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NON-STATIONARY PROBLEMS IN DYNAMICS OF A STRING ON AN ELASTIC FOUNDATION SUBJECTED TO A MOVING LOAD

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The dynamics of an infinite string on an elastic foundation subjected to a moving load is under investigation in this paper. The load is modelled by a moving concentrated force. Both analytical and numerical methods are used. Non-stationary problems are analyzed. In particular the wave process caused by the accelerating load passing through the sonic speed is investigated. It is shown that the load at the moment when its speed is equal to the critical velocity gives rise to a wave front travelling at the sonic speed along the string. The asymptotical solution describing this front for large values of time is obtained. It allows the investigation of the qualitative properties of the solution. This solution and the results obtained in the numerical simulation carried out are in a good agreement.

1. INTRODUCTION

The problems of the dynamics of systems with moving loads are usually considered in the statement: it is assumed that the load moves at a constant speed and the steady-state solution is sought. There are a lot of studies devoted to such stationary problems for different kinds of structures under the load and different kinds of loads [1, 2].

Slepyan [3], and Andrianov and Krisov [4] considered non-stationary wave processes in elastic waveguides caused by a load moving at a constant velocity. In reference [4] the limiting process to the stationary regime for the case of subsonic movement of the load on the string on an elastic bed is investigated. In some studies an accelerating load is considered, but they consider finite structures so the wave process caused by such a load is not discussed [5–8].

The purpose of this paper is to obtain the solutions of non-stationary problems describing qualitatively the non-stationary wave processes in an infinite mechanical system under a moving load. One of the interesting problems is a description of the behaviour of an elastic waveguide under a moving load which is accelerating and passing through the sonic speed. This problem may have applications in engineering since exceeding the critical velocity of the waveguide may take place in real mechanical systems [2]. However, the problems related to real engineering constructions are very complicated. To consider these problems one needs to study simple test problems that can be investigated in detail. The mathematical difficulties occurring even in this way are quite serious.

In reference [9] the passage through the critical velocity by a load moving at a constant speed on a string on an elastic foundation with variable (piecewise constant) parameters is investigated. Kaplunov and Muravskii consider the passage through the sonic speed by a uniformly accelerating load on a string [10] and on Timoshenko beam [11], respectively. In these latter papers the passage through the critical velocity with a small acceleration is studied and the asymptotes as the acceleration of the load becomes zero found for some special points moving along the string.

Problems for the simplest type of elastic waveguide, which is a string on Winckler's elastic foundation (see Figure 1) are considered. The moving load is modelled by the Dirac function. A method appropriate for the kind of problems under consideration is suggested. This method allows one to obtain the asymptotes for large values of time for the solutions. To illustrate the method the asymptotes for the non-stationary solution for the case when the load moves at a constant supercritical velocity is obtained. It is useful since real problems generally include a section where the load moves at constant supercritical velocity together with a section where the load accelerates. To describe the dynamical processes in the string for values of time, which are close to the moment of exceeding the critical velocity a numerical calculation is used. It is shown that a pronounced wave front rises at the moment the critical velocity is exceeded. Unlike reference [10] not only is the solution found on the front but also in a neighbourhood of the front.

2. MATHEMATICAL FORMULATION

The waves in the waveguide under the moving load are governed by the following equation:

$$u'' - \frac{1}{c^2}\ddot{u} - ku = \chi(t)\delta(x - l(t)). \tag{1}$$

The initial conditions are

$$u(x, t)|_{t=0} = 0, \quad \dot{u}(x, t)|_{t=0} = 0. \tag{2}$$

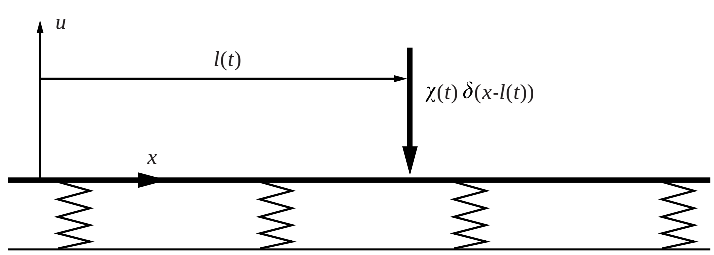

Figure 1. A string on an elastic foundation.

In the above equation, $u(x, t)$ is the displacement of a point on the string at the position x at time t; c is the sonic speed; k is the elastic coefficient of the foundation; $l(t)$ is the co-ordinate of the load; $\chi(t)$ is the load intensity, $\chi(t) \equiv 0$ for $t < 0$.

The initial value problem is considered: it is assumed that $u \equiv 0$ if $|x|$ is sufficiently large for given $t < \infty$.

3. THE GENERAL FORMULA FOR THE SOLUTION

A general formula is obtained which allows one to investigate the solution of the problem (1)–(2) numerically and in the simplest cases, analytically. Applying the Fourier transform in the x co-ordinate of equation (1) gives

$$\ddot{u}_F + c^2(\omega^2 + k)u_F = -c^2\chi(t)\, e^{i\omega l(t)} \tag{3}$$

The solution of equation (3) may be expressed as

$$u_F = -c\int_0^t \chi(\tau)\frac{\sin c\sqrt{\omega^2 + k}(t-\tau)}{\sqrt{\omega^2 + k}}\, e^{i\omega l(\tau)}\, d\tau. \tag{4}$$

Applying the inverse Fourier transform and changing the order of integration in the expression obtained gives

$$u = -\frac{c}{2\pi}\int_0^t\int_{-\infty}^{+\infty} \chi(\tau)\frac{\sin c\sqrt{\omega^2 + k}(t-\tau)}{\sqrt{\omega^2 + k}}\, e^{i\omega l(\tau)}\, e^{-i\omega x}\, d\omega\, d\tau \tag{5}$$

Calculating the internal integral yields

$$u = -\frac{c}{2}\int_0^t \chi(\tau)\theta\left(c - \frac{|x - l(\tau)|}{t-\tau}\right) J_0\left(\sqrt{k(c^2(t-\tau)^2 - (x - l(\tau))^2}\right) d\tau, \tag{6}$$

where $\theta(t)$ is the Heaviside function. Formula (6) is valid for all $l(t)$ and all $\chi(t)$. It was obtained in reference [10] for the case of a uniformly accelerating load.

