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STUDY OF A NON-LINEAR OSCILLATOR UNDER PARAMETRIC IMPULSIVE EXCITATION USING A NON-SMOOTH TEMPORAL TRANSFORMATION

V. N. Pilipchuk†, S. A. Volkova and G. A. Starushenko
Department of Mathematics, State Chemical and Technological University of Ukraine, Gagarin Ave., 8, Dnepropetrovsk 320005, Ukraine

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Solutions of differential equations of motion for mechanical systems with periodic impulsive excitation are represented in a special form which contains a standard pair of non-smooth periodic functions and possesses the structure of an algebra without division. This form is also suitable in the case of excitation with a periodic series of discontinuities of the first kind. All transformations are illustrated on the Duffing oscillator under a parametric non-equidistant pulsed forcing with a dipole-like shift of the impulses, although the technique can be applied to more general cases. An explicit form of analytical solutions has been obtained for periodic regimes. These solutions and numerical simulations indicate a principal role of the impulses' shift. Namely, the system performs periodic, multiperiodic and stochastic-like dynamical regimes if varying a parameter of the shift. The analytical approach is based on the limit of linear system under equidistant distribution of the impulses and asymptotically takes into account the dipole-like shift and non-linearity.

1. INTRODUCTION

Instantaneous impulses acting on a mechanical system can be simulated either by special conditions for the velocities and co-ordinates in the neighborhoods of the impulses location or by introducing Dirac's function into the equations. An advantage of the first approach is that the differential equations describing the system are the same as when there are no impulses acting [1]. However, these equations should be treated separately in each of the intervals between the impulses, and hence, instead of a single system, a whole sequence of systems must be considered. The second method gives a single set of equations over the whole time interval without introducing the above-mentioned conditions imposed on the variables. However, in this case a correct way of the analysis should be based on the theory of generalized functions (distributions) [2], which

†Currently at Mechanical Engineering, Wayne State University, Detroit, MI 48202, U.S.A.

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requires additional mathematical proofs in non-linear cases [3, 4]. Both techniques are fruitfully employed for different quantitative and qualitative investigations of mechanical systems under pulsed excitations [1–9]. In this paper, the external impulses acting on the system are taken into account by means of the special representation for periodic solutions including a standard pair of non-smooth periodic functions. Namely, substituting the representation into the differential equations of motion, the singular terms are eliminated in the equations. The transformation finally gives solutions in the form of a single analytic expression over the whole time interval. This means that one combines the advantages of both the ways mentioned above. In this connection, note reference [10] where solutions of the quasilinear systems under the action of an impulsive or discontinuous force (with a singularity at the point $t=a$) are found in the Caratheodory form [4] $x(t)=x_1(t)+x_2(t)H(t-a)$, where $H(t)$ is a standard single-step function. An analogous form of the solution is used to describe moving discontinuities in wave theory in reference [11]. In the case of systems with rigid constraints, special piecewise linear transformations have been constructed [12] which instantaneously rotate the co-ordinate axes at the instant of impact on a system. As a result of the transformation, the system is free from its bonds and the corresponding differential equations do not contain any "impact" terms whatsoever. Mechanical systems which are going to be considered here do not include any rigid constraints and hence do not belong to a class of the vibro-impact systems. That is why the representation used below for the solution instantaneously changes the direction and scale of time but not of the co-ordinates. By this means the co-ordinate transformation [12] deals with a function whereas the transformation of time transforms an independent variable (argument). As a result one needs another mathematical tool to manipulate with. Namely, after introduction of the saw-tooth time parameter, the co-ordinate acquires a special algebraic structure, and this enables one easily to use the representation associated with the solution of the differential equations. The representation is based on a proposition that an arbitrary periodic function $x(t)$ (the period of which is normalized to four) can be expressed as

$$\begin{gathered} x = X(\tau) + Y(\tau)e, \\ \tau = \tau(t;\Theta), \quad e = e(t;\Theta) = \frac{\partial \tau(t;\Theta)}{\partial t}, \end{gathered} \tag{1}$$

where τ is the saw-tooth piecewise-linear function of argument t and period equal to four (Figure 1):

$$\tau(t;\Theta) = \left\{ \begin{array}{ll} t/(1+\Theta), & -1-\Theta \leqslant t \leqslant 1+\Theta \\ (-t+2)/(1-\Theta), & 1+\Theta \leqslant t \leqslant 3-\Theta \end{array} \right\}, \tag{2}$$

where Θ $(-1<\Theta<1)$ is a parameter characterizing the slope (asymmetry) of the "saw". Note that the period has been normalized to four, not to 2π. A convenience of this choice is going to be discussed later. Figure 1 also

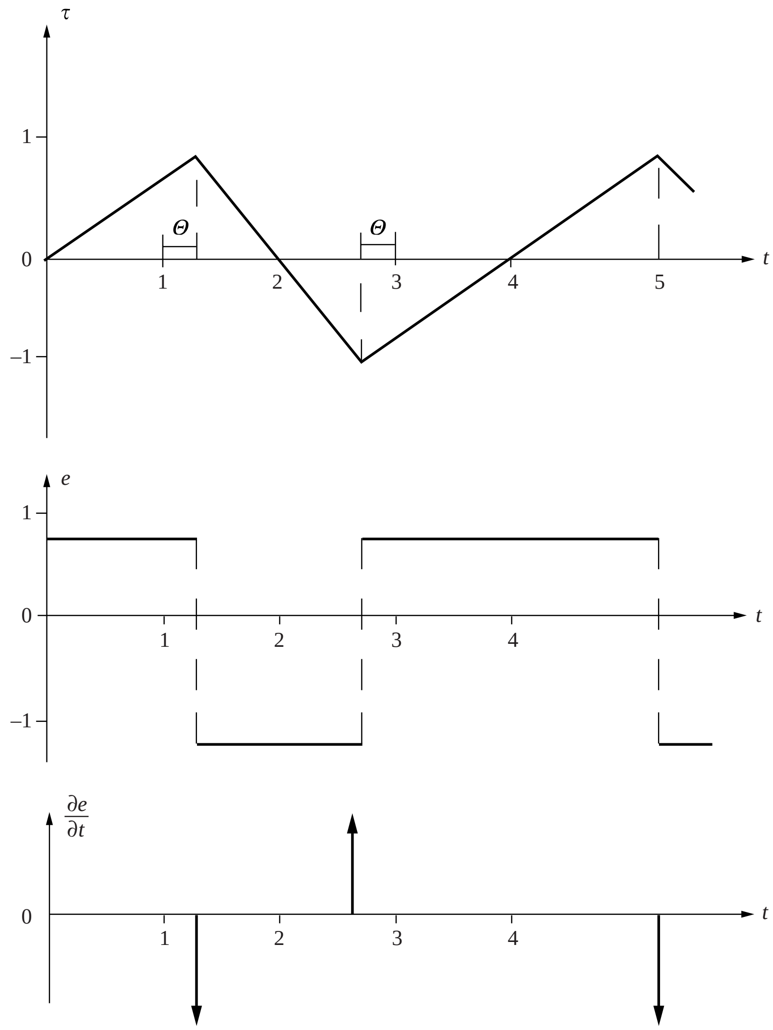


Figure 1. The asymmetric saw-tooth function and its two generalized derivatives. Non-zero slope of the saw-tooth function ($\Theta \neq 0$) creates a dipole-like shift of the impulses, i.e., a homogeneous periodic sequence is transformed into a periodic sequence of the double impulses.

schematically shows first and second generalized derivatives of the saw-tooth function. The second one consists of the Dirac functions.

The right-hand side of equation (1) includes two (X and Y) components, which could be easily expressed throughout $x(t)$ if this function would be known. If expression (1) represents an unknown solution, then both of the above mentioned components must be defined by solving a generally coupled boundary value problem (for example see section 3 below).

