR = N-1$, the present approach is the same as the one used by Shu and Du [12]. Accordingly, equation (23) should be applied at the interior points, $i = 2, \ldots, DL-1, DL+1, \ldots, DR-1, DR+1, \ldots, N-1$.

4.2. TRANSVERSE VIBRATION OF THIN, ISOTROPIC PLATES

The equation governing free vibration of rectangular plates can be expressed as

$$w_{xxxx} + 2\lambda^2 w_{xxyy} + \lambda^4 w_{yyyy} = \bar{\omega}^2 w, \tag{27}$$

where $\lambda = a/b$ denotes the aspect ratio, $\bar{\omega}^2 = \rho h a^4 \omega^2 / D$, D is the plate stiffness, h is the total plate thickness, ρ is the density, w is the modal deflection, and ω is the natural frequency of free vibration. Application of the GDQ method to equation (27) gives

$$\sum_{k=1}^{N} D_{ik}^{x} w_{kj} + (2\lambda^2) \sum_{m=1}^{N} B_{im}^{x} \sum_{k=1}^{M} B_{jk}^{x} w_{mk} + (\lambda^4) \sum_{k=1}^{M} D_{jk}^{y} w_{ik} = \bar{\omega}^2 w_{ij}. \tag{28}$$

Equation (28) should be applied at the interior points. In this study, three types of boundary conditions are considered:

Clamped edge:

$$w = 0, \ w_x = 0 \quad \text{at} \quad x = 0, 1, \tag{29a}$$

$$w = 0, \ w_y = 0 \quad \text{at} \quad y = 0, 1. \tag{29b}$$

The discretized form of equation (29) can be written as

$$w_{1,j} = 0, \ \sum_{k=1}^{N} A_{1k}^{x} w_{k,j} = 0; \ j = 1, 2, \ldots, M, \qquad \text{at} \ \ x = 0, \tag{30a}$$

$$w_{N,j} = 0, \ \sum_{k=1}^{N} A^x_{Nk} w_{k,j} = 0; \ j = 1, 2, \ldots, M, \qquad \text{at} \ \ x = 1, \tag{30b}$$

$$w_{i,1} = 0, \ \sum_{k=1}^{M} A^y_{1k} w_{i,k} = 0; \ i = 1, 2, \ldots, N, \qquad \text{at} \ \ y = 0, \tag{30c}$$

$$w_{i,M} = 0, \ \sum_{k=1}^{M} A^y_{Mk} w_{i,k} = 0; \ i = 1, 2, \ldots, N, \qquad \text{at} \ \ y = 1. \tag{30d}$$

Simply supported edge:

$$w = 0, \ w_{xx} = 0 \quad \text{at} \quad x = 0, 1, \tag{31a}$$

$$w = 0, \ w_{yy} = 0 \quad \text{at} \quad y = 0, 1. \tag{31b}$$

The discretized form of equation (31) is

$$w_{1,j} = 0, \ \sum_{k=1}^{N} B^x_{1k} w_{k,j} = 0; \ j = 1, 2, \ldots, M, \qquad \text{at} \ \ x = 0, \tag{32a}$$

$$w_{N,j} = 0, \ \sum_{k=1}^{N} B^x_{Nk} w_{k,j} = 0; \ j = 1, 2, \ldots, M, \qquad \text{at} \ \ x = 1, \tag{32b}$$

$$w_{i,1} = 0, \ \sum_{k=1}^{M} B^y_{1k} w_{i,k} = 0; \ i = 1, 2, \ldots, N, \qquad \text{at} \ \ y = 0, \tag{32c}$$

$$w_{i,M} = 0, \ \sum_{k=1}^{M} B^y_{Mk} w_{i,k} = 0; \ i = 1, 2, \ldots, N, \qquad \text{at} \ \ y = 1. \tag{32d}$$

