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CONTROL OF FLEXURAL WAVES ON A BEAM USING A TUNABLE VIBRATION NEUTRALISER

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This paper describes an analytical and experimental investigation into the use of a tunable vibration neutraliser to control the transmission of flexural propagating waves on an infinite Euler–Bernoulli beam. The paper investigates the way in which the physical properties of the neutraliser affect the attenuation of an incident flexural wave, and the frequency at which the neutraliser should be tuned to in order to achieve the maximum attenuation of this wave. Expressions are derived for the attenuation, the tuned frequency and the bandwidth of the device. A simple control parameter is also proposed that uses two signals measured with accelerometers on the beam at the base of the neutraliser. This facilitates a compact control device. The theoretical predictions are validated by a simple experiment.

1. INTRODUCTION

Beams are fundamental components of many engineering structures, and the control of vibration transmission along a beam is an important problem to be solved in the suppression of vibration and noise. Vibration propagates in several wave types, including flexural, compressional and torsional waves [1], but it is the control of flexural waves that is the subject of this paper. The suppression of flexural waves can be achieved by passive means using an impedance mismatch, such as a stiffness or a mass [1] or by active means, for example references [2–5]. In this paper a tunable vibration neutraliser is used. This is an adaptive-passive means of vibration control [6], where the stiffness of the device is adjusted so that its tuned frequency coincides with the frequency at which the propagating waves are forced along the beam. It is thus a tunable narrow-band control device. Such devices have been discussed by other workers, for example references [7–9], where they have been concerned with tuning the neutraliser so that it presents a maximum impedance at the frequency of interest. This is not the case when controlling the transmission of flexural waves on a beam because of the presence of evanescent waves at a discontinuity. If an undamped neutraliser is tuned so that its resonance frequency coincides with the forcing

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frequency then it would "pin" the beam at the neutraliser position at this frequency and consequently would only reduce the amplitude of a propagating flexural wave by 3 dB, as explained by Mead [10]. The tuning of a neutraliser to control the transmission of flexural waves on a beam has been discussed previously by Clark [11], and the purpose of this paper is to expand on this work. Parameters that could be measured and used in a control system to optimally tune the device to suppress an incident propagating wave are investigated.

An infinite beam rather than a finite beam has been chosen, because the object of the neutraliser is to suppress propagating waves rather than influence resonant behaviour. The term vibration neutraliser rather than vibration absorber is used in this paper to signify that the device is being used to control vibration at a troublesome excitation frequency rather than a resonance frequency [9]. The paper is set out as follows. Following the introduction, section 2 is devoted to the analytical study of the device, and simple expressions for the tuned frequency and the attenuation of the propagating waves are derived. In section 3 the tuning problem is discussed, and simulations are presented which show that a simple measurement of displacement and rotation of the beam at the position where the neutraliser is attached can be processed to give an indication of the tuning/mis-tuning of the device. This tuning parameter is novel, and although it is used in this work for an adaptive-passive device, it could also be used in a fully active system, thus avoiding the problems of measuring the amplitude of a propagating wave in the presence of a near-field wave as reported by Elliott and Billet [12]. In section 4 a simple experiment is reported that validates the theoretical predictions from sections 2 and 3. Finally, in section 5 some conclusions are drawn. There is also an Appendix in which some background theory is presented and the effects of mass and hysteretic spring-like discontinuities on an infinite beam are discussed.

2. STEADY-STATE BEHAVIOUR

It is demonstrated in the Appendix that an undamped spring fitted between a rigid foundation and a beam can completely suppress a flexural wave on an infinite beam at a single frequency. If the spring contains some damping then the flexural wave can no longer be completely suppressed, and the damping controls the degree of suppression. In practice it is rarely possible to fix the spring to a rigid foundation, and thus it has to react against an inertial mass. This combination of a spring and a mass constitutes a dynamic system (a neutraliser) which behaves in quite a different way than a spring alone. The aim of this section is to investigate the dynamic behaviour of the neutraliser fitted to the beam and to determine the condition when the neutraliser is "tuned", i.e., when a flexural wave propagating along the beam is suppressed.