Originally the representation (1) was proposed for the case of symmetric saw-tooth function [13], when $\Theta = 0$ and hence,

$$e^2 = 1. \tag{3}$$

The last expression provides a series of mathematically convenient properties. These properties, geometrical and physical meaning of the representation (1) were discussed in references [14, 15], when analyzing the oscillations of strongly non-linear mechanical systems. More details regarding the asymmetric version ($\Theta \neq 0$) can be found in reference [16–18]. This modified version leads to a more complicated "multiplication table" than (3), and as a result of the transformation, a more complicated system appears than one would have in the symmetric case. That is why this paper shows how to make use of the idea of an asymptotic expansion based on the limit $\Theta \to 0$.

2. A FORMULATION OF THE PROBLEM

In order to illustrate the transformations consider the Duffiing oscillator under a parametric pulsed excitation. Define a co-ordinate of the system, $x = x(t)$, which is described by the following differential equation of motion

$$\ddot{x} + \left[p + q \frac{\partial e(t; \Theta)}{\partial t} \right] x + \varepsilon x^3 = 0, \tag{4}$$

where $\varepsilon \ll 1$ is a small parameter; the dot denotes differentiation with respect to time, t; p and q are constant parameters. The parametric pulsed excitation is expressed by means of the generalized second derivative of a saw-tooth piecewise-linear function $\tau(t; \Theta)$ of argument t and period equal to four (Figure 1)

$$\frac{\partial e(t; \Theta)}{\partial t} = \frac{\partial^2 \tau(t; \Theta)}{\partial t^2} = \frac{2}{1 - \Theta^2} \sum_{k=-\infty}^{\infty} [\delta(t + 1 + \Theta - 4k) - \delta(t - 1 - \Theta - 4k)], \tag{5}$$

where Θ ($-1 < \Theta < 1$) is a parameter characterizing the slope of the "saw". This way of describing the impulsive excitation corresponds to a basic idea of the non-smooth temporal transformation and plays a special role in derivations of the next sections. Note that the asymmetry of the saw-tooth function stays for extending the method on the non-equidistant series of impulses. Regarding the model chosen, note that the oscillator (4) itself has a general enough meaning, because it presents a well known model of non-linear mechanics (that is a Duffiing oscillator) under parametric excitation by a linear component of the restoring force characteristic. The following example illustrates one of many possible mechanical problems in which the oscillator (4) may occur. More examples of mechanical systems under parametric impulsive excitation can be found in reference [6].

Example 1. *Consider a simply supported, axially loaded beam on a linear–cubic elastic foundation. The partial differential equation of motion is*

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial y^4} + T \frac{\partial^2 w}{\partial y^2} + \alpha w + \beta w^3 = 0, \tag{6}$$

where $w = w(t, y)$ *is the lateral displacement of the beam center line*; ρA, EI *and*

α, β are constants characterizing properties of the beam and the foundation, respectively. The axial impulsive loading is defined as $T = T_0(\partial e(\omega t;\, \Theta)/\partial(\omega t))$, $T_0 = const$. The boundary conditions of the simple support are

$$w(t,0) = w(t,l) = 0, \qquad \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0} = \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=l} = 0.$$

Approximating the centerline function as

$$w = x(t)\sin\frac{\pi}{l}y$$

and applying the Galerkin technique to equation (6), one obtains ordinary differential equation (4) *with respect to the modal "coefficient", $x(t)$. In the case considered, one has the following expressions for the parameters of the system*

$$p = \frac{1}{\rho A}\left[\left(\frac{\pi}{l}\right)^4 EI + \alpha\right], \qquad q = -\frac{T_0}{\rho A}\left(\frac{\pi}{l}\right)^2, \qquad \varepsilon = \frac{3}{4}\frac{\beta}{\rho A}.$$

Figure 2 shows a possible scheme for the impulsive axial loading for the case $\Theta = 0$, when the "saw" becomes symmetric.

As follows from Floquet theory [19] the periodic solutions have a special meaning for linear differential equations with periodic coefficients. In fact, the periodic solutions as a rule separate regions of stability and instability in a space of parameters, and hence gives one enough information about the space structure. Because of this, based on the model (4) the next section presents an asymptotic process giving the periodic solutions of weakly non-linear systems under the parametric impulsive excitation. A role of the generating system will be to play a linearized ($\varepsilon = 0$) one with the excitation generated by the symmetric saw-tooth function ($\Theta = 0$). Supposing that the slope Θ of the saw-tooth function is of order ε, then put $\Theta = \varepsilon\theta$, where θ is of order 1. A first order approximate solution will take into account both the non-linearity and the asymmetry of the system associated with non-equidistant character of the impulsive loading. Different non-linear systems under the equidistant impulsive loading are considered in reference [17]. Regarding the model (4), the asymptotic technique enables one to avoid the operations with complicated special functions (see section 4). The technique can also be applied to more general cases when

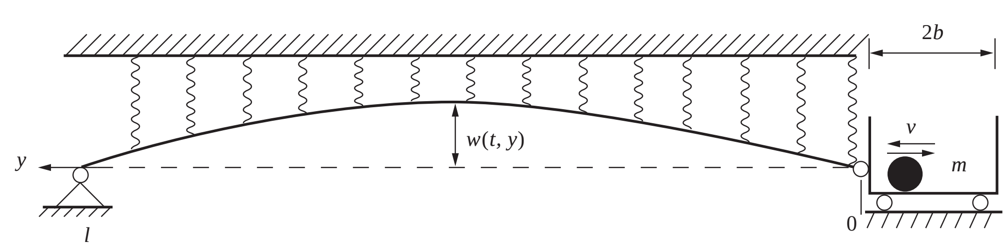


Figure 2. One of possible mechanical treatments of the impulsive axial loading. In this case the parameters of the loading are expressed as: $\omega = v/b$, $T_0 = (mv^2/b)t$.

solutions are not represented by special functions. The asymptotic technique will be applied not to the system (4) directly but to the transformed one, which does not contain singular terms and hence can be considered outside distribution theory. Note that the equality in the original equation (4) should be correctly understood in the sense of a distribution with respect to the variable t.

Non-periodic solutions will be studied numerically. An influence of the parameter asymmetry, Θ, on a global structure of the solutions' manifold will be questioned.

3. ANALYTICAL STUDY

3.1. THE NON-SMOOTH TRANSFORMATION OF THE SYSTEM

It was mentioned above, that system (4) will be transformed first in order to eliminate the singular Dirac's functions. The case in point is the non-smooth temporal transformation in a manifold of periodic regimes. To provide this transformation, a periodic solution with the period $T=4$ is represented due to (1) by the following expression

$$x(t) = X(\tau) + Y(\tau)e; \qquad \tau = \tau(t;\Theta), \qquad e = e(t;\Theta), \tag{7}$$

where X and Y are functions to be determined, notations for τ and e follow (1).

Note that the derivative $e = \partial\tau(t;\,\Theta)/\partial t$ is a piecewise constant function. Hence, it can be verified that

$$e^2 = \alpha + \beta e, \tag{8}$$

where $\alpha = 1/(1-\Theta^2)$ and $\beta = -2\Theta\alpha$.

In the symmetric case ($\Theta = 0$) the relation reduces to equation (3).

One should take the "multiplication rule" (8) into account when manipulating with representation (7).