Free edge:

$$w_{xx} + \nu\lambda^2 w_{yy} = 0, \ w_{xxx} + (2 - \nu)\lambda^2 w_{xyy} = 0 \qquad \text{at } x = 0, 1, \tag{33a}$$

$$\lambda^2 w_{yy} + \nu w_{xx} = 0, \ \lambda^2 w_{yyy} + (2 - \nu) w_{xxy} = 0 \qquad \text{at } x = 0, 1, \tag{33b}$$

$$w_{xy} = 0 \qquad \text{at corner of two adjacent free edges.} \tag{33c}$$

Using the GDQ method, equation (33) can be discretized as

$$\sum_{k=1}^{N} B^x_{1,k} w_{k,j} + (\nu\lambda^2) \sum_{k=1}^{M} B^y_{j,k} w_{1,k} = 0, \tag{34a}$$

$$\sum_{k=1}^{N} C_{1,k}^{x} w_{k,j} + (2-\nu)\lambda^{2} \sum_{m=1}^{N} A_{1,m}^{x} \sum_{k=1}^{M} B_{j,k}^{y} w_{m,k} = 0, \tag{34b}$$

for the edge of $x=0$,

$$\sum_{k=1}^{N} B_{N,k}^{x} w_{k,j} + (\nu\lambda^{2}) \sum_{k=1}^{M} B_{j,k}^{y} w_{N,k} = 0, \tag{34c}$$

$$\sum_{k=1}^{N} C_{N,k}^{x} w_{k,j} + (2-\nu)\lambda^{2} \sum_{m=1}^{N} A_{N,m}^{x} \sum_{k=1}^{M} B_{j,k}^{y} w_{m,k} = 0, \tag{34d}$$

for the edge of $x=1$,

$$\lambda^{2} \sum_{k=1}^{M} B_{1,k}^{x} w_{i,k} + (\nu) \sum_{k=1}^{N} B_{i,k}^{x} w_{k,1} = 0, \tag{34e}$$

$$\lambda^{2} \sum_{k=1}^{M} C_{1,k}^{y} w_{i,k} + (2-\nu) \sum_{m=1}^{N} B_{i,k}^{x} \sum_{k=1}^{M} A_{1,k}^{y} w_{m,k} = 0, \tag{34f}$$

for the edge of $y=0$,

$$\lambda^{2} \sum_{k=1}^{M} B_{M,k}^{y} w_{i,k} + \nu \sum_{k=1}^{N} B_{i,k}^{x} w_{k,M} = 0, \tag{34g}$$

$$\lambda^{2} \sum_{k=1}^{M} C_{M,k}^{y} w_{i,k} + (2-\nu) \sum_{m=1}^{N} B_{i,m}^{x} \sum_{k=1}^{M} A_{M,k}^{y} w_{m,k} = 0, \tag{34h}$$

for the edge of $y=1$. At four corner points, equation (33c) can be applied. For example, at the corner of $x=1$, $y=1$, the discretized equation is

$$\sum_{k1=1}^{N} A_{N,k1}^{x} \sum_{k2=1}^{M} A_{M,k2}^{y} w_{k1,k2} = 0. \tag{35}$$

From equations (30), (32) and (34), it can be seen that for all types of plate edge conditions, there are two discretized boundary conditions at each edge. Among them, one can be applied at the boundary point itself, and the other should be applied at an interior point. In the work of Shu and Du [12, 13], the interior point is chosen as the neighboring point to the boundary. In this study, the discretized derivative condition equations at $y=0$ and $y=1$ are still applied at the neighboring points to the boundary along the line of $j=2$ and $j=M-1$, respectively. Along the x direction, the discretized derivative condition at $x=0$ is applied at interior points along the line of $i=DL$, and the discretized derivative condition at $x=1$ is applied at interior points along the line of $i=DR$. Again,

DL and DR are changeable, $2 \leqslant DL \leqslant N/2$, $N/2 \leqslant DR \leqslant N-1$. The method of implementing the boundary conditions in references [12, 13] is also used in this study. Since the discretized boundary condition equations are applied at some interior points, equation (28) should be applied at the interior points, $i = 2, \ldots, DL-1, DL+1, \ldots, DR-1, DR+1, \ldots, N-1, j = 3, \ldots, M-2$. To obtain the frequencies of free vibration for beams and plates, the same approach is adopted as shown in references [12, 13]. In this study, the effect of DL and DR on the accuracy of the numerical solution will be investigated. It is expected from the study that an optimal value of DL and DR can be found.