4. THE MOVEMENT AT A CONSTANT SUPERCRITICAL SPEED

Now consider the simplest example of application of the expression (6). Let $\chi(t) = \theta(t)$, $v(t) \equiv \dot{l}(t) = \text{const} > c$. The notation $\xi = x - vt$ will be used.

The formula (6) gives that $u = 0$ if $\xi > 0$, i.e., all waves excited in the string lag behind the load.

In accordance with equation (6) one finds

$$u = -\frac{c\theta(-\xi)}{2}\int_{\frac{|\xi|}{c+v}}^{\min\left(t, \frac{|\xi|}{v-c}\right)} J_0\left(\sqrt{k(c^2\tau^2 - (v\tau + \xi)^2}\right) d\tau. \tag{7}$$

If $t \to \infty$ and ξ is fixed then

$$u = u^* = \frac{\theta(-\xi)}{\sqrt{k(1-\beta^2)}} \sin \sqrt{k_0}\xi, \quad k_0 = \left|\frac{k}{1-\beta^2}\right|, \tag{8}$$

where u^* is the solution of the following problem:

$$\begin{cases} u^{*\prime\prime} - \dfrac{1}{c^2}\ddot{u}^* - ku^* = \delta(x - vt), \\ u^* = u^*(x - vt), \\ u^* = 0, \quad x - vt > 0. \end{cases} \tag{9}$$

Thus the load radiates running waves.

One can see from equation (7) that the steady-state solution at the position ξ in the moving co-ordinate system is formed in the finite time $|\xi|/(v-c)$.

The formula (7) does not allow one to visualize clearly the behaviour of the non-stationary solution if $\xi < t(c-v)$. It is interesting to investigate the solution for large values of t. Consider the equation

$$u'' - \frac{1}{c^2}\ddot{u} - 2\gamma\dot{u} - ku = \theta(t)\delta(x - vt), \tag{10}$$

in the moving co-ordinate system (ξ; t) (in this section $(\ )' \equiv \mathrm{d}/\mathrm{d}\xi$):

$$(1-\beta^2)u'' + \frac{2v}{c^2}\dot{u}' - 2\gamma\dot{u} - \frac{1}{c^2}\ddot{u} - ku + 2\gamma v u' = \theta(t)\delta(t), \tag{11}$$

where $\gamma > 0$ is the coefficient of friction. In accordance with the limit absorption principle add the term $-2\gamma\dot{u}$ into equation (10) to be able to apply the Fourier transform in a classical sense in the ξ co-ordinate. In order to solve equation (10) for the case $\gamma = 0$, one can pass to the limit as $\gamma \to 0$:

$$u(x, t)|_{\gamma=0} = \lim_{\gamma \to 0} u_\gamma(x, t), \tag{12}$$

where u_γ satisfies equation (10). Taking the Fourier transform of equation (11) gives

$$\ddot{u}_F + 2(\mathrm{i}\omega v + c^2\gamma)\dot{u}_F + c^2(k - (\beta^2 - 1)\omega^2 + 2\mathrm{i}\gamma v\omega)u_F = -c^2. \tag{13}$$

Now consider the stationary solution of equation (13)

$$u_F^* = \frac{1}{(\beta^2 - 1)\omega^2 - 2\mathrm{i}\gamma v\omega - k}. \tag{14}$$

Using the inverse transform and passing to the limit gives the steady-state solution of equation (10) without friction:

$$u^* = \frac{1}{2\pi(\beta^2 - 1)} \lim_{\varepsilon \to +0} \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega\xi}\,\mathrm{d}\omega}{\omega^2 - 2\mathrm{i}\varepsilon\omega - k_0}. \tag{15}$$

The Jordan lemma is used to calculate the integral (15):

$$\int_{-\infty}^{+\infty} F(\omega)\,\mathrm{d}\omega = \lim_{R\to\infty}\int_{(\Gamma_R)} F(\omega)\,\mathrm{d}\omega. \tag{16}$$

Here Γ_R is the closed contour in the upper or lower half-plane of the complex plane depending on the sign of ξ (Figure 2). Then the residue theorem is applied. There are two poles ($\hat{\omega}_\pm = \mathrm{i}\varepsilon \pm \sqrt{k_0}$) inside of the contour if $\xi < 0$ and no poles if $\xi > 0$. The result is expression (8).

Another method of inversion of u_F^* is more efficient. It will be used for more complicated cases. Consider its application to the inversion of equation (14) as an example. One has

$$u^* = \frac{1}{4\pi\sqrt{k(\beta^2-1)}}\lim_{\varepsilon\to+0}\int_{-\infty}^{+\infty}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega\xi}}{\omega-\mathrm{i}\varepsilon-\sqrt{k_0}} - \frac{\mathrm{e}^{-\mathrm{i}\omega\xi}}{\omega-\mathrm{i}\varepsilon+\sqrt{k_0}}\right)\mathrm{d}\omega. \tag{17}$$

The formula obtained can be interpreted as a result of the action of distributions $1/(\omega \pm \sqrt{k_0} - \mathrm{i}0)$ on the test function $\mathrm{e}^{-\mathrm{i}\omega\xi}$. Using Sohotskiy's formulae [12]

$$\frac{1}{\omega \mp \mathrm{i}0} = \pm\mathrm{i}\pi\delta(\omega) + Vp\frac{1}{\omega}, \tag{18}$$

one obtains

$$u^* = \frac{1}{4\pi\sqrt{k(\beta^2-1)}}\Bigg(\mathrm{i}\pi\left(\mathrm{e}^{-\mathrm{i}\sqrt{k_0}\xi} - \mathrm{e}^{\mathrm{i}\sqrt{k_0}\xi}\right) + Vp\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{-\mathrm{i}\sqrt{k_0}\xi}\,\mathrm{e}^{-\mathrm{i}\omega\xi}\,\mathrm{d}\omega}{\omega} - Vp\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{\mathrm{i}\sqrt{k_0}\xi}\,\mathrm{e}^{-\mathrm{i}\omega\xi}\,\mathrm{d}\omega}{\omega}\Bigg). \tag{19}$$

In equation (18) $Vp\ 1/\omega$ denotes the distribution such that the result of its action on a test function $\varphi(\omega)$ is the Cauchy principal value of the integral $\int \varphi(\omega)\omega^{-1}\,\mathrm{d}\omega$.