Consider an infinite beam with a neutraliser fitted, which is shown as a discontinuity in Figure 1. The complex amplitude of the incident flexural wave is denoted A_i and the amplitudes of the transmitted and reflected waves A_t and A_r, respectively. K_d represents the dynamic stiffness of the neutraliser and K_b the

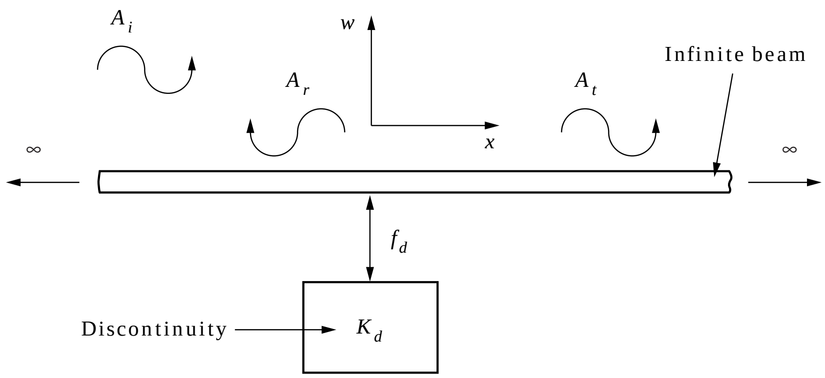

Figure 1. An infinite Euler–Bernoulli beam with a discontinuity.

dynamic stiffness of the beam. The dynamic stiffness of the neutraliser is given by [9]:

$$K_d = \frac{-\omega^2 m(1 + \mathrm{j}\eta)}{1 - \Omega^2 + \mathrm{j}\eta}, \tag{1}$$

where $\Omega = \omega/\omega_n$ and $\omega_n^2 = k/m$ where k and m are the the stiffness and mass of the neutraliser, respectively, and η is the loss factor of the neutraliser's hysteretically damped spring. The dynamic stiffness of a beam is given by [10]:

$$K_b = -2EIk_f^3(1 - \mathrm{j}), \tag{2}$$

where E and I are the Young's modulus and second moment of area of the beam, respectively. k_f is the flexural wave number of the beam and is given by $k_f = (\rho A/EI)^{1/4}\omega^{1/2} = 2\pi/\lambda$ where ρ, A and λ are the density, cross-sectional area and flexural wavelength of the beam, respectively. It is shown in the Appendix that the ratios of the transmitted and reflected waves to the incident wave are given by:

$$\frac{A_t}{A_i} = \frac{4 - \varepsilon - \mathrm{j}\eta\varepsilon}{4 - \varepsilon(1 - \eta) - \mathrm{j}\varepsilon(1 + \eta)}, \quad \frac{A_r}{A_i} = \frac{-\varepsilon(1 + \mathrm{j}\eta)}{\varepsilon(1 + \eta) + \mathrm{j}(4 - \varepsilon)}, \tag{3a, b}$$

where

$$\varepsilon = \frac{K_d}{EIk_f^3}. \tag{4}$$

The condition for tuning the *undamped* neutraliser can be determined by setting $\eta = 0$ in equation (1), combining this with equations (2), (3a) and (4) and setting the modulus of A_t/A_i to zero. One finds that the tuned frequency satisfies:

$$\Omega_t^2 = 1 + \frac{\mu_t}{4}, \tag{5}$$

where the subscript t denotes the tuned frequency and has been reported previously by Clark [11]. μ is a non-dimensional variable that relates the mass of the neutraliser m compared to the mass in one flexural wavelength of the beam and is given by:

$$\mu = \frac{m\omega^2}{EIk_f^3} = \frac{mk_f}{\rho A} = 2\pi \frac{m}{\rho A \lambda},$$

therefore

$$\mu = 2\pi \frac{\text{Mass of neutraliser}}{\text{Mass of one wavelength of the beam}}. \tag{6}$$

Equation (5) can also be written as:

$$\Omega_t^2 = 1 + \frac{\mu_n}{4} \Omega_t^{1/2}, \tag{7}$$

where μ_n is given by equation (6) evaluated at the natural frequency of the neutraliser. It can be seen that there is not a simple analytical solution for the tuned frequency, however, approximations can be made when μ_n is either small or large. When μ_n is small Ω_t is close to unity, and in this situation equation (7) becomes:

$$\Omega_t \approx \left(1 + \frac{\mu_n}{4}\right)^{1/2}. \tag{8}$$

When $\mu_n/4 \gg 1$ then equation (7) can be approximated to:

$$\Omega_t \approx \left(\frac{\mu_n}{4}\right)^{2/3}. \tag{9}$$

Equations (7–9) are plotted in Figure 2. To find the regions where the approximations are valid, one can combine equations (8) and (9) and solve iteratively for $\mu_n/4$. This results in a value of $\mu_n/4$ of approximately 2·63. The error in using the approximations is a maximum at this point and is about 14%. Thus, when $\mu_n/4 > 2{\cdot}63$ equation (9) can be used and when $\mu_n/4 < 2{\cdot}63$ then equation (8) can be used to give reasonable approximations to the tuned frequency. It can be seen that when $\mu_n/4$ is less than about 0·1 then $\omega_t \approx \omega_n$. For a neutraliser with small damping, i.e., $\mu \ll 1$, the frequency at which the neutraliser is tuned is the same as the undamped neutraliser and is thus given by equation (7).