5. RESULTS AND DISCUSSION

In the present study, the shifted Chebyshev–Gauss–Lobatto points are adopted as the basic mesh points as shown below:

$$\xi_i = \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right), \qquad i = 1, 2, \ldots, N, \tag{36a}$$

$$\eta_j = \frac{1}{2}\left(1 - \cos\left(\frac{j-1}{N-1}\pi\right)\right), \qquad j = 1, 2, \ldots, N. \tag{36b}$$

To obtain more accurate results of SS–SS, F–SS and F–F beams and their combinations for plates, the grid points yielded by formula (36) need to be further stretched towards the boundary [13]. The following stretch formulation [2] is used,

$$x_i = (1-\alpha)(3\xi_i^2 - 2\xi_i^3) + \alpha\xi_i, \qquad i = 1, 2, \ldots, N, \tag{37a}$$

$$y_j = (1-\beta)(3\eta_j^2 - 2\eta_j^3) + \beta\eta_j, \qquad j = 1, 2, \ldots, N, \tag{37b}$$

where α and β are stretching parameters along the x and y directions. The smaller the values of α and β, the closer the mesh points to the boundary. In this study, values are chosen of $\alpha = \beta = 0{\cdot}8$ for C–C, $\alpha = \beta = 0{\cdot}6$ for C–SS, $\alpha = \beta = 0{\cdot}4$ for SS–SS, and $\alpha = \beta = 0$ for F–F boundary conditions, respectively. The detailed study on the stretching of mesh points will be provided in a separate paper. It should be indicated that when a one-dimensional problem such as a beam is considered, only equation (37a) is used.

5.1. ERROR DISTRIBUTION OF DERIVATIVE APPROXIMATION

The truncation error distributions of the first, second, third, and fourth order derivatives for a one-dimensional problem namely, $e^{(1)}(x_i)$, $e^{(2)}(x_i)$, $e^{(3)}(x_i)$, $e^{(4)}(x_i)$, are studied for different types of grids. $e^{(1)}(x_i)$, $e^{(2)}(x_i)$, $e^{(3)}(x_i)$, $e^{(4)}(x_i)$ are defined in the previous section. Table 1 shows the truncation error distributions for $N = 9$, $\alpha = 1$. For this case, the grid is actually the Chebyshev–Gauss–Lobatto grid. It can be seen clearly from Table 1 that all the error distributions are symmetric under the present symmetric mesh point distribution. It was found that the error of the first order derivative approximation is lowest among four

TABLE 1

Error distributions of the 1*st*, 2*nd*, 3*rd* and 4*th order derivative approximations for the Chebyshev–Gauss–Lobatto Grid* ($N=9$, $\alpha=1{\cdot}0$)

x_i	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$e^{(1)}(x_i)$	1·3E-9	6·7E-10	6·7E-10	6·7E-10	6·7E-10	6·7E-10	6·7E-10	6·7E-10	1·3E-9
$e^{(2)}(x_i)$	1·2E-7	9·8E-9	3·2E-9	1·9E-9	1·3E-9	1·9E-9	3·2E-9	9·8E-9	1·2E-7
$e^{(3)}(x_i)$	4·8E-6	1·6E-6	3·8E-7	2·2E-7	1·8E-7	2·2E-7	3·8E-7	1·6E-6	4·8E-6
$e^{(4)}(x_i)$	1·2E-4	5·9E-5	3·9E-6	1·3E-6	7·2E-7	1·3E-6	3·9E-6	5·9E-5	1.2E-4

respective derivatives. As the order of derivative is increased, the error of approximation is greatly increased. As one knows, the order of derivatives involved in the boundary conditions is at least one order lower than the order of derivatives involved in the governing equation. For example, the clamped condition only involves the first order derivatives, the simply-supported condition involves the second order derivatives, while the free condition involves both the second and third order derivatives, but the governing equation involves the fourth order derivatives. From the information shown in Table 1, it can be concluded that the numerical error for the boundary condition approximation is less than that for the governing equation approximation. In other words, the boundary condition approximation is more accurate than the governing equation approximation. The information also gives us a hint that the numerical errors of highest order derivative approximation in the differential equation have a dominant effect on the accuracy of numerical solution since the numerical error of high order derivative approximation is much larger than that of low order derivative approximation. Another important observation from Table 1 is that except for the first order derivative, the numerical errors at mesh points near the boundary for other derivatives are much larger than those at mesh points near the center of the domain.

On the other hand, it should be noted that the numerical error of a discretized equation system is a major cause for lower accuracy of the numerical solution. Obviously, the fewer the numerical errors of the equation system, the more accurate the numerical solution. If for some case, the discretized governing equations are required to be replaced by other more accurate equations at some points, these points should be chosen in such a way that the numerical errors of discretized governing equations are largest at selected points. In so doing, the total numerical error of the equation system is reduced when the discretized governing equations with larger numerical errors are replaced by other equations with fewer numerical errors. As a consequence, the accuracy of the numerical solution is improved.