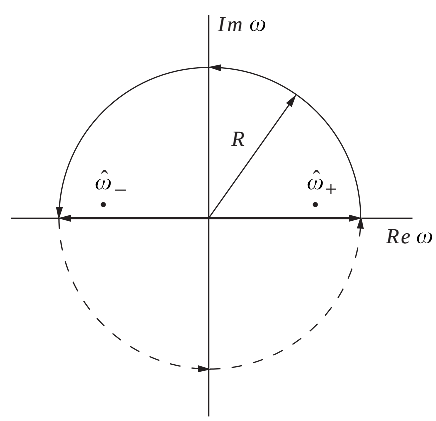

Figure 2. The contour Γ_R. — $\xi < 0$; - - -, $\xi > 0$.

Usually Sohotskiy's formulae are proved in supposition that test functions are infinitely differentiable functions of compact support. However, the sufficient conditions for validity of formulae (18) are quite weaker. In particular one can prove that Sohotskiy's formulae are valid for sufficiently smooth test functions $\Psi(\omega)$, such that the integrals $\int_{\pm R}^{\pm\infty} \Psi(\omega)\,\omega^{-1}\,\mathrm{d}\omega\ (R > 0)$ exist in proper or improper sense.

Calculating the integral in equation (19) gives expression (8) again. With this method of inversion one does not need to investigate the behaviour of the integrand in equation (15) in the whole complex plane but only on the real axis.

Now apply this method to find the non-stationary solution. The solutions of equation (13) are given by

$$\begin{aligned} u_F &= C^+\ \mathrm{e}^{\mathrm{i}(-\omega v+\sqrt{\omega^2+k})t} + C^-\ \mathrm{e}^{\mathrm{i}(-\omega v-c\sqrt{\omega^2+k})t} + u_F^* \\ &\equiv \sum_{(\pm)} C^{\pm}\ \mathrm{e}^{\mathrm{i}(-\omega v\pm\sqrt{\omega^2+k})t} + u_F^* \equiv u_F^+ + u_F^- + u_F^*, \end{aligned} \tag{20}$$

where constants

$$C^{\pm} = \frac{\sqrt{\omega^2+k}\pm\omega v}{2\sqrt{\omega^2+k}(k-(\beta^2-1)\omega^2)} \tag{21}$$

are found from the initial conditions (2).

Applying the inverse Fourier transform of $u_F^{\pm}$ and using Sohotskiy's formulae (18) as before one obtains

$$u^+ + u^- = -\frac{\sin\sqrt{k_0}\xi}{2\sqrt{k(\beta-1)}} - \frac{\Sigma}{4\pi(\beta^2-1)}, \tag{22}$$

where

$$\Sigma = \sum_{(\pm)} Vp\int_{-\infty}^{+\infty} \frac{\sqrt{\omega^2+k}\pm\omega\beta}{\sqrt{\omega^2+k}(\omega^2-k_0)}\ \mathrm{e}^{\mathrm{i}(\pm\sqrt{\omega^2+k}-\omega\beta)ct-\mathrm{i}\omega\xi}\ \mathrm{d}\omega. \tag{23}$$

Let $x = Wt$. To investigate the behaviour of Σ in the co-ordinate system moving at a velocity W, search for the principal part of the asymptotic expansion of Σ as $t \to \infty$. The expression (23) can be rewritten as

$$\Sigma = \sum_{(\pm)} Vp\int_{-\infty}^{+\infty} \frac{\sqrt{\omega^2+k}\pm\omega\beta}{\sqrt{\omega^2+k}(\omega^2-k_0)}\ \mathrm{e}^{\mathrm{i}(\pm c\sqrt{\omega^2+k}-\omega W)t}\ \mathrm{d}\omega. \tag{24}$$

According to the stationary phase method, if $t \to \infty$ one can integrate only in small neighbourhoods $(\pm\sqrt{k_0}-\varepsilon;\ \pm\sqrt{k_0}+\varepsilon)$ of singularities. The part omitted has an order $O(t^{-1})$. Hence,

$$\Sigma = \sum_{(\pm)} Vp\int_{\pm\sqrt{k_0}-\varepsilon}^{\pm\sqrt{k_0}+\varepsilon} \frac{\sqrt{\omega^2+k}\pm\omega\beta}{\sqrt{\omega^2+k}(\omega^2-k_0)}\ \mathrm{e}^{\mathrm{i}(\pm c\sqrt{\omega^2+k}-\omega W)t}\ \mathrm{d}\omega + O(t^{-1}). \tag{25}$$

Since $\varepsilon > 0$ is arbitrary small, the integrand in equation (25) can be approximated in the following way ($\omega = \pm k_0 + \delta$):

$$\Sigma = \frac{1}{\sqrt{k_0}} \sum_{(\pm)} \left(\pm \, \mathrm{e}^{\mp \mathrm{i}\sqrt{k_0}\xi} Vp \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{e}^{\mathrm{i}(c_g\delta \pm \frac{c_g}{2\sqrt{k_0}}(1-\beta^{-2})\delta^2 + o(\delta^2))t - \delta x)}}{\delta + o(\delta)} \, \mathrm{d}\delta \right) + O(t^{-1}), \quad (26)$$

where $c_g = c^2/v$ is the value of group velocity such that it corresponds to the value of phase velocity to be equal to the load velocity [3].

Omitting the terms $O(\delta^2)$ one finds the approximate expression for u, which is valid as $t \to \infty$ for x such that $x - c_g t \to \infty$:

$$\begin{aligned} u = u^+ + u^- + u^* &\simeq -\frac{\sin\sqrt{k_0}\xi}{2\sqrt{k(1-\beta^2)}} - \frac{\mathrm{e}^{-\mathrm{i}\sqrt{k_0}\xi} - \mathrm{e}^{\mathrm{i}\sqrt{k_0}\xi}}{4\pi\sqrt{k(1-\beta^2)}} \int_{-\varepsilon}^{+\varepsilon} \frac{\mathrm{i}\sin((c_g t - x)\delta)}{\delta} \, \mathrm{d}\delta \\ &= \frac{\sin\sqrt{k_0}\xi}{\sqrt{k(1-\beta^2)}} \theta\left(\frac{x}{t} - \frac{c^2}{v}\right)\theta\left(v - \frac{x}{t}\right). \end{aligned} \quad (27)$$