From the above discussion it is evident that to suppress flexural waves on an infinite beam, a neutraliser is generally not tuned so that its natural frequency coincides with the forcing frequency. If an undamped neutraliser is tuned to meet this condition then it would "pin" the beam at this frequency and this would only result in a 3-dB reduction in the transmitted wave as reported by Mead [10]. The neutraliser has to present a spring-like impedance to the beam at the tuned frequency. Because a neutraliser has a mass-like impedance below its

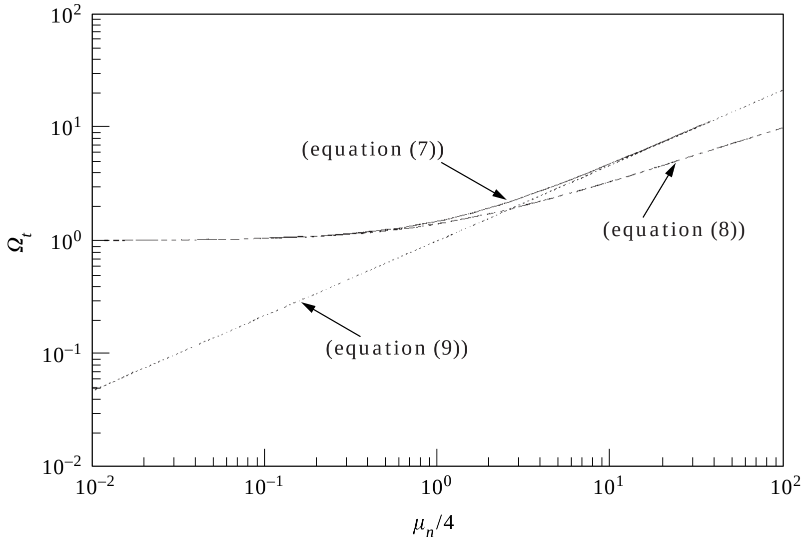

Figure 2. Graphical representation of the approximations for the tuned frequency.

natural frequency and a spring-like impedance above its natural frequency, the excitation frequency where control is achieved will always be above the natural frequency of the neutraliser, as can be seen from equation (5).

To determine the amplitude of the reflected and transmitted waves when the neutraliser is tuned, equations (3)–(5) can be combined. Provided that $\eta \ll 1$, the modulus of the ratio of the transmitted to the incident wave is given by:

$$\left|\frac{A_t}{A_i}\right|_{tuned} = \frac{1 + \frac{\mu_t}{4}}{1 + \frac{\mu_t}{4\eta}} \tag{10}$$

If $\mu_t/4 \ll 1$, which would generally be the case for a practical neutraliser, then the attenuation is only a function of the ratio μ_t/η, i.e:

$$\left|\frac{A_t}{A_i}\right|_{tuned} \approx \frac{1}{1 + \frac{\mu_t}{4\eta}}. \tag{11}$$

If the damping ratio is much smaller than the mass ratio then equation (11) reduces to:

$$\left|\frac{A_t}{A_i}\right|_{tuned} = \frac{4\eta}{\mu_t}. \tag{12}$$

Equations (11) and (12) are plotted as function of μ_t/η in Figure 3. If the mass of the neutraliser is very large compared to the size of the beam, then the situation is very much like a spring connected to a rigid foundation as discussed

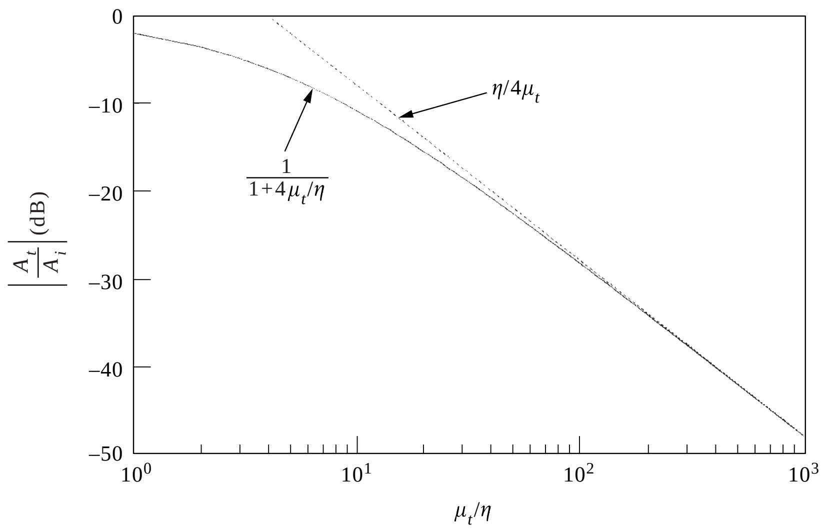

Figure 3. Amplitude of the transmitted flexural wave to the incident wave as a function of mass to damping ratio.