Another relation which will be taken into account is

$$e\frac{\partial e}{\partial t} = \frac{\beta}{2}\frac{\partial e}{\partial t}. \tag{9}$$

This relation can be obtained by a formal differentiation of both sides of equality (8). Regarding the left-hand side of equality (9) one should note that from the point of view of the generalized functions (distributions), a product of the Dirac δ-function and function which has a discontinuity at a "point of localization" of a δ-impulse does not have a definite meaning in the general case. The left-hand side of equality (9) includes this kind of product. However, based on a result of work [3], it can be shown [16] that the equality (9) holds in the sense of the distributions theory. Continuous solutions will be sought. It implies that the first derivative of representation (7) with respect to time, t, does not contain any singular terms. The first formal derivative is

$$\dot{x} = \alpha Y' + (X' + \beta Y')e + Y\frac{\partial e}{\partial t}, \tag{10}$$

where the prime denotes differentiation with respect to τ. The last term on the right-hand side consists of the periodic sequence of δ-functions and has to be eliminated by imposing the following condition

$$Y|_{\tau=\pm 1} = 0. \tag{11}$$

Indeed roots of equations $\tau(t;\, \Theta) = \pm 1$ coincide with the "points of localization" of periodic singular function $\partial e(t;\, \Theta)/\partial t$ (5) (Figure 1), hence this term vanishes under condition (11).

Substituting (7) and (10) into (4) and taking into account (8) and (9) gives

$$\begin{aligned}&[(1-\Theta^2)X'' - 2\Theta Y'' + (1-\Theta^2)^2 pX + \varepsilon(1-\Theta^2)^2 R_f]\\ &\quad + [(1+3\Theta^2)Y'' - 2\Theta(1-\Theta^2)X'' + (1-\Theta^2)^2 pY + \varepsilon(1-\Theta^2)^2 I_f]e\\ &\quad + (1-\Theta^2)[(X'+qX) - 2\Theta Y' - \Theta^2(X'+qX) + qYe]\frac{\partial e}{\partial t} = 0,\end{aligned} \tag{12}$$

where

$$I_f = 3X^2Y - \frac{6\Theta}{1-\Theta^2}XY^2 + \frac{1+3\Theta^2}{(1-\Theta^2)^2}Y^3,$$

$$R_f = X^3 + \frac{3}{1-\Theta^2}XY^2 - \frac{2\Theta}{(1-\Theta^2)^2}Y^3.$$

The last (singular) term in (12) is both due to the differentiation of (10) and because of the parametric impulsive excitation, which is described by function $\partial e(t;\, \Theta)/\partial t$ in the original equation of motion (4). This singular term is eliminated analogously to (10) by imposing another condition

$$(X' + qX)|_{\tau=\pm 1} = [2\Theta Y' + \Theta^2(X' + qX)]|_{\tau=\pm 1}. \tag{13}$$

On equating the remaining terms in the brackets in (12) separately to zero, one obtains the set of equations with respect to X- and Y-components of the solution as

$$\begin{aligned}(1-\Theta^2)X'' - 2\Theta Y'' + (1-\Theta^2)^2 pX &= -\varepsilon(1-\Theta^2)^2 R_f,\\ (1+3\Theta^2)Y'' - 2\Theta(1-\Theta^2)X'' + (1-\Theta^2)^2 pY &= -\varepsilon(1-\Theta^2)^2 I_f.\end{aligned} \tag{14}$$

The boundary condition (13) is now written as

$$(X' + qX)|_{\tau=\pm 1} = [2\Theta Y' + \Theta^2(X' + qX)]|_{\tau=\pm 1} \tag{15}$$

So starting from the original equation of motion (4) and considering the manifold of periodic solutions, one has obtained the boundary value problem (11), (14), (15) in a domain $-1 \leqslant \tau \leqslant 1$. In new terms the system does not contain any singular functions, and hence its solutions can be considered from the classical point of view. The pulsed excitation appears here as an additional term with coefficient q in boundary condition (15). This term vanishes if the excitation is equal to zero, $q = 0$.

A disadvantage of the transformation is that its "real" (X) and "imaginary" (Y) components are coupled in the inertia terms due to the asymmetry (Θ), and in the stiffness terms due to non-linearity. In the symmetric case ($\Theta = 0$), only the non-linear coupling remains, and the boundary value problem reads

$$\begin{aligned} X'' + pX &= -\varepsilon(X^3 + 3XY^2), \qquad (X' + qX)|_{\tau=\pm 1} = 0, \\ Y'' + pX &= -\varepsilon(3X^2 Y + Y^3), \qquad Y|_{\tau=\pm 1} = 0. \end{aligned} \tag{16}$$

When $\varepsilon = 0$, this problem is linear and decoupled, and possesses a standard family of eigen-values p and eigen-functions X and Y. Based on this linear symmetric case, an asymptotic process for small but non-zero non-linearity and asymmetry (ε and Θ) will be given below.

3.2. THE ASYMPTOTIC EXPANSIONS

To construct the asymptotic process one represents unknown components of the solution, X, Y, and parameter p in power series form with respect to ε:

$$\begin{aligned} X &= X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + \dots, \\ Y &= Y_0(\tau) + \varepsilon Y_1(\tau) + \varepsilon^2 Y_2(\tau) + \dots, \end{aligned} \tag{17}$$

and

$$p = \lambda^2 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots, \tag{18}$$

where $X_0(\tau)$, $X_1(\tau)$, ...; $Y_0(\tau)$, $Y_1(\tau)$, ... and λ, p_1, p_2, ... are functions and constants to be determined.

An iterative process is going to be constructed similar, in principal, to the asymptotic technique for quasilinear eigen-value problems [20].

Substituting expansions (17), (18) into equations (14) and boundary conditions (11), (15), setting $\Theta = \varepsilon\theta$ and matching the coefficients of the respective powers of ε, one obtains the sequence of boundary-value problems. The zeroth order (generating) problem consists of homogeneous equations

$$X_0'' + \lambda^2 X_0 = 0, \qquad Y_0'' + \lambda^2 Y_0 = 0 \tag{19}$$

and homogeneous boundary conditions

$$(X_0' + qX_0)|_{\tau=\pm 1} = 0, \qquad Y_0|_{\tau=\pm 1} = 0. \tag{20}$$

This problem admits two sets of eigen-functions and eigen-values. The first one is

$$X_{0j} = A_0\varphi_j(\tau), \qquad Y_{0j} = C_0\psi_j(\tau), \tag{21}$$

where A_0, C_0 are arbitrary constants and the following notations for normalized eigen-functions are introduced

$$\begin{aligned} &\varphi_j(\tau) = \sqrt{\frac{2}{q^2+\lambda_j^2}}\,(q\cos\lambda_j\tau + \lambda_j\sin\lambda_j\tau), \\ &\psi_j(\tau) = \sqrt{2}\cos\lambda_j\tau \\ &\text{for } \lambda_j = j\frac{\pi}{2}, \qquad j = 1, 3, 5, \ldots \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\varphi_j(\tau) = \sqrt{\frac{2}{q^2+\lambda_j^2}}\,(\lambda_j\cos\lambda_j\tau - q\sin\lambda_j\tau), \\ &\psi_j(\tau) = \sqrt{2}\sin\lambda_j\tau \\ &\text{for } \lambda_j = j\pi, \qquad j = 1, 2, 3, \ldots \end{aligned} \tag{23}$$

These normalized functions satisfy the conditions

$$\begin{aligned} \langle\varphi_i\varphi_j\rangle &\equiv \frac{1}{2}\int_{-1}^{1}\varphi_i(\tau)\varphi_j(\tau)\,\mathrm{d}\tau = \delta_{ij}, \\ \langle\psi_i\psi_j\rangle &= \delta_{ij}, \end{aligned} \tag{24}$$

where δ_{ij} is the Kronecker symbol.