However, expression (27) does not describe well the behaviour of the solution if $|c_g t - x| \simeq 0$ (in this case the quadratic term of the exponent becomes the principle one). Keeping this term one can obtain a more precise expression for the asymptotes:

$$\begin{aligned} u = &\frac{1}{2\sqrt{k(\beta^2-1)}} \left(\sin\sqrt{k_0}\xi \left(-1 + \left(\mathrm{C}\left(\frac{\kappa}{\sqrt{2\pi\mu}}\right) + \mathrm{S}\left(\frac{\kappa}{\sqrt{2\pi\mu}}\right) \right) \operatorname{sign}\kappa \right) \right. \\ &\left. - \cos\sqrt{k_0}\xi \left(\left(\mathrm{C}\left(\frac{\kappa}{\sqrt{2\pi\mu}}\right) - \mathrm{S}\left(\frac{\kappa}{\sqrt{2\pi\mu}}\right) \right) \operatorname{sign}\kappa \right) + 2\sin\sqrt{k_0}\xi\theta(-\xi) \right), \end{aligned} \quad (28)$$

where

$$\kappa = \frac{x - c_g t}{\sqrt{t}}, \quad \mu = \frac{c^4(\beta^2-1)^{3/2}}{2v^3\sqrt{k}}, \quad (29)$$

$$\mathrm{C}(z) = \int_0^z \cos\frac{\pi}{2} t^2 \, \mathrm{d}t, \quad \mathrm{S}(z) = \int_0^z \sin\frac{\pi}{2} t^2 \, \mathrm{d}t, \quad (30)$$

($\mathrm{S}(z)$ and $\mathrm{C}(z)$ are Fresnel integrals).

Thus, there exist two pronounced wave fronts for large values of t. The first one is the front under the load travelling at the velocity v. The second one travels at the velocity c_g. In Figure 3 the graph of solution (28) in the neighbourhood of the second front is presented.

Slepyan [3] obtained similar results by solving this problem with another method. However, it is believed that his solution is not quite accurate since he used the asymptotically non-equivalent approximation in certain steps of the construction for the solution. Having corrected the inaccuracies, a formula is obtained which concides with equation (28).

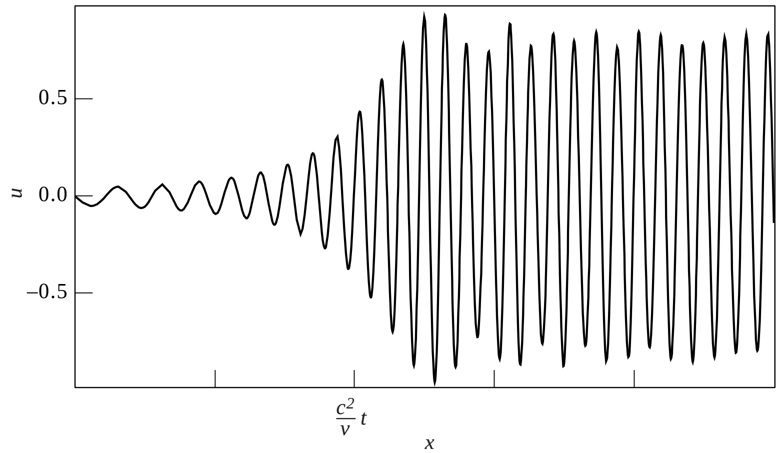

Figure 3. Non-stationary solution for the case $v = \text{const} > c$; the value of t is large. The neighbourhood of the second wave front.

5. THE PASSAGE THROUGH THE CRITICAL VELOCITY

It is interesting to consider the wave process given rise by the load passing through the critical velocity. Indeed, in the case $\chi(t) = \theta(t)$, $v(t) = c$ general formula (6) yields

$$u = -\sqrt{\frac{ct - \frac{|\xi|}{2}}{2k|\xi|}}\mathrm{J}_1\left(\sqrt{2k|\xi|\left(ct - \frac{|\xi|}{2}\right)}\right)\theta\left(t - \frac{|\xi|}{2c}\right)\theta(-\xi). \tag{31}$$

For $\xi \to -0$

$$u = -\frac{ct}{2} + o(1). \tag{32}$$

Thus, if the load moves at the critical velocity during a finite time interval then the solution of that problem is discontinuous. The question of interest is what would happen if the load moving at gradually increasing velocity overcomes instantly the sound speed.

Numerical calculation of the integral in formula (6) shows that at the moment $t = t_0$ when the load is moving at the critical velocity the pronounced "pit" under the load is beginning to lag behind it (Figure 4). Here ξ is the co-ordinate in the co-ordinate system moving together the load; $t_1 - t_0 > 0$ is a small time interval. Later this "pit" transforms into a pronounced wave front running at the critical velocity c. There are intensive oscillations behind the front (Figure 5). An attempt will be made to describe the evolution of this front for the large values of t.

Suppose that $v(t)$ varies as follows:

$$v(t) = \begin{cases} at, & 0 \leqslant t < T \\ aT, & t \geqslant T \end{cases}. \tag{33}$$

Since the problem is linear the solution can be represented as a superposition of

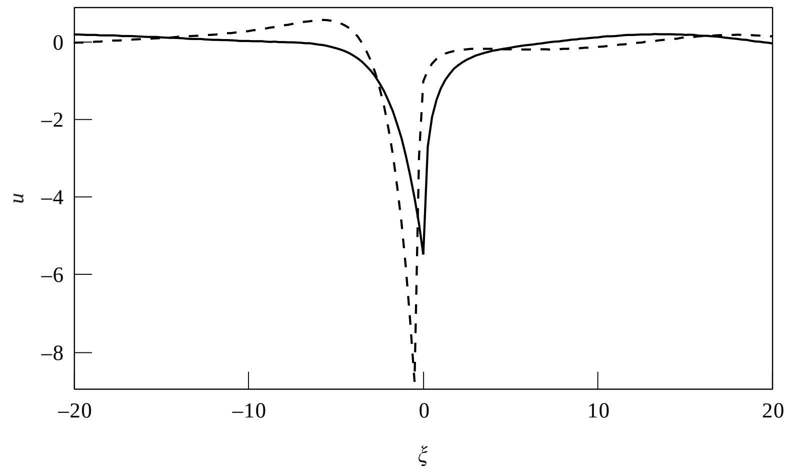

Figure 4. The "pit" under the load. —, $t = t_0$; - - -, $t = t_1$.

the following two problems. In the first one the load appears at the moment $t = 0$, moves with constant acceleration a and disappears at $t = T$. In the second one the load appears at the moment $t = T$ and moves at the constant supersonic speed. The second problem was considered above. Consider now the first problem.