in the Appendix. If μ_t is $\gg 1$ and $\eta \ll 1$ then equation (10) reduces to:

$$\left|\frac{A_t}{A_i}\right|_{tuned} = \eta, \tag{13}$$

which is the same as that derived for a hysteretic spring in the Appendix. The ratio of the transmitted wave to the incident wave as a function of frequency are shown in Figure 4 for different values of η and $\mu_t = 0{\cdot}1$. Outside this narrow bandwidth, the neutraliser is ineffective and the reason for this can be explained by considering the dynamic stiffness of the neutraliser. Below the natural frequency, the dynamic stiffness of a neutraliser is mass-like and Figure A1(a) shows that at low frequencies this has negligible effect on flexural waves. At high frequencies, above the neutraliser's natural frequency, the dynamic stiffness is spring-like. At one particular frequency the stiffness has a profound effect on the transmitted flexural wave, and at frequencies greater than this the stiffness has little effect on the transmitted wave, as can be seen by examining Figure A1(b).

In accordance with the normal convention the bandwidth of the neutraliser is defined as the range of frequencies (normalised to the tuned frequency) over which the modulus of the transmitted wave is within 3 dB of the minimum. It is convenient to examine two different frequency regimes which roughly coincide with those shown in Figure 2: (1) the mass of the neutraliser is much less than the mass of one wavelength of the beam at the natural frequency of the neutraliser ($\mu_n/4 \ll 2{\cdot}63$); (2) the mass of the neutraliser is much greater than the mass of one wavelength of the beam at the natural frequency of the neutraliser ($\mu_n/4 \gg 2{\cdot}63$).

In case 1 the tuned frequency is given by equation (8) and the ratio of the amplitude of the transmitted wave to the incident wave at this frequency is given

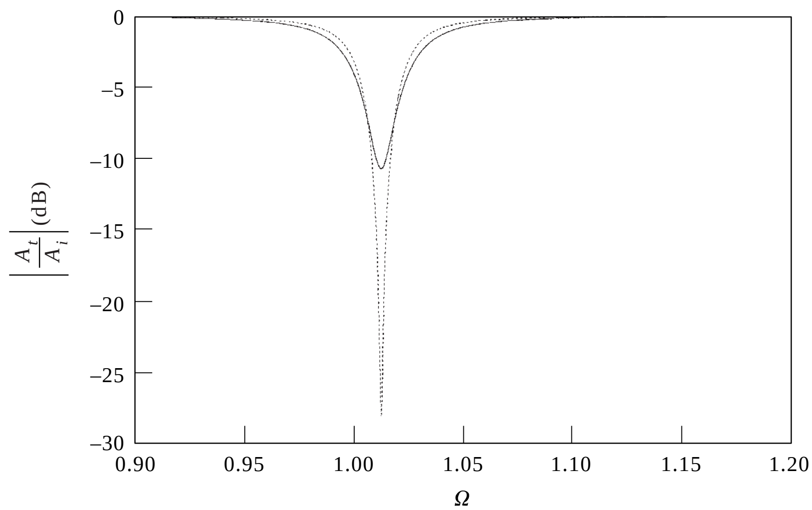

Figure 4. Ratio of the transmitted flexural wave to the incident wave as a function of non-dimensional frequency. —, $\mu_t = 0{\cdot}1$, $\eta = 0{\cdot}01$; ------, $\mu_t = 0{\cdot}1$, $\eta = 0{\cdot}001$.

by equation (11). Taking the square of the modulus of equation (3a) and setting this equal to twice the square of equation (11) and solving for Ω, one can find the upper and lower frequencies of the effective bandwidth of the neutraliser. Thus,

$$\left|\frac{4-\varepsilon-\mathrm{j}\eta\varepsilon}{4-\varepsilon(1-\eta)-\mathrm{j}\varepsilon(1+\eta)}\right|^2 = 2\left(\frac{1}{1+\dfrac{\mu_t}{4\eta}}\right)^2, \tag{14}$$

where ε is given by equation (4). By solving this equation, and assuming $\eta \ll 1$ and $\eta/\mu_t \ll 1$ it is found that the bandwidth is given by:

$$BW = \frac{\omega_2-\omega_1}{\omega_t} = \eta. \tag{15}$$

It can be seen that although the neutraliser is tuned to an off-resonant condition such that the impedance that is presented to the beam is predominantly spring-like, the damping in the neutraliser still controls the bandwidth as with a neutraliser tuned to resonance.