Following the method of perturbation for weakly non-linear eigen-value problems [20], one must choose a definite eigen-function to deal with a definite non-linear solution. Take for example an arbitrary solution of the first set (22). Taking into account (21), the next step of the asymptotic process gives the following equations

$$\begin{aligned} X_1'' + \lambda_j^2 X_1 &= 2\theta Y_{0j}'' - p_1 X_{0j} - X_{0j}^3 - 3X_{0j}Y_{0j}^2, \\ Y_1'' + \lambda_j^2 Y_1 &= 2\theta X_{0j}'' - p_1 Y_{0j} - Y_{0j}^3 - 3Y_{0j}X_{0j}^2, \end{aligned} \tag{25}$$

under the boundary conditions

$$\begin{aligned} (X_1' + qX_1)|_{\tau=\pm 1} &= 2\theta Y_{0j}'|_{\tau=\pm 1}, \\ Y_1|_{\tau=\pm 1} &= 0. \end{aligned} \tag{26}$$

The solution can be found as expansions on the zeroth order eigen-function

$$\begin{aligned} X_1 &= \sum_{k=1,3,5,\ldots}\alpha_{1k}\varphi_k(\tau) + X_1^*(\tau), \\ Y_1 &= \sum_{k=1,3,5,\ldots}\beta_{1k}\psi_k(\tau), \end{aligned} \tag{27}$$

where α_{1k} and β_{1k} are constant coefficients to be defined,

$$X_1^*(\tau) = 2\sqrt{2}C_0\theta\cos\lambda_j\tau \quad \text{for} \quad \lambda_j = j\frac{\pi}{2}, \qquad j = 1, 3, 5, \ldots,$$

$$X_1^*(\tau) = 2\sqrt{2}C_0\theta\sin\lambda_j\tau \quad \text{for} \quad \lambda_j = j\pi, \qquad j = 1, 2, 3, \ldots.$$

The expression for X_1 includes term $X_1^*(\tau)$ in order to satisfy non-homogenous boundary condition (26), where the non-homogeneity is due to the zeroth order approximation (each of eigen-functions $\varphi_k(\tau)$, $\psi_k(\tau)$ satisfy homogeneous boundary condition (20)).

Substituting (27) into (25) gives

$$\sum_{k=1,3,5,\ldots} \alpha_{1k}(\lambda_j^2 - \lambda_k^2)\varphi_k = -2\theta C_0\lambda_j^2\psi_j - p_1A_0\varphi_j - A_0^3\varphi_j^3 - 3A_0C_0^2\varphi_j\psi_j^2,$$

$$\sum_{k=1,3,5,\ldots} \beta_{1k}(\lambda_j^2 - \lambda_k^2)\psi_k = -2\theta A_0\lambda_j^2\varphi_j - p_1C_0\psi_j - C_0^3\psi_j^3 - 3C_0A_0^2\psi_j\varphi_j^2.$$

Multiplying the first equation by φ_k and the second one by ψ_k and integrating with respect to τ in domain $-1 \leqslant \tau \leqslant 1$ gives

$$\begin{aligned} \alpha_{1k} &= \frac{-2\theta C_0\lambda_j^2\langle\psi_j\varphi_k\rangle - p_1A_0\langle\varphi_j\varphi_k\rangle - A_0^3\langle\varphi_j^3\varphi_k\rangle - 3A_0C_0^2\langle\varphi_j\psi_j^2\varphi_k\rangle}{\lambda_j^2 - \lambda_k^2}, \\ \beta_{1k} &= \frac{-2\theta A_0\lambda_j^2\langle\varphi_j\psi_k\rangle - p_1C_0\langle\psi_j\psi_k\rangle - C_0^3\langle\psi_j^3\psi_k\rangle - 3C_0A_0^2\langle\psi_j\varphi_j^2\psi_k\rangle}{\lambda_j^2 - \lambda_k^2}, \end{aligned} \tag{28}$$

where $k \neq j$.

If $k = j$ the numerators of expressions (28) must be equal to zero. It gives algebraic non-linear equations for A_0, B_0 of the form

$$p_1A_0 + bC_0 = f(A_0, C_0), \qquad bA_0 + p_1C_0 = g(A_0, C_0), \tag{29}$$

where

$$b = 2\theta\lambda_j^2\langle\varphi_j\psi_j\rangle = -\frac{2\theta\lambda_j^2 q}{\sqrt{\lambda_j^2 + q^2}},$$

$$f(A_0, C_0) = -\langle\varphi_j^4\rangle A_0^3 - 3\langle\psi_j^2\varphi_j^2\rangle A_0C_0^2 = -\frac{3}{2}A_0^3 - \frac{3}{2}\frac{\lambda_j^2 + 3q^2}{\lambda_j^2 + q^2}A_0C_0^2,$$

$$g(A_0, C_0) = -\langle\psi_j^4\rangle C_0^3 - 3\langle\psi_j^2\varphi_j^2\rangle A_0^2C_0 = -\frac{3}{2}C_0^3 - \frac{3}{2}\frac{\lambda_j^2 + 3q^2}{\lambda_j^2 + q^2}A_0^2C_0.$$

Equations (29) indicate that the first order correction to the eigen-value, p_1, depends on both the asymmetry of the impulses location and the non-linearity. To obtain an explicit form of the dependencies one should add and subtract equations (29). As a result the equations take the form

$$\left[p_1 + b + \frac{3}{2}\left(A_0^2 + C_0^2 + \frac{2q^2}{\lambda_j^2 + q^2} A_0 C_0\right)\right](A_0 + C_0) = 0,$$
$$\left[p_1 - b + \frac{3}{2}\left(A_0^2 + C_0^2 - \frac{2q^2}{\lambda_j^2 + q^2} A_0 C_0\right)\right](A_0 - C_0) = 0. \tag{30}$$

These equations have two non-trivial solutions if the following relations hold $A_0 \pm C_0 = 0$. It gives two branches for parameter p as

$$A_0 = \pm C_0 = A$$
$$p = \lambda_j^2 - \varepsilon\left(\pm \frac{2\theta\lambda_j^2 q}{\sqrt{\lambda_j^2 + q^2}} + 3\frac{\lambda_j^2 + 2q^2}{\lambda_j^2 + q^2} A^2\right) + O(\varepsilon^2). \tag{31}$$

The related periodic solutions are

$$x = A\left[\sqrt{\frac{2}{q^2 + \lambda_j^2}}(q\cos\lambda_j\tau + \lambda_j \sin\lambda_j\tau) \pm e\sqrt{2}\cos\lambda_j\tau\right] + O(\varepsilon)$$

$$\text{for} \quad \lambda_j = j\frac{\pi}{2}, \qquad j = 1, 3, 5, \ldots$$

and

$$x = A\left[\sqrt{\frac{2}{q^2 + \lambda_j^2}}(\lambda_j\cos\lambda_j\tau - q \sin\lambda_j\tau) \pm e\sqrt{2}\sin\lambda_j\tau\right] + O(\varepsilon)$$

$$\text{for} \quad \lambda_j = j\pi, \qquad j = 1, 2, 3, \ldots$$

where $\tau = \tau(t;\, \varepsilon\theta)$, $e = e(t;\, \varepsilon\theta)$.

Expression (31) indicates a branching of curves $p = p(q)$ on a plane of the system parameters, pq, when a non-zeroth slope ($\theta \neq 0$) of the saw-tooth function appears. The branching generate instability regions on the parameters plane and strongly affects on the system dynamics.

To justify the above written asymptotic solution both the non-linearity of the system and the shift of the impulses associated with parameter Θ have to be small enough. Results of the numerical simulations for all domains $0 < \Theta < 1$ will be discussed in section 5.

4. EXACT SOLUTION IN TERMS OF ELLIPTIC FUNCTIONS

Note that between the points of localization of impulses equation (4) takes the form of a Duffing oscillator and hence admits a general solution in terms of special Jacobi functions [21]. The corresponding way of analysis will now be illustrated. To simplify derivations assume that $\Theta = 0$. In this case the resulting boundary value problem (after elimination of the impulses from the system) admits a family of solutions on which $Y \equiv 0$, and the original co-ordinate consists of the X-component only, $x = X(\tau)$. Setting $\Theta = 0$ and $Y \equiv 0$ in equations (11), (15) and (14), one obtains the reduced boundary value problem as follows

$$X'' + pX + \varepsilon X^3 = 0, \qquad (X' + qX)|_{\tau=\pm 1} = 0. \tag{32, 33}$$

The general solution of equation (32) can be written in terms of Jacobi functions as

$$X = \alpha \,\mathrm{cn}(u\tau + v|m), \tag{34}$$

where m is the parameter; α, u and v are constants to be defined due to the differential equation and boundary conditions.