5.1. THE ANALYTICAL SOLUTION

The principal part of asymptotic expansion of u as $t \to \infty$ will be sought. Applying the double Fourier transform of equation (1) according to the limit absorption principle gives:

$$u = \lim_{\varepsilon \to +0} \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{F(\Omega, \omega)\, \mathrm{e}^{-\mathrm{i}(\Omega t + \omega x)}}{\Omega^2 - c^2(\omega^2 + k) + 2\mathrm{i}\varepsilon\Omega} \, \mathrm{d}\Omega \, \mathrm{d}\omega, \tag{34}$$

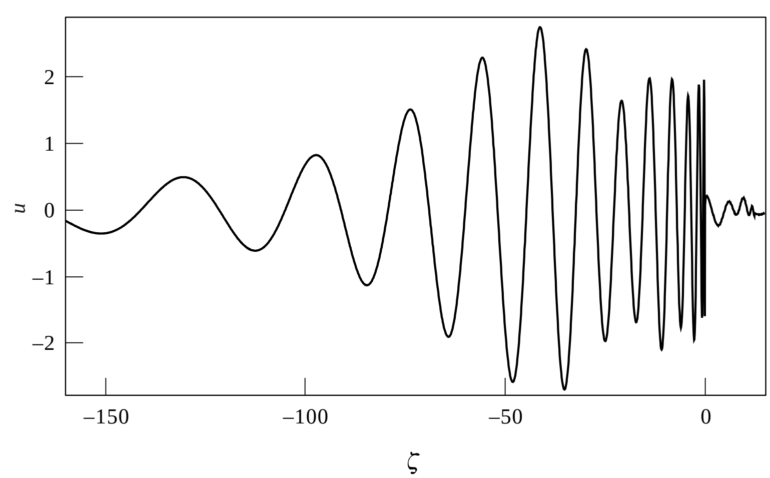

Figure 5. Passage through the critical velocity. The solution of the problem for large t.

where $F(\Omega, \omega)$ is

$$F(\Omega, \omega) = c^2 \int_0^T \mathrm{e}^{\mathrm{i}\left(\frac{at^2}{2}\omega + t\Omega\right)}\, \mathrm{d}t = -\left.\frac{c^2\sqrt{\pi}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\,\mathrm{e}^{-\mathrm{i}\frac{\Omega^2}{2a\omega}}\,\mathrm{erf}\left(\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}\dfrac{(at\omega + \Omega)}{\sqrt{2a\omega}}\right)}{\sqrt{2a\omega}}\right|_{t=0}^{t=T}, \quad (35)$$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\, \mathrm{d}t. \quad (36)$$

The integral representation (34) will be used to find the asymptotes. The main difficulty is the sufficiently complicated behaviour of the function $F(\Omega, \omega)$. The basic idea of our method is the following. It is known that a load moving uniformly at velocity $v > c$ radiates waves with phase velocity equal to the load velocity. It suggests that the wave packet radiated at the moment of overcoming the critical velocity involves the waves with phase velocities aproximately equal to c (and to be greater than c). The values of frequencies $\pm\Omega = \mp c\omega$ as $\omega \to \infty$ correspond to these values of phase velocity. Therefore, it is hoped that the integral (34) as $t \to \infty$ is determined by values of the integrand in the domain where $\pm\Omega = \mp c\omega$, $\omega \to \infty$. But for these values Ω, ω one can easily calculate the asymptotes of $F(\Omega, \omega)$ and use it instead of the exact expression. Keeping this idea in mind proceed to the exact formulae.

Let $x = x_0 + Wt$. The parameter W determines the velocity of a moving co-ordinate system in terms of which the solution will be investigated.

Further one needs to know the behaviour of $F(\Omega, \omega)$ for large Ω, ω. One can investigate it making use of the stationary phase method. It is easy find F to vanish in infinity in plane $(\Omega; \omega)$ as $h^{-1/2}$ at the worst, where $h = \sqrt{\Omega^2 + \omega^2}$. It may be shown that the integral (34) exists in a Lebesgue's sense if $\varepsilon \neq 0$ and the external integral in equation (34) converges uniformly in t and ε. In order to transform the limit of the internal integral value to an integral in the sense of a Cauchy principal value, use Sohotskiy's formulae (18) again:

$$\begin{aligned} u &= \lim_{\varepsilon\to+0} \frac{1}{4\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{F(\Omega, \omega)\,\mathrm{e}^{-\mathrm{i}(\Omega t + \omega(x_0 + Wt))}}{(\Omega - c\sqrt{\omega^2 + k} - \mathrm{i}\varepsilon)(\Omega + c\sqrt{\omega^2 + k} - \mathrm{i}\varepsilon)}\, \mathrm{d}\Omega\, \mathrm{d}\omega \\ &= \frac{1}{8\pi^2 c} \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega(x_0 + Wt)}}{\sqrt{\omega^2 + k}} \sum_{(\pm)} \pm\left(\mathrm{e}^{\pm\mathrm{i}ct\sqrt{\omega^2 + k}}\left(\pi\mathrm{i}F\left(\mp\sqrt{\omega^2 + k}, \omega\right) - \Phi(\omega, t)\right)\right) \mathrm{d}\omega, \end{aligned} \quad (37)$$

where

$$\Phi(\omega, t) = Vp \int_{-\infty}^{+\infty} \frac{F(\mp c\sqrt{\omega^2 + k} + \Omega, \omega)\,\mathrm{e}^{-\mathrm{i}\Omega t}}{\Omega}\, \mathrm{d}\Omega. \quad (38)$$

The asymptotes of the kind of integrals similar to integral (38) is considered in reference [13]. It can be calculated by the stationary phase method. Now rewrite (38) as follows:

$$\Phi(\omega, t) = Vp \int_{-\infty}^{+\infty} (\eta(\Omega) + (1 - \eta(\Omega))) \frac{F(\mp\sqrt{\omega^2 + k} + \Omega, \omega)\, \mathrm{e}^{-\mathrm{i}\Omega t}}{\Omega}\, \mathrm{d}\Omega = I_1 + I_2, \tag{39}$$

where $\eta(\Omega)$ is an infinitely differentiable function, the support for which lies in a small neighbourhood of zero. Taking into account the asymptotical and differential properties of F one finds that $I_2 = O(t^{-\infty})$. The integral I_1 may be calculated by integration on a small neighbourhood of zero in which the integrands can be replaced by its approximations. The calculation yields

$$\Phi(\omega, t) = -\mathrm{i}\pi F(\mp c\sqrt{\omega^2 + k}, \omega) + O(t^{-\infty}), \quad t \to \infty. \tag{40}$$

The result is

$$u = \frac{\mathrm{i}}{4\pi c} \sum_{(\pm)} \left(\pm \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega x_0}\ \mathrm{e}^{\mathrm{i}(-\omega W \pm c\sqrt{\omega^2+k})t} F(\mp c\sqrt{\omega^2 + k}, \omega)}{\sqrt{\omega^2 + k}}\, \mathrm{d}\omega \right) + O(t^{-\infty}). \tag{41}$$

The asymptotes of the solution in a neighbourhood of the wave front under consideration will be obtained from this solution.