In the Appendix, the bandwidth for a hysteretically damped spring fitted between ground and the beam is calculated and found to be $4\eta/3$. Thus, a neutraliser that has a large mass compared to the wavelength of the beam should, in the limit, have a similar bandwidth to this. To check this, one can conduct an analysis similar to that above. For a large mass ratio the amplitude of the transmitted wave at the tuned frequency is governed by the loss factor only and is given by equation (13) rather than equation (11). Thus, in this case, equation (14) becomes:

$$\left|\frac{4-\varepsilon-\mathrm{j}\eta\varepsilon}{4-\varepsilon(1-\eta)-\mathrm{j}\varepsilon(1+\eta)}\right|^2=2\eta^2. \tag{16}$$

If this equation is solved to determine the upper and lower frequencies, then provided $\eta \ll 1$, one gets:

$$\Omega_{1,2}^2=1+\frac{\mu_t}{4}(1\pm\eta). \tag{17}$$

Now as μ_t is large one can make a similar approximation to that which was done in going from equation (8) to equation (9), and hence giving:

$$\left(\frac{\omega_2}{\omega_n}\right)^{3/2}-\left(\frac{\omega_1}{\omega_n}\right)^{3/2}=\frac{\mu_n}{2}\eta. \tag{18}$$

To obtain the bandwidth of the device one needs to normalise the difference in upper and lower frequencies by the tuned rather than the natural frequency of the neutraliser. This can be achieved by dividing equation (18) by equation (9) to give:

$$\left(\frac{\omega_2}{\omega_t}\right)^{3/2}-\left(\frac{\omega_1}{\omega_t}\right)^{3/2}=2\eta. \tag{19}$$

By using a similar technique to that shown in the Appendix it is found that the bandwidth of the device is given by:

$$BW=\frac{\omega_2-\omega_1}{\omega_t}=\frac{4}{3}\eta \tag{20}$$

as required. Thus, it can be seen that the bandwidth of a neutraliser used to attenuate flexural propagating waves on a beam, is controlled predominantly by the damping in the neutraliser. If the mass of the neutraliser is much smaller than the mass of one flexural wavelength of the beam at the tuned frequency of the neutraliser, then the bandwidth is equal to the neutraliser's loss factor. However, if the mass of the neutraliser is much larger than the mass of one wavelength of the beam at the tuned frequency of the neutraliser, then the bandwidth is increased by a third.

3. TUNING THE NEUTRALISER

It was shown in the previous section that a neutraliser used to control flexural waves on a beam is a narrow-band device. It would be useful to be able to adjust the tuned frequency of the neutraliser if the disturbance changes its excitation frequency with time. Previous work by Brennan *et al.* [13] on tunable neutralisers that were designed to present a maximum impedance to the host structure at the operating frequency, showed that a relatively simple algorithm could be employed to tune the device. The control algorithm simply forced the base of the neutraliser and the neutraliser mass to move in quadrature. In the application discussed in this paper, however, the phase angle required between the base of the neutraliser and the neutraliser mass changes with the size of the neutraliser and frequency of operation. Thus, the simple algorithm described by Brennan *et al.* cannot be used. In this section ways of tuning a neutraliser to control the transmission of flexural waves on a beam are explored and one method proposed that will be discussed in some detail.

Perhaps the simplest way to control flexural waves on an infinite beam is to place a sensor which measures lateral motion of the beam some distance downstream of the neutraliser and to tune the neutraliser until the output from this sensor is a minimum. This method relies on the sensor output not being contaminated by the near-field wave being generated by the neutraliser [12]. Because the amplitude of a near-field wave is negligibly small about one wavelength from its source this would mean that the sensor would have to be placed quite some distance from the neutraliser to control low frequency waves. One of the advantages of using a tunable neutraliser is that it is possible to use a compact control system local to the neutraliser. Because a control algorithm that uses phase is very simple, the phase relationship between the measurable quantities local to the neutraliser is investigated to see whether there is a relationship that is invariant of the neutraliser mass.