Substituting equation (34) into equation (32) gives

$$\begin{aligned} X'' + pX + \varepsilon X^3 &= \alpha \,\mathrm{cn}(u\tau + v|\, m)[\, p - n^2 + 2mu^2 - (\alpha^2\varepsilon - 2mu^2)\,\mathrm{cn}^2(u\tau + v\,|\, m)] \\ &= 0. \end{aligned}$$

This expression leads to

$$p = u^2 - \varepsilon\alpha^2, \qquad m = \frac{\varepsilon\alpha^2}{2u^2}. \tag{35}$$

Substituting equation (34) into the boundary condition (33) and combining the related expressions after a transformation gives

$$\begin{aligned} &\mathrm{cn}(u\,|\,m)[um\,\mathrm{sn}^2(u\,|\,m)\,\mathrm{sn}(v\,|\,m)\,\mathrm{cn}^2(v\,|\,m)\,\mathrm{dn}(v\,|\,m) \\ &\quad - u\,\mathrm{dn}^2(u\,|\,m)\,\mathrm{sn}(v\,|\,m)\,\mathrm{dn}(v\,|\,m) + q\,\mathrm{cn}(v\,|\,m) \\ &\quad - qm\,\mathrm{cn}(v\,|\,m)\,\mathrm{sn}^2(u\,|\,m)\,\mathrm{sn}^2(v\,|\,m)] = 0 \end{aligned} \tag{36}$$

and

$$\begin{aligned} &\mathrm{sn}(u\,|\,m)\,\mathrm{dn}(u\,|\,m)[u\,\mathrm{dn}^2(v\,|\,m)\,\mathrm{cn}(v\,|\,m) \\ &\quad - um\,\mathrm{cn}^2(u\,|\,m)\,\mathrm{sn}^2(v\,|\,m)\,\mathrm{cn}(v\,|\,m) + q\,\mathrm{sn}(v\,|\,m)\,\mathrm{dn}(v\,|\,m) \\ &\quad - qm\,\mathrm{dn}(v\,|\,m)\,\mathrm{sn}^2(u\,|\,m)\,\mathrm{sn}^3(v\,|\,m)] = 0. \end{aligned} \tag{37}$$

Equations (36) and (37) split into the two sets of equations. The first of them is

$$\mathrm{cn}(u\,|\,m) = 0,$$

$$u\,\mathrm{cn}(v\,|\,m)\,\mathrm{dn}^2(v\,|\,m) + q\,\mathrm{sn}(v\,|\,m)\,\mathrm{dn}(v\,|\,m) - mq\,\mathrm{dn}(v\,|\,m)\,\mathrm{sn}^3(v\,|\,m) = 0.$$

In this case one has $u = (2n+1)\,\mathrm{K}(m)$, $n = 0, 1, 2, \ldots$, where $\mathrm{K}(m)$ is a complete elliptic integral of the first kind. The coefficient of linear rigidity of the oscillator is expressed by means of equation (35) as

$$p = (2n+1)^2 \mathrm{K}^2(m) - \varepsilon\alpha^2, \qquad n = 0, 1, \ldots$$

The second set of equations is

$$sn(u) = 0$$
$$q\,\mathrm{cn}(v \mid m) - u\,\mathrm{sn}(v \mid m)\,\mathrm{dn}(v \mid m) = 0.$$

This respectively gives $u = 2n\mathrm{K}(m)$, $n = 1, 2, \ldots$ and

$$p = 4n^2 \mathrm{K}^2(m) - \varepsilon\alpha^2, \qquad n = 1, 2, \ldots$$

The reduced boundary value problem (32), (33) can be solved by means of the asymptotic process of section 3.1. Taking $\Theta = 0$ and realizing two steps of the process give the following expression for the coefficient of linear rigidity

$$p = \lambda_j^2 - \varepsilon\tfrac{3}{2}A_0^2 + O(\varepsilon^2) \tag{38}$$

and the related periodic solutions as

$$x = A_0\left[\sqrt{\frac{2}{q^2 + \lambda_j^2}}(q\cos\lambda_j\tau + \lambda_j\sin\lambda_j\tau)\right] + O(\varepsilon); \qquad \lambda_j = j\frac{\pi}{2}, \qquad j = 1, 3, 5, \ldots$$

and

$$x = A_0\left[\sqrt{\frac{2}{q^2 + \lambda_j^2}}(\lambda_j\cos\lambda_j\tau - q\sin\lambda_j\tau)\right] + O(\varepsilon); \qquad \lambda_j = j\pi, \qquad j = 1, 2, 3, \ldots$$

To compare these approximate solutions with the exact solution one should obtain a relationship between parameters of the two kind of solutions. Equating the right-hand sides of equations (35) and (38) gives

$$\alpha^2 = \frac{1}{\varepsilon}(u^2 - \lambda_j^2) + \frac{3}{2}A_0^2.$$

Figure 3 shows a good enough agreement between the exact and asymptotic solutions.

5. NUMERICAL ANALYSIS

In this section some numerical results will be presented to illustrate a global structure of the solutions manifold and location of the periodic regimes on the manifold. Numerical simulations take much less time after the system has been represented in an appropriate form by some standard analytical steps.

Expression (5) shows that the positive and negative impulses of the external parametric force are located at points $t_k^+ = 3 - \Theta + 4k$, and $t_k^- = 1 + \Theta + 4k$, respectively, for any integer k. Introducing notations $q_k^\pm$ instead of $\pm 2\alpha q$, the system (4) can be written in a more general form