Now, consider the vibration analysis of beams and plates. The governing equation of the problem involves the second and the fourth order derivatives. It is known from the above analysis that the numerical error of the fourth order derivative approximation is the major numerical error in the discretized governing equation, and numerical errors of the discretized governing equation at points near the boundary are much larger than those at points near the center.

It is also noticed that the numerical error of the discretized boundary condition equation is smaller than that of the discretized governing equation. For our approach to implement the boundary condition, the discretized derivative condition should be used as the discretized equation for some interior points. In other words, at selected interior points, the discretized governing equation should be replaced by the discretized boundary condition equation. From the above analysis, the optimal location of this replacement should be the position where the numerical error of discretized governing equation is largest. From the information in Table 1, it is found that the optimal location of replacement is the interior point just next to the boundary. In other words, the optimal choice of DL and DR is $DL = 2$ and $D = N - 1$.

The error distributions of respective derivatives for different grids have also been studied by changing the parameter α in equation (37a). It was found that as α is decreased from 1 to 0, the numerical error of derivatives at points near the boundary is decreased while the numerical error at points near the center of the domain is increased. This can be observed from Table 2 where $N = 9$, $\alpha = 0$. The error distribution for this case seems to be flatter than the original Chebyshev–Gauss–Lobatto grid. It can be seen from Table 2 that the numerical error of higher order derivative approximation is still bigger than that of lower order derivative approximation, and the numerical error of the fourth order derivative approximation at points near the boundary is larger than that at points near the center of the domain. Therefore, the conclusion that $DL = 2$ and $DR = N - 1$ are the optimal choice remains the same. In our approach, the boundary condition is exactly discretized at the boundary. Since the derivative approximation at points near the boundary is greatly improved when the grid is stretched (reducing α), it is believed that the numerical error of discretized boundary condition equation can be reduced when α is decreased. As a result, the accuracy of numerical results can be improved.

5.2. EFFECT OF SELECTING POINT ON ACCURACY OF NUMERICAL SOLUTION

In this section the effect of changing the replacement point on the accuracy of the numerical solution will be investigated, and the conclusion drawn in the above error distribution analysis validated. The free vibrations of beams and plates are studied. To show the effect of replacement point on the accuracy of the DQ solution, the relative deviation between the present DQ solution and the

TABLE 2

Error distributions of the 1st, 2nd, 3rd and 4th order derivative approximations for grid with $N = 9$, $\alpha = 0{\cdot}0$

x_i	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$e^{(1)}(x_i)$	5·6E-11	4·9E-11	3·6E-10	1·5E-9	2·5E-9	1·5E-9	3·6E-10	4·9E-11	5·6E-11
$e^{(2)}(x_i)$	3·0E-8	2·1E-8	1·7E-8	1·9E-8	4·9E-9	1·9E-8	1·7E-8	2·1E-8	3·0E-8
$e^{(3)}(x_i)$	2·4E-6	2·0E-6	2·2E-7	3·6E-7	4·1E-7	3·6E-7	2·2E-7	2·0E-6	2·4E-6
$e^{(4)}(x_i)$	8·3E-5	7·6E-5	1·9E-5	5·9E-6	1·6E-6	5·9E-6	1·9E-5	7·6E-5	8·3E-5

TABLE 3

Relative deviations of fundamental frequencies of various beams ($N = 9$)

DL	DR	SS-SS	C-SS	C-C	F-F
2	8	2·0E-5	1·5E-5	7·4E-5	7·6E-4
3	7	7·8E-3	9·6E-3	3·4E-3	5·9E-2
4	6	1·1E-1	9·8E-2	1·0E-1	3·8E-1
2	7	4·3E-3	6·2E-4	1·9E-3	1·2E-2
3	8	4·3E-3	9·4E-3	1·9E-3	3·0E-2
4	8	6·5E-2	7·8E-2	6·5E-2	2·3E-1

reference results given by other researchers is defined as follows,

$$e_k = \left| \frac{\text{Reference data} - DQ \text{ solution}}{\text{Reference data}} \right|, \tag{38}$$