One possible tuning parameter is the displacement of the beam where the neutraliser is fitted, referred to the slope of the beam at this position. Using a finite difference technique would only require two transducers and would be relatively compact. The displacement at the position on the beam where the neutraliser is attached is given by:

$$\frac{w(0)}{A_i} = \frac{1}{1 + \dfrac{K_d}{K_b}}, \tag{21}$$

where the dynamic stiffnes of the neutraliser K_d, and beam K_b are given by equations (1) and (2), respectively. Provided that the neutraliser is compact so that it has negligible rotational inertia, the slope at this position is unaffected by the neutraliser and is thus given by:

$$\frac{w'(0)}{A_i} = -\mathrm{j}k_f. \tag{22}$$

Dividing equation (21) by equation (22) gives the ratio of the displacement to the rotation of the beam at the point where the neutraliser is attached. It is, in fact, better to write this in terms of the product of the displacement and the flexural wavenumber divided by the rotation, i.e.

$$\frac{k_f w(0)}{w'(0)} = \frac{-\mathrm{j}}{1 + \dfrac{K_d}{K_b}}. \tag{23}$$

If the damping in the neutraliser is set to zero and the neutraliser tuned according to equation (5), equation (23) becomes:

$$\left.\frac{k_f w(0)}{w'(0)}\right|_{tuned} = -1, \tag{24}$$

which is entirely real, which means there is a phase angle of 180° between the displacement and rotation at the root of the neutraliser when it is tuned to the forcing frequency. Thus, it appears that equation (24) could be used as a tuning criterion. To visualise this the real and imaginary parts of equation (23) are plotted as a function of non-dimensional frequency in Figure 5(a) for a value of μ_n of 0·1. It can be seen from Figure 5(a) that although the real part of the ratio of the displacement to the rotation could be used there could be some problems.

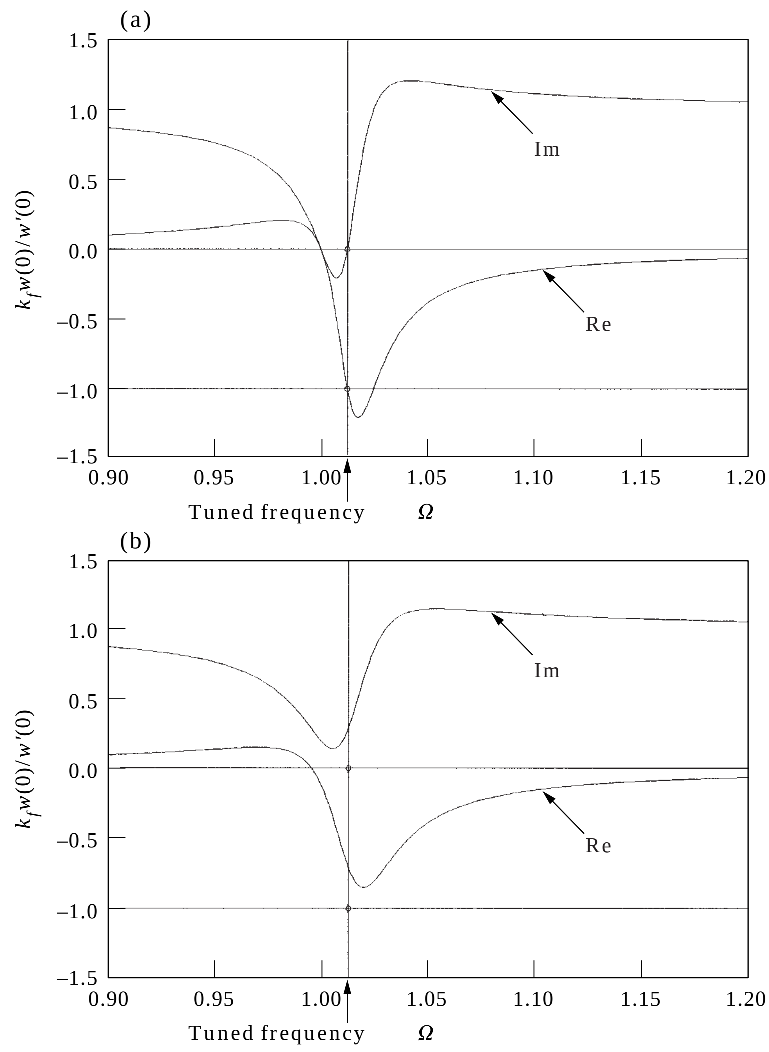

Figure 5. Real and imaginary parts of equation (23) plotted as a function of non-dimensional frequency; $\mu_n = 0{\cdot}1$. (a) An undamped neutraliser, $\eta = 0$. (b) With damping in the neutraliser, $\eta = 0{\cdot}01$.