5.2. THE ASYMPTOTES OF THE SOLUTION BEHIND THE WAVE FRONT

For the kind of integrals as follows

$$I(t) = \int_a^b f(\omega)\, \mathrm{e}^{\mathrm{i}\varphi(\omega)t}\, \mathrm{d}\omega, \tag{42}$$

in the case when $f(\omega)$ and $\varphi(\omega)$ are sufficiently smooth functions and $\varphi(\omega)$ has one and only one stationary point ω_* in the integration interval, for $t \to +\infty$ the asymptotical formula [13]

$$I = \mathrm{e}^{\mathrm{i}\varphi(\omega_*)}\ \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\operatorname{sign}\varphi''(\omega_*)} \sqrt{\frac{2\pi}{|\varphi''(\omega_*)|t}} f(\omega_*) + O(t^{-1}) \tag{43}$$

is right.

Let $x_0 = -c^2/(2a)$. For this choice x_0 in the case $W = c$ the origin of the moving co-ordinate system and the wave front under consideration are moving together. Denote

$$f_\pm(\omega) = \pm \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega x_0} F(\mp c\sqrt{\omega^2 + k}, \omega)}{4\pi c\sqrt{\omega^2 + k}}, \tag{44}$$

$$\varphi_\pm(\omega) = -\omega W \pm c\sqrt{\omega^2 + k}. \tag{45}$$

Let $|W| < c$, i.e., the moving co-ordinate system under consideration lags behind the wave front. In this case in both of the two integrals in equation (41) the phases $\varphi_{\pm}(\omega)$ have single stationary points $\pm\omega_*$:

$$\omega_* = W\sqrt{\frac{k}{c^2 - W^2}}. \tag{46}$$

Applying formula (43) to (41) results in (the contributions from the ends of integration interval $\pm\infty$ have the asymptotic order $O(t^{-\infty})$, thus one can obtain more precise estimation than in equation (43))

$$\begin{aligned} u &= \sum_{(\pm)} \int_{-\infty}^{+\infty} f_{\pm}(\omega)\, \mathrm{e}^{\mathrm{i}\varphi_{\pm}(\omega)t}\, \mathrm{d}\omega \\ &= \sum_{(\pm)} \mathrm{e}^{\mathrm{i}\varphi_{\pm}(\pm\omega_*)}\, \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\operatorname{sign}\varphi''_{\pm}(\pm\omega_*)} \sqrt{\frac{2\pi}{|\varphi''_{\pm}(\pm\omega_*)|t}}\, f_{\pm}(\omega_*) + O(t^{-3/2}). \end{aligned} \tag{47}$$

Using formula (47) and making the hard calculation one can find the following expression for the principal term of the asymptotes for u:

$$\begin{aligned} u = -\frac{c}{\sqrt{2Wakt}} \Bigg(&\cos\left(\sqrt{k(c^2 - W^2)}\left(t - \frac{c^2}{2aW}\right) + \frac{\pi}{4}\right) \mathrm{S}\left(\frac{a\tau\omega_* - c\sqrt{\omega_*^2 + k}}{\sqrt{\pi a\omega_*}}\right) \\ &+\sin\left(\sqrt{k(c^2 - W^2)}\left(t - \frac{c^2}{2aW}\right) + \frac{\pi}{4}\right) \mathrm{C}\left(\frac{a\tau\omega_* - c\sqrt{\omega_*^2 + k}}{\sqrt{\pi a\omega_*}}\right)\Bigg)\Bigg|_{\tau=0}^{\tau=T} + O(t^{-3/2}). \end{aligned} \tag{48}$$

Choosing $W \simeq c$ one obtains the solution in a small neighbourhood of the wave front. In this case formula (48) reduced to

$$u \simeq -\frac{c}{\sqrt{Wakt}} \cos\left(\sqrt{k(c^2 - W^2)}\left(t - \frac{c}{2a}\right)\right). \tag{49}$$

Formula (49) can be easily obtained directly proceeding from (41). One can find that $\omega_* \simeq +\infty$ if $W \simeq c$, $W < c$. On calculating the asymptotes of u one can use the asymptotes of $F(\mp c\sqrt{\omega_*^2 + k},\ \pm\omega_*)$ as $\omega_* \to +\infty$ instead of the exact formula for it. Now

$$\begin{aligned} F(\mp c\sqrt{\omega_*^2 + k},\ \pm\omega_*) &= c^2 \int_0^T \mathrm{e}^{\pm\mathrm{i}\left(\frac{at^2}{2}\omega_* - ct\sqrt{\omega_*^2 + k}\right)}\, \mathrm{d}t \\ &= c^2 \int_0^T \mathrm{e}^{\pm\mathrm{i}\left(\frac{at^2}{2} - ct\right)\omega_*}\, \mathrm{e}^{\pm\mathrm{i}ct(\omega_* - \sqrt{\omega_*^2 + k})}\, \mathrm{d}t. \end{aligned} \tag{50}$$