where e_k is the relative deviation of the frequency for the kth mode. It is indicated that the "Reference data" are the results of Blevin [15] for beams, and the results of Leissa and Narita [l7] for the F-F-F-F plate, and the results of Leissa [l6] for other plate configurations. The results of Leissa [16] are the analytic solutions for plates with a pair of opposite SS boundary conditions. The average error is defined as

$$\varepsilon = \frac{1}{n} \sum_{k=1}^{n} e_k, \tag{39}$$

The free vibrations of SS-SS, C-C, SS-C, F-F beams and SS-SS-SS-SS, C-SS-C-SS and F-F-F-F square plates are chosen as test examples in this study. Table 3 displays the relative deviations of the fundamental frequencies of SS-SS, C-C, SS-C and F-F beams with respect to the change of DL and DR values. The total number of mesh points used is $N = 9$. It is observed from Table 3 that the errors increase evidently as the value of DL increases and the value of DR decreases in all cases. This shows that the closer the replacement points to the center, the

TABLE 4

Relative deviations of the first five frequencies of the C-C beam ($N = 15$, $\alpha = 0{\cdot}8$)

DL	DR	e_1	e_2	e_3	e_4	e_5	ε
2	14	6·3E-7	5·2E-8	4·1E-6	1·7E-4	2·1E-4	7·7E-5
3	13	1·7E-5	1·2E-4	3·6E-4	2·2E-3	1·0E-2	2·3E-3
4	12	9·9E-4	6·4E-3	1·7E-2	3·1E-2	2·8E-2	1·7E-2
5	11	1·1E-2	4·5E-2	6·3E-2	4·0E-2	4·7E-3	3·3E-2
6	10	4·4E-2	6·6E-2	5·0E-2	2·1E-1	3·7E-2	8·2E-2
7	9	8·5E-2	1·4E-1	1·5E-1	1·9E-1	2·0E-1	1·5E-1
2	13	9·0E-6	5·7E-5	1·9E-4	9·7E-4	5·1E-3	1·3E-3
2	11	5·8E-3	2·3E-2	3·3E-2	2·0E-2	2·7E-3	1·7E-2
3	11	5·8E-3	2·3E-2	3·3E-2	2·1E-2	2·9E-3	1·7E-2

TABLE 5

Relative deviations of the first five frequencies of the SS-SS beam ($N = 15$, $\alpha = 0{\cdot}4$)

DL	DR	e_1	e_2	e_3	e_4	e_5	ε
2	14	4·0E-8	1·0E-7	2·0E-6	2·3E-4	1·4E-3	3·4E-4
3	13	1·3E-4	5·1E-4	1·1E-3	1·9E-3	6·0E-3	1·9E-3
4	12	2·5E-3	9·9E-3	1·9E-2	3·0E-2	3·4E-2	1·9E-2
5	11	1·4E-2	4·2E-2	4·6E-2	1·2E-2	3·4E-2	3·0E-2
6	10	4·4E-2	5·0E-2	7·6E-2	2·3E-1	2·2E-3	8·4E-2
7	9	7·2E-2	1·9E-1	1·4E-1	9·2E-3	3·0E-1	1·6E-1
2	13	6·2E-5	2·5E-4	5·6E-4	8·4E-4	3·8E-3	1·1E-3
2	11	7·3E-3	2·1E-2	2·3E-2	7·3E-3	1·7E-2	1·5E-2
3	11	7·2E-3	2·1E-2	2·4E-2	8·4E-3	1·4E-2	1·5E-2

lower the accuracy of the numerical solution. Clearly, $DL = 2$ and $DR = 8$ $(N - 1)$ are the optimal choice. It was found that the solution of $DL = 3$ and $DR = 8$ is superior to that of $DL = 3$ and $DR = 7$ in various beams examined. This seems to show that whether the replaced points are symmetric or not has little effect on the accuracy of the numerical solution.

Tables 4–6 list the relative deviations of the first five frequencies for C-C, SS-SS and F-F beams, in which 15 mesh points are used. The solutions of $DL = 2$ and $DR = N - 1 = 14$ are obviously the most accurate among all results of each case. The worst solutions in all cases were found when $DL = 7$ and $DR = 9$. The general tendency is that the accuracy decreases as the replacement positions move toward the center of variable domain. However, there is an exceptional case, in which the accuracy of the fifth frequency of the C-C beam for $DL = 5$ and $DR = 11$ is higher than that for $DL = 4$ and $DR = 12$. In addition, it is observed that the solutions of lower frequencies are usually more accurate than those of higher frequencies. As is shown in Table 6, the accuracy of the F-F beam is more sensitive to the change of replacement points. For this case, the solutions have no practical significance when DL is increased from 4 to 7 while DR is decreased from 12 to 9.