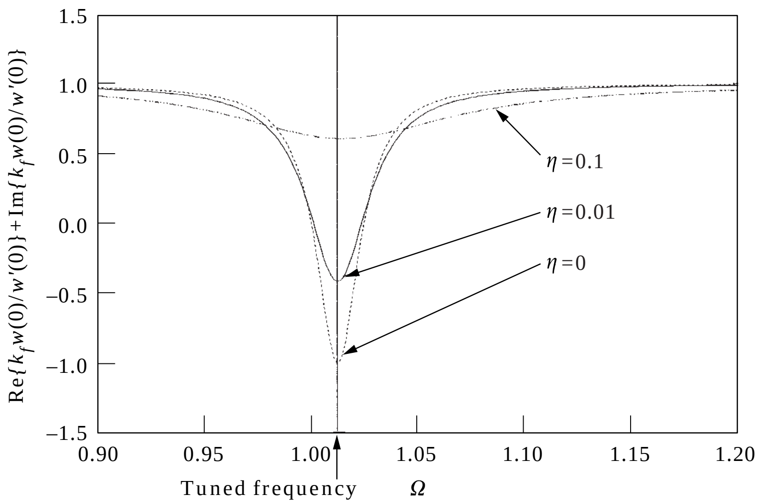

Figure 6. Sum of the real and imaginary parts of equation (23) plotted as a function of non-dimensional frequency for various values of loss factor; $\mu_n = 0{\cdot}1$.

The real part of this function has a value of -1 when the neutraliser is tuned, but it also has a value of -1 when $\Omega^2 = 1 + \mu/2$, which means that there is potential for the neutraliser to be mis-tuned with this control strategy. It should be noted that the imaginary part of the function has similar characteristics to the real part but it is zero when $\Omega^2 = 1$ and when $\Omega^2 = 1 + \mu/4$.

To investigate the effects of damping in the neutraliser the real and imaginary parts of equation (23) are plotted in Figure 5(b) for a loss factor of $\eta = 0{\cdot}01$. Comparing Figures 5(a) and (b), it can be seen that damping has a significant effect on both the real and imaginary parts of equation (23). With a damped neutraliser the real part does not cross the negative unity line and the imaginary part no longer crosses the zero line. To overcome the problems of ambiguity and to form a ratio (which one wishes to use as a tuning parameter) which is insensitive to neutraliser damping, the real and imaginary parts of $k_f w(0)/w'(0)$ can be combined. This forms a function whose derivative with respect to frequency is similar to the derivative of the square of the modulus of the wave transmission ratio, which means that the function is a minimum when the device is tuned. The sum of the real and imaginary parts (which is defined as the objective function) are plotted for various damping values in Figure 6. When there is no damping in the neutraliser, this function has a minimum value of -1 when the neutraliser is tuned to completely suppress the incident propagating wave. Even when there is damping in the neutraliser, the objective function has a minimum when the neutraliser is tuned, provided that $\eta \ll 1$. Thus, one has a parameter that has a minimum when the neutraliser is tuned, and it is insensitive to the mass and damping of the neutraliser.

To initially tune the neutraliser, an open-loop control algorithm can be used, as in the case of a conventional tuned neutraliser [13]. This open-loop tuning could, for example, tune the neutraliser to its natural frequency and then a fine

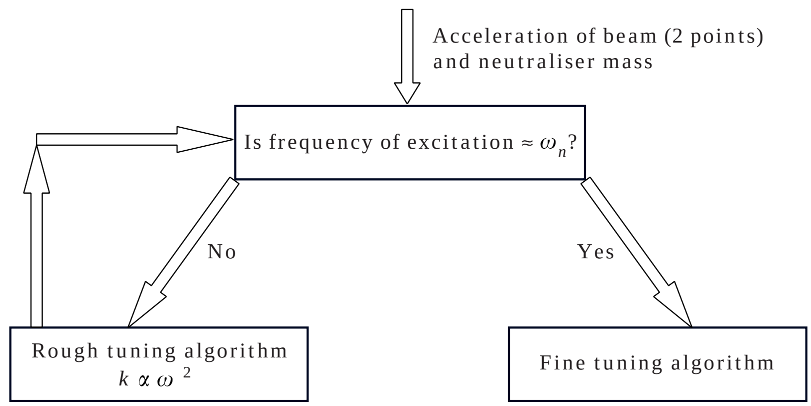

Figure 7. Outline of control algorithm to tune a neutraliser fitted to a beam to control propagating flexural waves.

closed-loop control algorithm could be used based on the objective function discussed above. A gradient descent algorithm could be used for fine tuning such as one of those described by Kuo and Morgan [14]. The proposed algorithm is drawn as a block diagram in Figure 7. The rough tuning algorithm could be based on a look-up table where, for a given operating frequency, the required stiffness of the neutraliser would be known, having been previously measured off-line. The rotation of the beam could be measured using two accelerometers and employing finite-difference techniques.