In this case a direct application of formula (43) to calculate the asymptotes of the function $F(\mp c\sqrt{\omega_*^2+k},\ \pm\omega_*)$ is not possible because one has an integrand of a more common kind than in equation (43). However, the asymptotes can still be calculated. To do this take notice of $\omega_* - \sqrt{\omega_*^2+k}\to 0$ as $\omega_*\to +\infty$. The function $\mathrm{e}^{\pm\mathrm{i}ct(\omega_*-\sqrt{\omega_*^2+k})}$ can be expanded in a convergent series of negative powers of ω_*. Changing the order of integration and summation one obtains the series of integrals each of them has one and only one stationary point $t_* = c/a$—the moment of time in which the load passes through the critical velocity. Now one can apply the formula (43) for each integral. After this the series is summed

$$\begin{aligned}F(\mp c\sqrt{\omega_*^2+k},\ \pm\omega_*) &= c^2\int_0^T \mathrm{e}^{\pm\mathrm{i}\left(\frac{at^2}{2}-ct\right)\omega_*}\sum_{k=0}^{\infty} c_k(t)\omega_*^{-k}\,\mathrm{d}t\\ &= \sum_{k=0}^{\infty} c^2\omega_*^{-k}\int_0^T \mathrm{e}^{\pm\mathrm{i}\left(\frac{at^2}{2}-ct\right)\omega_*} c_k(t)\,\mathrm{d}t\\ &= c^2\sqrt{\frac{2\pi}{|\omega_*|a}}\,\mathrm{e}^{-\mathrm{i}\frac{c^2}{2a}\omega_*}\,\mathrm{e}^{\mathrm{i}(\mp\sqrt{\omega_*^2+k}+\omega_*)\frac{c^2}{a}}\,\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{4}}\\ &\quad + O(\mathrm{e}^{\mathrm{i}(\mp\sqrt{\omega_*^2+k}+\omega_*)\frac{c^2}{a}}\omega_*^{-1}).\end{aligned} \tag{51}$$

The asymptotic expansion obtained should be interpreted as an asymptotic expansion in asymptotic scale $h_k = \mathrm{e}^{\mathrm{i}(\mp\sqrt{\omega_*^2+k}+\omega_*)c^2/a}\omega_*^{-k}$. Substituting equation (51) into equation (44), and equation (44) and (45) into equation (47) one gets formula (49).

In order to find the value of $u(\zeta,\ t)$ in the moving together with the wave front co-ordinate system one needs to substitute $W = c + \zeta/t$ into equation (48) (or into equation (49)). For $\zeta = 0$ this yields

$$u|_{\zeta=0} = -\sqrt{\frac{c}{akt}} + O(t^{-3/2}). \tag{52}$$

Formula (52) is in a good agreement with the results obtained in reference [10].

5.3. THE ASYMPTOTICS OF THE SOLUTION BEFORE THE WAVE FRONT

For $|W| > c$ the integrals in equation (41) have no stationary points, so $u = O(t^{-\infty})$ because the integrands are smooth functions and contributions from the ends of integration interval have the asymptotic order $O(t^{-\infty})$. This result is easy to understand: choosing $|W| > c$ the behaviour of the solution in the co-ordinate system moving at a velocity greater than c is investigated. Hence, u will be equal to zero in finite time at any point of this co-ordinate system because u is not equal to zero only for $x \in (-ct;\ ct)$.

The question of interest is does a right neighbourhood exist in which $u(\zeta,\ t)$ has the asymptotic order $O(t^{-1/2})$ (this is order of u in a left neighbourhood). Put $W = c$ and $x_0 = -c^2/(2a) + \Delta(t)$ in equation (41) (the analysis before corresponds to the case $\Delta(t) = (W-c)t$). Let $\Delta(t) = \delta t^{\alpha},\ \delta > 0$. To answer this question find the maximal $\alpha = \alpha_0$ such that the value of u calculated from equation (41) has the order $O(t^{-1/2})$ for small δ. After respective calculations (the asymptotes of such integrals are consdered in reference [14]) one can find that $\alpha_0 = -1$. This means that the neighbourhood required exists and its size decreases as $O(t^{-1})$.

So let $\Delta(t) = \delta/t$. For $W = c$ the stationary points of the integrals in equation (41) are $\pm\infty$ and the value of u is asymptotically determined by the behaviour of integrands in equation (41) in the neighbourhoods of $\pm\infty$. Based on this one can replace the integrands by an approximate one. In addition one can integrate on arbitrary neighbourhoods $(\pm R;\ \pm\infty)$ (for $F(\mp c\sqrt{\omega^2 + k}, \omega)$ using asymptotical estimation (51)):

$$u = \frac{\mathrm{i}c}{2\sqrt{2\pi a}} \sum_{(\pm)} \left(\int_{\pm R}^{\pm\infty} \frac{\mathrm{e}^{\mathrm{i}\left(\pm\frac{\pi}{4} - \omega\Delta - \frac{k}{2\omega}t\right)}}{\sqrt{\pm\omega}|\omega|(1 + O(\omega^{-1}))}\, \mathrm{d}\omega \right) + o(t^{-1/2}). \tag{53}$$

One can transform formula (53) changing the variable as follows: $\omega = \tau^{-1} t (ck/2\delta)^{1/2}$, and taking into account that $t \to \infty$:

$$u = -\frac{c^{3/4}\delta^{1/4}}{(2k)^{1/4}(\pi a t)^{1/2}} \int_0^{+\infty} \sin\left(\sqrt{\frac{\delta c k}{2}}(\tau - \tau^{-1}) + \frac{\pi}{4} \right) \tau^{-1/2}\, \mathrm{d}\tau + o(t^{-1/2}). \tag{54}$$

After the calculation of the integral in equation (54) one gets

$$u = -\sqrt{\frac{c}{akt}}\, \mathrm{e}^{-\sqrt{2ck\delta}} + o(t^{-1/2}). \tag{55}$$

One can take notice that both asymptotes obtained ((49) and (55)) have equal values on the wave front so that the string does not lose the continuity.

5.4. COMPARISON OF ANALYTICAL AND NUMERICAL RESULTS

It is possible to investigate the dynamical processes in the string by the numerical calculation of the integral in formula (6). All numerical results listed further were obtained using the following values of problem parameters: $c = 1$, $k = 0{\cdot}1$, $a = 0{\cdot}01$, $T = 150$. In all figures cited below the solid line corresponds to the numerical solution and the dashed line corresponds to the analytical solution.

In Figure 6 the string displacement on the front is shown (the analytical solution is given by formula (52)). The numerical solution oscillates in the neighbourhood of the analytical one, and its oscillations decrease with time.

In Figure 7 the displacements of points on the string in a small neighbourhood of the wave front are displayed for $t = 400$ (ζ is the co-ordinate in the co-ordinate system moving together with the wave front under consideration). Here the analytical solution is given by formulae (49) and (55) to the left and to the right of the wave front, respectively.