TABLE 6

Relative deviations of the first five frequencies of the F-F beam ($N = 15$, $\alpha = 0{\cdot}0$)

DL	DR	e_1	e_2	e_3	e_4	e_5	ε
2	14	6·5E-7	3·1E-6	8·3E-5	5·6E-3	1·0E-2	3·1E-3
3	13	1·3E-3	2·2E-3	3·2E-3	1·1E-2	6·3E-3	4·8E-3
4	12	3·8E-2	6·4E-2	7·0E-2	7·5E-2	1·2E-1	7·4E-2
5	11	8·4E-3	1·1E-1	1·1E-1	3·6E-2	3·1E-1	1·3E-1
6	10	2·3E-1	1·3E-1	2·2E-1	1·9E-1	2·4E-1	2·0E-1
7	9	3·8E-1	2·1E-1	3·8E-1	4·0E-1	6·0E-1	3·9E-1
2	13	6·6E-4	1·1E-3	1·6E-3	8·3E-3	8·1E-3	9·4E-3
2	11	4·0E-2	5·2E-2	4·5E-2	3·7E-1	5·7E-1	2·2E-1
3	11	4·0E-2	5·3E-2	2·4E-2	3·8E-1	5·7E-1	2·1E-1

TABLE 7
Relative deviations of the first five frequencies of the C-SS-C-SS plate ($N = 15$, $\alpha = 0{\cdot}8$, $\beta = 0{\cdot}4$)

DL	DR	e_1	e_2	e_3	e_4	e_5	ε
2	14	5·2E-6	1·1E-6	2·4E-7	3·0E-6	8·9E-7	2·1E-6
3	13	1·0E-4	1·5E-4	2·6E-5	2·0E-4	1·8E-4	1·3E-4
4	12	5·3E-3	8·0E-3	4·4E-3	1·2E-2	8·0E-3	7·6E-3
5	11	3·7E-3	1·5E-3	3·8E-2	3·0E-2	4·8E-5	1·5E-2
6	10	2·6E-2	1·5E-2	5·8E-2	3·9E-2	1·9E-2	3·2E-2
7	9	3·0E-2	2·8E-3	1·7E-1	6·7E-2	1·4E-1	8·3E-2
2	13	1·0E-4	1·1E-4	3·7E-5	1·5E-4	9·7E-5	9·9E-5
2	11	4·6E-3	5·4E-3	2·1E-2	2·1E-2	5·5E-3	1·2E-2
3	11	4·6E-3	5·4E-3	2·2E-2	2·1E-2	5·5E-3	1·2E-2

TABLE 8
Relative deviations of the first five frequencies of the SS-SS-SS-SS plate ($N = 15$, $\alpha = \beta = 0{\cdot}4$)

DL	DR	e_1	e_2	e_3	e_4	e_5	ε
2	14	2·7E-5	4·4E-7	6·6E-7	4·4E-7	1·2E-6	5·8E-6
3	13	5·9E-3	2·9E-4	7·1E-3	5·1E-3	6·4E-4	3·8E-3
4	12	2·4E-2	4·5E-3	2·0E-2	6·5E-3	1·3E-2	1·4E-2
5	11	3·1E-2	2·7E-2	3·3E-2	3·3E-3	3·8E-2	2·7E-2
6	10	6·2E-2	4·1E-2	1·2E-1	6·6E-2	3·5E-2	6·4E-2
7	9	1·2E-1	2·3E-2	4·2E-1	2·2E-1	1·4E-1	1·8E-1
2	13	5·4E-3	1·6E-4	6·6E-3	4·7E-3	3·3E-4	3·4E-3
2	11	2·3E-2	1·5E-2	2·1E-2	1·6E-3	1·9E-2	1·6E-2
3	11	2·3E-2	1·5E-2	2·2E-2	2·3E-3	1·9E-2	1·6E-2

TABLE 9
Relative deviations of the first five frequencies of the F-F-F-F plate ($N = 15$, $\alpha = \beta = 0{\cdot}0$, $\nu = 0{\cdot}3$)