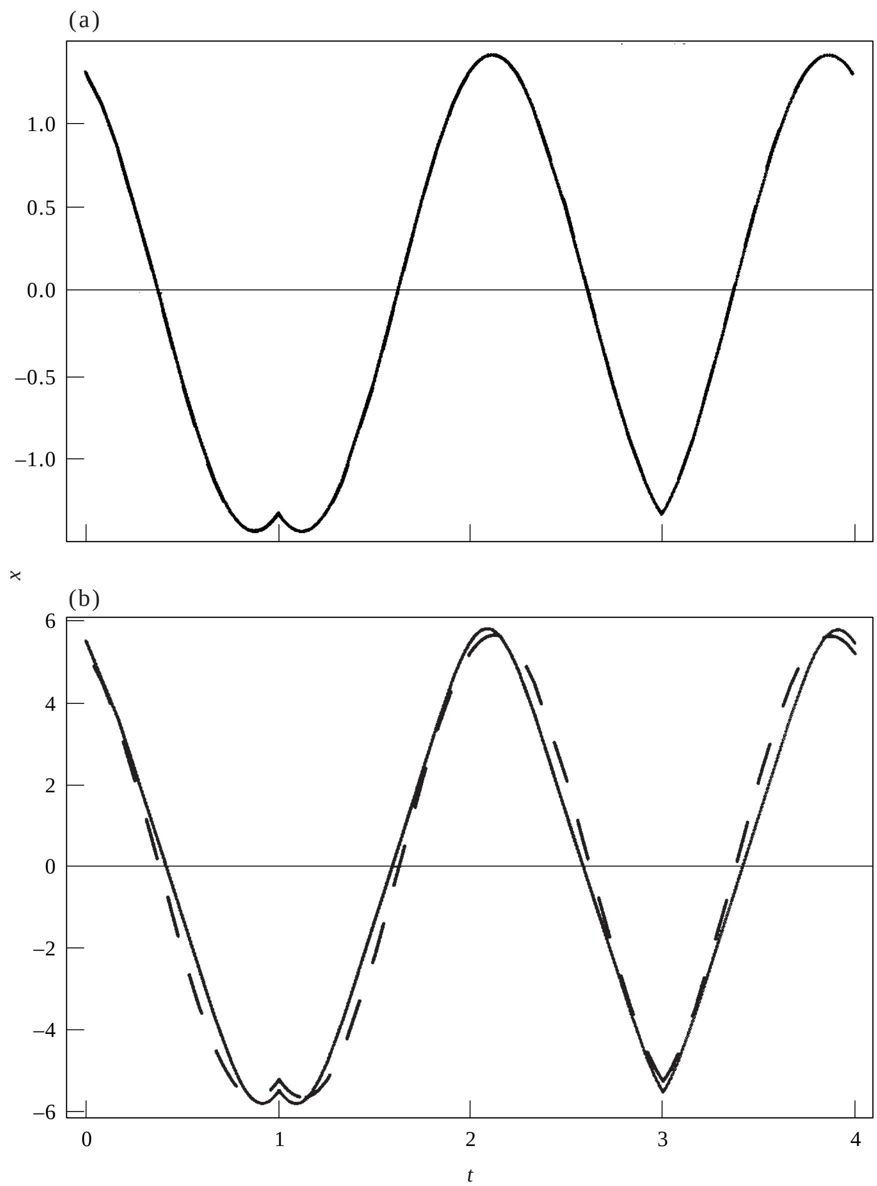


Figure 3. Exact periodic solution in terms of Jacobi's functions (the solid gray line) and asymptotic approach (the dashed line). Parameters of calculations have been taken as: $\varepsilon = 0{\cdot}5$, $q = 1{\cdot}3$, $p = 1$, $\lambda_j = \pi$, $u = 2\mathrm{K}(m)$; (a) $A_0 = 1$, (b) $A_0 = 4$.

$$\ddot{x} + \left\{ p + \sum_{k=-\infty}^{k=\infty} [q_k^+ \delta(t - t_k^+) + q_k^- \delta(t - t_k^-)] \right\} x + \varepsilon x^3 = 0, \tag{39}$$

where each of the impulses has its own amplitude. Also assume that the amplitudes $q_k^\pm$ of the impulses do not necessarily depend on the asymmetry parameter Θ as the form of the original equation (4) implies.

First, conditions for co-ordinates and velocities at the point $t_k^{\pm}$ will be obtained. The condition of continuity of the co-ordinate is

$$x(t_k^{\pm} - 0) = x(t_k^{\pm} + 0) = x(t_k^{\pm}). \tag{40}$$

The velocity is a discontinuous function at points $\{t_k^{\pm}\}$. To obtain the condition for "jumps" of the velocity one should integrate equations (39) in a domain $t_k^{\pm} - \epsilon < t < t_k^{\pm} + \epsilon$, $\epsilon > 0$ and then take a limit $\epsilon \to 0$. It follows from a definition of the Dirac's function that

$$\int_{t_k^{\pm}-\epsilon}^{t_k^{\pm}+\epsilon} x(t)\delta(t - t_k^{\pm})\,\mathrm{d}t = x(t_k^{\pm}).$$

Taking this expression and expression (40) into account gives

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\bigg|_{t=t_k^{\pm}+0} - \frac{\mathrm{d}x}{\mathrm{d}t}\bigg|_{t=t_k^{\pm}-0}\right) + q_k^{\pm} x(t_k^{\pm}) = \mathbf{0}. \tag{41}$$

Expressions (40) and (41) completely describe the system in the neighborhoods of points $\{t_k^{\pm}\}$, where the impulses are located. Between the impulses the system (39) is reduced to

$$\ddot{x} + \omega_0^2 x + \varepsilon x^3 = 0 \tag{42}$$

where notation $\omega_0^2 = p$ has been introduced.

To avoid numerical calculations between the impulses a simple analytical approach will be employed. The system will be replaced with an average one. The averaging will be done in terms of the action and angle variables. To introduce these variables, take the Hamiltonian function of the system (42) as

$$H = \frac{1}{2}\dot{x}^2 + \omega_0^2 \frac{x^2}{2} + \varepsilon \frac{x^4}{4}, \tag{43}$$

where x and $\dot{x}$ are considered as the Hamiltonian co-ordinates and momenta, respectively.

The action I and angle φ variables can be introduced by means of the relationships

$$x = \sqrt{\frac{2I}{\omega_0}} \cos\varphi, \qquad \dot{x} = -\sqrt{2I\omega_0}\sin\varphi. \tag{44}$$

Substituting (44) into (43), one obtains

$$H = I\omega_0 + \varepsilon \frac{I^2}{\omega_0^2}\cos^4\varphi. \tag{45}$$

The equations of motion associated with (45) are

$$\dot{\varphi} = \frac{\partial H}{\partial I}, \qquad \dot{I} = -\frac{\partial H}{\partial \varphi}$$

or

$$\dot{I} = \varepsilon \frac{I^2}{\omega_0^2}\left(\sin 2\varphi + \frac{1}{2}\sin 4\varphi\right), \qquad \dot{\varphi} = \omega_0 + \varepsilon \frac{I}{\omega_0^2}\left(\frac{3}{4} + \cos 2\varphi + \frac{1}{4}\cos 4\varphi\right).$$

Let $\Delta^{\pm} I$ and $\Delta^{\pm}\varphi$ be jumps of variables when passing the kth impulse. As a result of the impulse action one has transmission: $\{I, \varphi\} \to \{I + \Delta^{\pm} I, \varphi + \Delta^{\pm}\varphi\}$. Expressions for $\Delta^{\pm} I$ and $\Delta^{\pm}\varphi$ follow from (40) and (41). Taking into account (44) gives

$$\sqrt{I + \Delta^{\pm} I}\ \cos(\varphi + \Delta^{\pm}\varphi) = \sqrt{I}\cos\varphi,$$

$$\sqrt{I + \Delta^{\pm} I}\ \sin(\varphi + \Delta^{\pm}\varphi) = \sqrt{I}\sin\varphi + \frac{q_k^{\pm}}{\omega_0}\sqrt{I}\cos\varphi.$$

Combining these equations squared gives

$$\Delta^{\pm} I = 2I\frac{q_k^{\pm}}{\omega_0}\sin\varphi\cos\varphi + I\left(\frac{q_k^{\pm}}{\omega_0}\right)^2\cos^2\varphi. \tag{46}$$

Multiplying the first equation by $\sin(\varphi + \Delta^{\pm}\varphi)$ and the second one by $\cos(\varphi + \Delta^{\pm}\varphi)$ and equating the right-hand sides leads to expression

$$\Delta^{\pm}\varphi = \arctan\left(\frac{\dfrac{q_k^{\pm}}{\omega_o}\cos^2\varphi}{1 + \dfrac{q_k^{\pm}}{\omega_0}\sin\varphi\cos\varphi}\right). \tag{47}$$

Expressions (46) and (47) define the jumps of variables when passing the kth impulse.

Between the impulses the system considered will be also replaced by an averaged one. The last step enables one to use a simple enough analytical approach in terms of trigonometric functions between impulses. As a result, the problem will be reduced to standard map calculations.

Averaging terms of order ε with respect to the fast phase φ gives

$$\dot{I} = 0, \qquad \dot{\varphi} = \omega_0 + \varepsilon\frac{3I}{4\omega_0^2} \tag{48}$$

and admits a direct analytical integration.