In Figure 8 the displacements of points on the string behind the wave front are shown ($t = 900$). The analytical solution is given by equation (48).

The graphs presented show the asymptotical solution obtained to describe well the dynamical processes in the string and to allow one to see clearly the principle qualitative properties of the problem solutions.

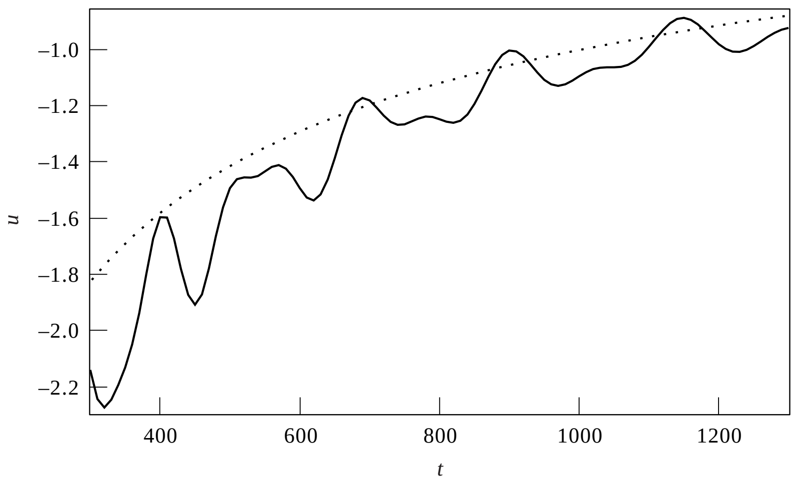

Figure 6. The displacements of the string on the wave front.

6. CONCLUSION

The analytical and numerical investigations carried out allow one to describe qualitatively the behaviour of the string under the action of the moving load passing through the sound speed.

On the time interval where the load moves with acceleration, when the velocity of the load is less than the critical velocity there is a "pit" on the string. The deflection of the string vanishes rapidly in moving away from the load. The graph of deflection of the string is qualitatively similar to the graph of the stationary solution for the case of subsonic speed. Starting from the moment t_0, when the velocity of the load passes through the critical value c this "pit" begins to lag behind the load (Figure 4), moving at the velocity c. Intensive oscillations begin behind the load whereas before the load the solution vanishes exponentially in the space co-ordinate after a long time. Thus, some time after t_0,

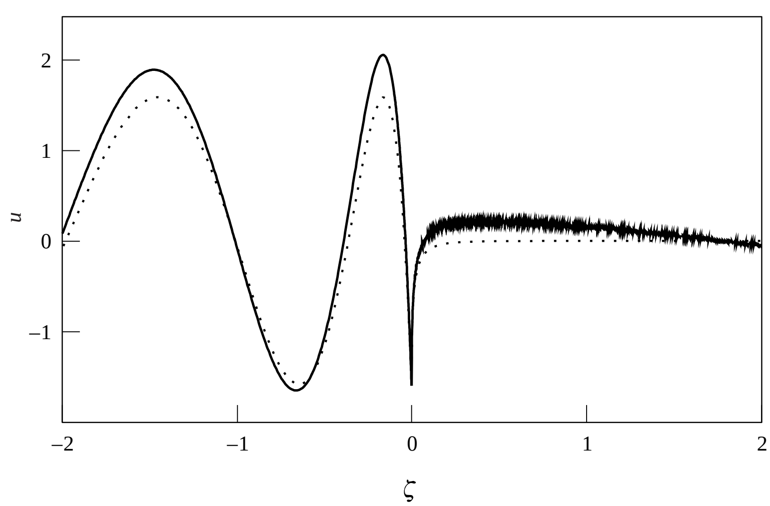

Figure 7. The displacements of the string in a neighbourhood of the wave front.

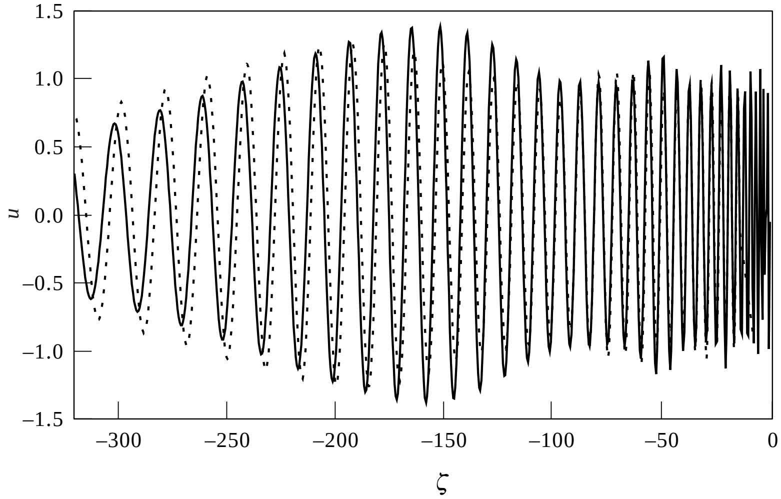

Figure 8. The oscillations of the string behind the front.

the pronounced wave front on the string running at the velocity c is formed. The value of displacement of the point on the front decreases as $O(t^{-1/2})$. It is as large as the acceleration of the load is small at the moment of passing through the critical velocity (52).

It should be noted that the method presented in this paper allows one to consider the problem with the load moving with non-constant acceleration. In this case one needs to make harder calculations. However, the result will be the same in essence, since the asymptotes (49) and (55) are determined only by the contribution from the moment t_0 of passing the critical velocity. It is only significant that $\ddot{l}(t_0) \neq 0$. It is quite possible to obtain the formulae analogous to equations (49) and (55). It is important that the value of deflection on the front will be expressed by formula (52), where $a = \ddot{l}(t_0)$. If one considers a degenerate case $(\ddot{l}(t_0) = 0)$ one finds the principal term of the asymptotes to vanish more slowly than $t^{-1/2}$ as $t \to \infty$.

It is easy to analyze the solution of the aggregate problem that also includes the section where the load is moving at a supersonic speed. As it has been shown before for $t > T$ the approximate solution of the second problem is given by formula (27) (where one needs to substitute $t - T$ instead of t). Hence, after some time, the wave front generated on the first section of movement outruns the back front of the sine-shaped quasi-stationary wave described by equation (27).

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