DL	DR	e_1	e_2	e_3	e_4	e_5	ε
2	14	7·4E-5	4·5E-5	5·3E-4	4·6E-4	4·6E-4	3·1E-4
3	13	1·2E-1	1·4E-3	2·8E-1	1·8E-1	3·7E-1	1·9E-1
4	12	3·0E-1	5·8E-2	2·8E-1	4·2E-1	4·2E-1	3·0E-1
5	11	2·7E-1	5·2E-1	2·3E-1	2·1E-1	3·8E-1	3·2E-1
6	10	1·1E+0	4·4E-1	1·1E+0	4·3E-1	5·5E-1	7·1E-1
7	9	1.2E+0	5·3E-1	2·7E-1	2·0E-1	2·0E-1	4·9E-1
2	13	5·5E-3	4·9E-3	2·8E-1	1·0E-1	3·8E-1	1·8E-1
2	11	5·0E-1	7·5E-2	1·2E-1	3·9E-1	1·4E-1	2·5E-1
3	11	7·0E-1	2·8E-1	1·3E-1	3·9E-1	3·9E-1	3·1E-1

Tables 7–9 compare the relative errors of the first five frequencies of C-SS-C-SS, SS-SS-SS-SS and F-F-F-F plates under various values of DL and DR. The mesh size used is 15×15. In general, the least accurate DQ solutions of all these cases were found when $DL = 7$ and $DR = 9$. The accuracy of numerical results for $DL = 2$ and $DR = \mathrm{N} - 1 = 14$ is obviously higher than other cases. This again confirms that the optimal replacement position should be the point just next to the boundary. In general, the movement of replacement points towards the center of the domain will decrease the accuracy of the numerical solution. One can also find that the solutions are not sensitive to whether the replacement point are placed in a symmetric or asymmetric manner. In some cases, for example, the C-SS-C-SS plate as shown in Table 7, the solutions of higher frequencies have better accuracy than lower frequencies when $DL = 2$ and $DR = 14$. As can be seen from Tables 7–9, the average relative errors of the first five frequencies under $DL = 2$ and $DR = 14$ are very low in the present DQ computation. As discussed in the error distribution analysis, the stretching of mesh points towards the boundary can reduce the error of derivative approximation at points near the boundary. Therefore, the boundary condition can be discretized more accurately. As a consequence, the accuracy of numerical results can be improved. This is particularly true for the F-F-F-F plate configuration. As shown by Shu and Du [13], for the F-F-F-F plate, the α value should be reduced to 0 in order to obtain accurate numerical solutions. It can be seen from Table 9 that even for this case, the optimal replacement points are still at $DL = 2$ and $DR = N - 1 = 14$.

6. CONCLUSIONS

In the early work of implementing the boundary conditions in the free vibration analysis of beams and plates, the discretized governing equation at some interior points need to be replaced by the discretized boundary condition equation. This paper discusses the selection of optimal replacement points from the error distribution analysis of derivative approximation. From the error distribution analysis, it was found that the optimal replacement positions are the interior points immediately adjacent to the boundary. This conclusion has been confirmed by numerical experiments for free vibration analysis of beams and plates with clamped, simply-supported and free boundary conditions. The conclusion may provide a practical guidance in applying the DQ method to general high order boundary value problems.

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APPENDIX: NOMENCLATURE

$c_{ij}^{(N)}$	GDQ weighting coefficients of the Nth order derivative
$p_i(x)$	Lagrangian interpolated polynomial
$R^{(N)}(x_i)$	truncation error of GDQ approximation of the Nth order derivative at x_i discrete point
$e^{(N)}(x_i)$	truncation error coefficient of GDQ approximation of the Nth order derivative at x_i discrete point
A_{ij}, B_{ij}, C_{ij}, D_{ij}	GDQ weighting coefficients of the 1*st*, 2*nd*, 3*rd* and 4*th* order derivatives
W	modal deflection
ω	natural frequency of free vibration

l	length of the beam
ρ	density of the beam
I	constant area moment of inertia about the neutral axis
E	elastic modulus
DL	index number of left replacement point of analog governing equations
DR	index number of right replacement point of analog governing equations
λ	aspect ratio of the plate
ν	Poisson ratio
D	plate stiffness
h	total plate thickness
α, β	stretching parameters of grid spacing along x and y directions
e_k	relative deviation of frequency for the kth mode
ε	average relative error or deviation