Let I, φ and $\bar{I}, \bar{\varphi}$, be action and angle variables calculated after an arbitrary positive impulse and arbitrary negative impulse, respectively (Figure 4). After a negative impulse one has

$$\bar{\varphi} = \varphi + \varphi_{dist} + \Delta^{-}\varphi|_{\varphi\to\varphi+\varphi_{dist}}, \qquad \bar{I} = I + \Delta^{+} I|_{\varphi\to\varphi+\varphi_{dist}}, \tag{49}$$

where

$$\varphi_{dist} = \left(\omega_0 + \varepsilon\frac{3I}{4\omega_0^2}\right)(2 + 2\Theta).$$

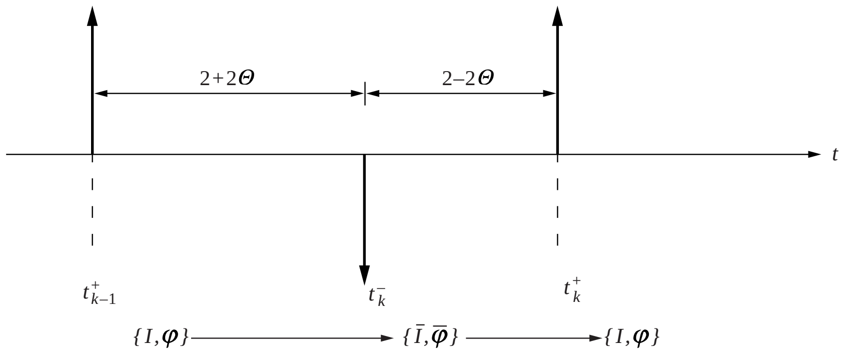


Figure 4. A scheme of the action and angle transformation during a single period of the pulsed excitation.

After an arbitrary positive impulse one has

$$\varphi = \bar{\varphi} + \bar{\varphi}_{dist} + \Delta^{+}\varphi|_{\varphi\to\bar{\varphi}+\bar{\varphi}_{dist}}, \qquad I = \bar{I} + \Delta^{+}I|_{\varphi\to\bar{\varphi}+\bar{\varphi}_{dist}}, \tag{50}$$

where

$$\bar{\varphi}_{dist} = \left(\omega_0 + \varepsilon\frac{3I}{4\omega_0^2}\right)(2 - 2\Theta).$$

Equations (49) and (50) give a complete variation of the action and angle for one period of external force as following maps: $\bar{\varphi} = \bar{\varphi}(\varphi, I)$, $\bar{I} = \bar{I}(\varphi, I)$ and $\varphi = (\bar{\varphi}, \bar{I})$, $I = I(\bar{\varphi}, \bar{I})$. Figures 5 and 6 illustrate results of the numerical analysis of the maps depending on the asymmetry parameter, Θ. One should understand the diagrams as follows. For a certain parameter Θ measured along the horizontal axes, starting from $\Theta = 0$, one has a sequence of points along the vertical axes I showed after each iteration of the mapping, that is after each circle of the impulsive excitation until the angle variable reaches a certain magnitude has been chosen as $\varphi = 300$. Then, parameter Θ is increased by a small step ($\Delta\Theta = 0{\cdot}001$) and a new series of iterations is implemented under the same initial data for I and φ. If all iterations of the series appear at the same point, i.e., the action variable I remains constant, one has a periodic motion of the amplitude $\sqrt{2I/\omega_0}$ in terms of the original co-ordinate. Several points should be treated as a modulated multiperiodic regime. The mapping may also produce an irregular random-like sequence of points along I-axes. In this case, one should expect a complicated oscillation of a randomly varying amplitude. The diagrams show that the mapping frequently forms a "continuous" family of curves on the plane $\Theta - I$. The periodic regimes are associated with "knots" of the diagrams. For instance, Figure 6(c) gives an impression about the knot's neighborhood. The knot's location on the Θ-axes can be approximately detected if using the linear related system ($\varepsilon = 0$). It can be done both directly and by means of the saw-tooth transformations [16]. Using the present notations, an equation for Θ can be written as

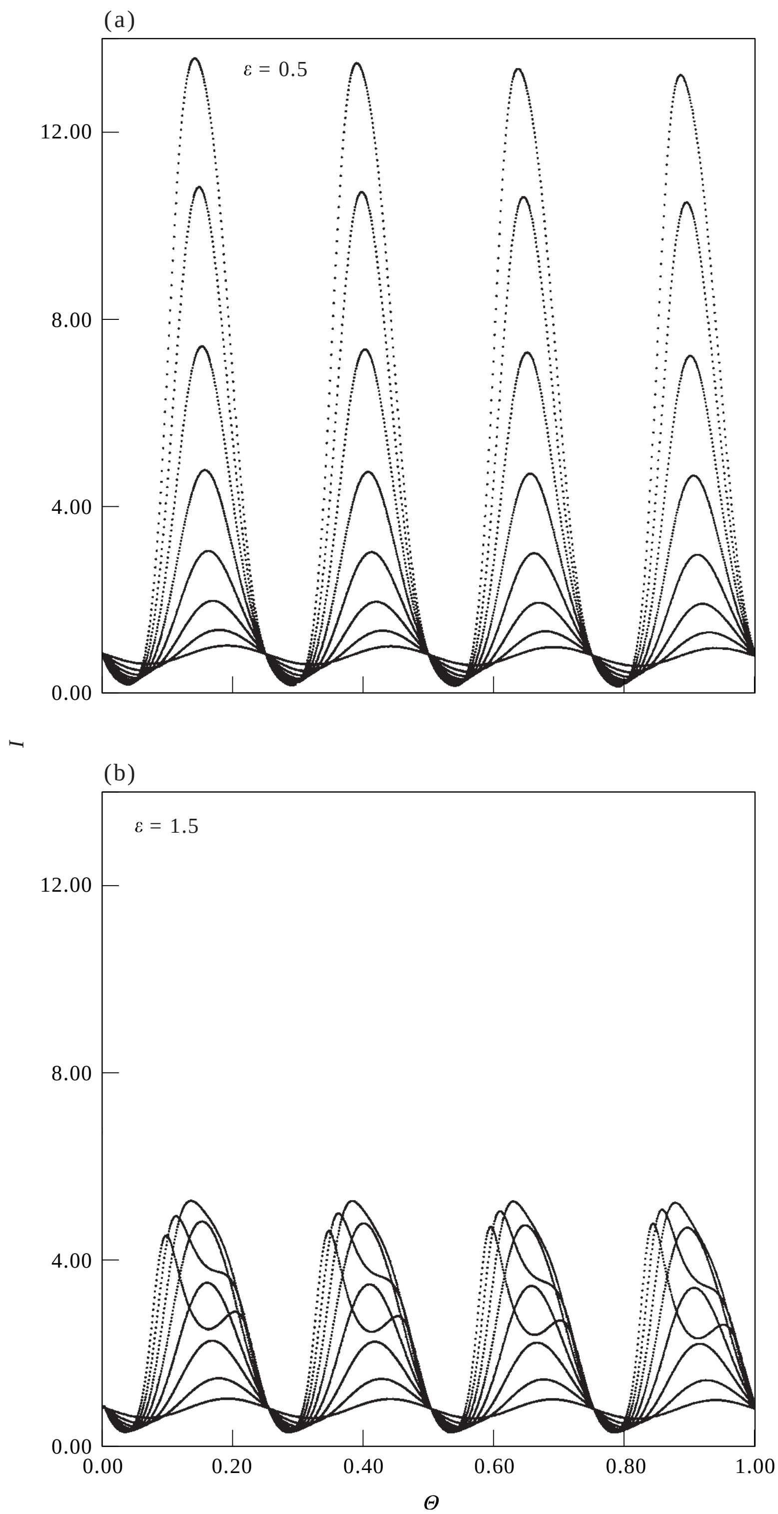
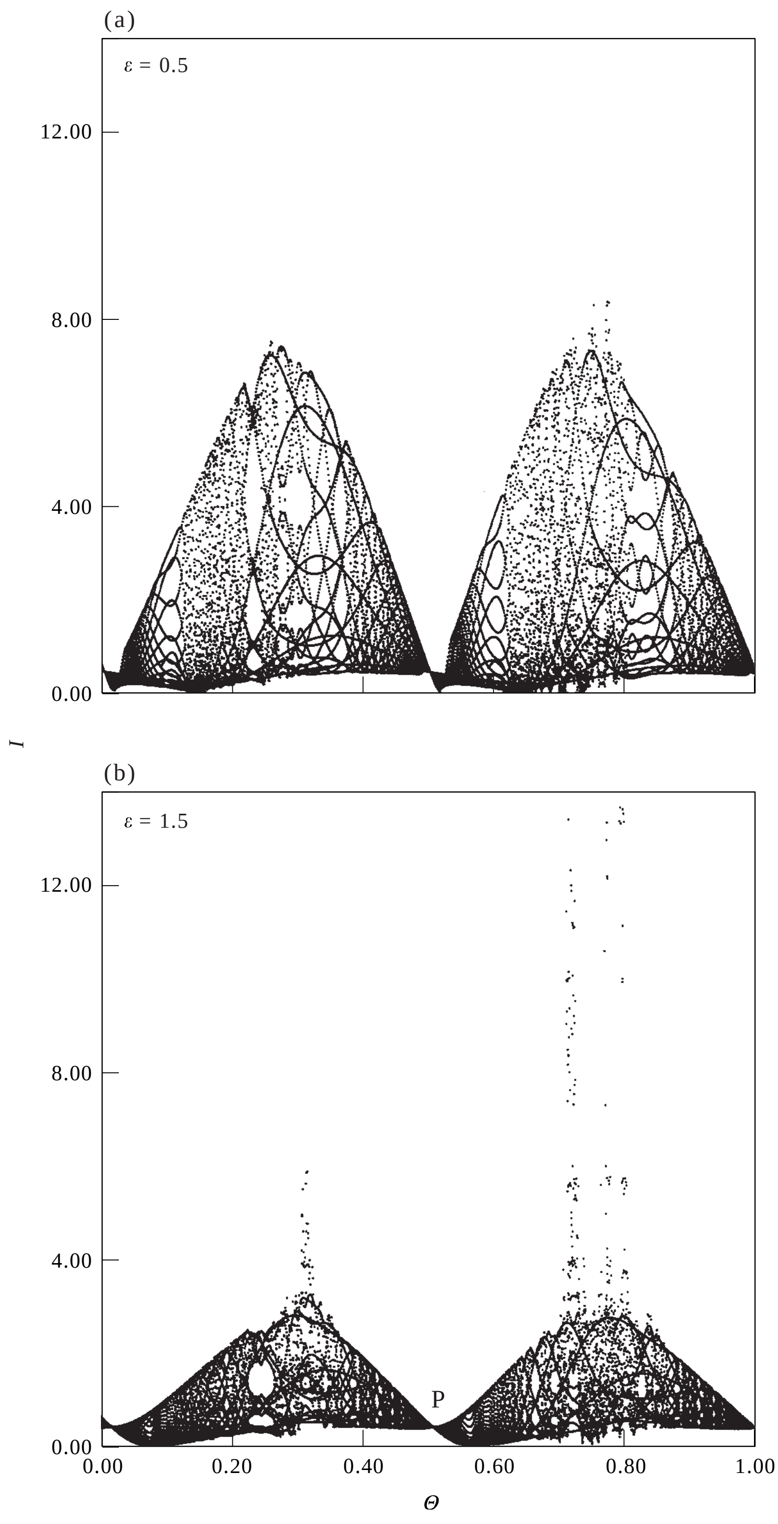


Figure 5. Diagram of the solution's structure for the following parameters: $|q_k^{\pm}| = \pi/2$, $\omega_0 = 2\pi$; (a) $\varepsilon = 0{\cdot}5$, (b) $\varepsilon = 1{\cdot}5$. Knots of the diagrams indicate a periodicity of solutions for the parameter of the shift related, Θ.


(a)
ε = 0.5
12.00
8.00
4.00
0.00
I
(b)
ε = 1.5
12.00
8.00
4.00
P
0.00
0.00
0.20
0.40
0.60
0.80
1.00
Θ

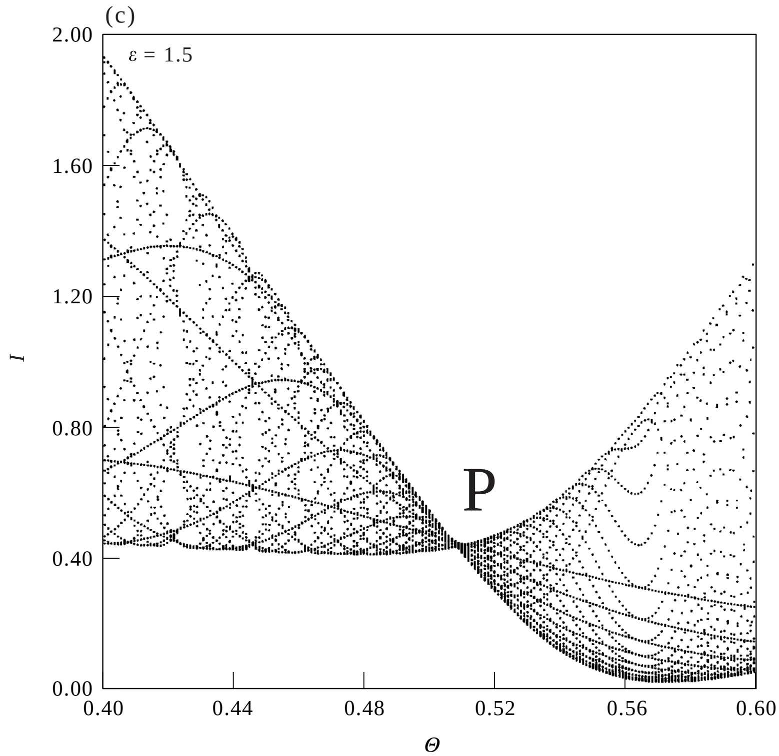


Figure 6. Diagram of the solution's structure for the following parameters: $|q_k^{\pm}| = \pi/2$, $\omega_0 = \pi$; (a) $\varepsilon = 0{\cdot}5$, (b) $\varepsilon = 1{\cdot}5$, (c) magnitude portion of the diagram at the neighborhood of periodic solution for $\varepsilon = 1{\cdot}5$.

$$\frac{q^2}{4}(\cos 4\omega_0\Theta - \cos 4\omega_0) = \cos 4\omega_0 - 1$$

where $q = |q_k^{\pm}|/\omega_0$.

Figures 5(a, b) show an evolution of the diagram for $\omega_0 = 2\pi$ when the parameter of non-linearity is growing from $\varepsilon = 0{\cdot}5$ to $\varepsilon = 1{\cdot}5$. Figures 6(a–c) show the evolution for $\omega_0 = \pi$. Analyzing the diagrams one can conclude that the stochastic-like regions are caused by parametric instability. The non-linear cubic term first of all decreases amplitudes of oscillations.

To this end, one should keep in mind that the diagrams have been obtained by using an approximate (averaged) model, and this may bring some doubts regarding extension of the numerical results on the original system. In the linear case ($\varepsilon = 0$), however, the analytical approach becomes exact and hence one can verify that the basic qualitative features of the diagrams are not produced by errors of the approximation.

6. CONCLUSIONS

In this paper the non-smooth temporal transformation has been applied to construct a family of periodic solutions of a weakly non-linear system under the parametric impulsive excitation. The transformation eliminates singular terms and reduces in the equation of motion to a standard weakly non-linear boundary value problem. To solve this problem asymptotic expansions were applied. As a result explicit form analytical solutions in terms of elementary functions have been obtained for small asymmetry of the distribution of impulses' sequences, (the dipole-like shift of each two neighboring impulses). The solutions and numerical simulations show a principal role of the shifts of the impulses' sequences. Namely, periodic, multiperiodic and stochastic-like regimes may be realized if varying the shift parameter. From a point of view of the asymptotic process, a small shift leads to branching of curves of the periodic solutions on the parameters plane and hence generates instability regions.

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