4. EXPERIMENTAL WORK

To validate the approximate formulae developed in sections 1 and 2 that describe the important properties of the device, some experimental work was conducted. A beam-like vibration neutraliser made from an aluminium strip with small brass masses attached, and with fixed characteristics, was designed and fitted to a 6 m × 6 mm × 50 mm steel beam that had sandboxes fitted to each end to act as anechoic terminations. The neutraliser was designed so that the mass ratio at the tuned frequency μ_t would be about 0·1. The experimental set-up is shown in Figure 8. Accelerometers were placed on the brass masses on the neutraliser and at positions 1, 2 and 3 shown in Figure 8. Accelerometers 1 and 2 were spaced 100 mm apart and were positioned equidistant either side of the neutraliser. An electrodynamic shaker, positioned 700 mm from one of the sand boxes, was used to generate a dynamic force. The beam was excited with band-limited random noise over a frequency range 100 to 500 Hz, and the frequency response functions between the current applied to the shaker (which is proportional to the force applied) and the accelerometers were measured.

The measured data were post-processed to give the required results. All the results are presented in the frequency domain. The frequency response between the applied force and accelerometers 1 and 3 with and without the neutraliser fitted are presented in Figures 9(a) and (b), respectively. Each of the results is

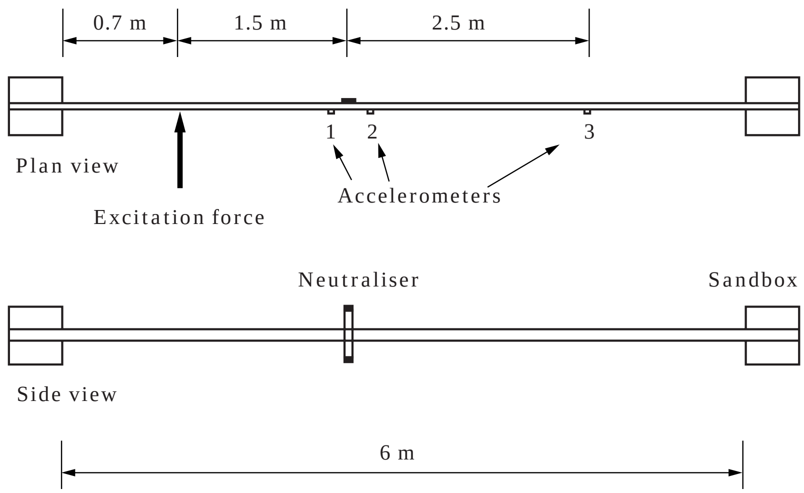

Figure 8. Experimental set-up to test the operation and tuning criterion for a tunable vibration neutraliser fitted to a beam.

normalised to the minimum acceleration measured at positions 1 and 3, respectively, so that the attenuation at this position and frequency can be easily distinguished.

It can be seen from Figures 9(a) and (b) that the minima in acceleration at points 1 and 3 differ in frequency. The frequency at which the acceleration of point 1 is a minimum occurs at the natural frequency of the neutraliser at about 189 Hz. A reduction in the motion of the beam at this frequency of about 32 dB has been achieved by the neutraliser. At this frequency the neutraliser is attempting to pin the beam and not block the propagating flexural waves. Examination of Figure 9(b) shows that the minimum in the acceleration at point 3 occurs at about 191 Hz. This is the tuned frequency that was discussed in section 2. The reduction in the amplitude of the propagating flexural wave at this frequency is about 16 dB.

One can compare the attenuation of the flexural propagating waves with that predicted in section 2 using equation (11) and the approximation given by equation (12). It should be noted, that in order to predict the attenuation, a measure of the damping in the neutraliser is required. This was achieved by measuring the transmissibility across the neutraliser using accelerometer 1 and an accelerometer fitted to one of the neutraliser masses, and noting that the amplitude of transmissibility at resonance is approximately equal to the loss factor η. This turned out to be 0·004 and the mass ratio between the neutraliser and the mass of one wavelength of the beam at 190·25 Hz (given by equation (6)) is 0·083. The ratio μ_r/η is therefore 20·75. Referring to Figure 3 it can be seen that for the particular arrangement used in the experimental work, the approximation given in equation (12) is quite accurate. In fact the attenuation achieved in the experiment was 16·56 dB and the attenuation predicted by equations (11) and (12) were 15·83 and 14·30 dB, respectively. It should be noted that the bandwidth of the neutraliser was only about 0·75 Hz, and that if this was required to be increased, then in order to maintain the same attenuation in the flexural waves, the mass of the neutraliser would have to be increased proportionately.

To validate the tuning algorithm described in section 3, the following function was formed using experimental data:

$$\alpha = \frac{\ddot{w}_1}{\ddot{w}_2 - \ddot{w}_1}, \tag{25}$$

where the subscripts refer to the accelerometer positions shown in Figure 8, w denotes lateral displacement of the beam and a dot denotes differentiation with respect to time. The real and imaginary parts of equation (25) were summed to

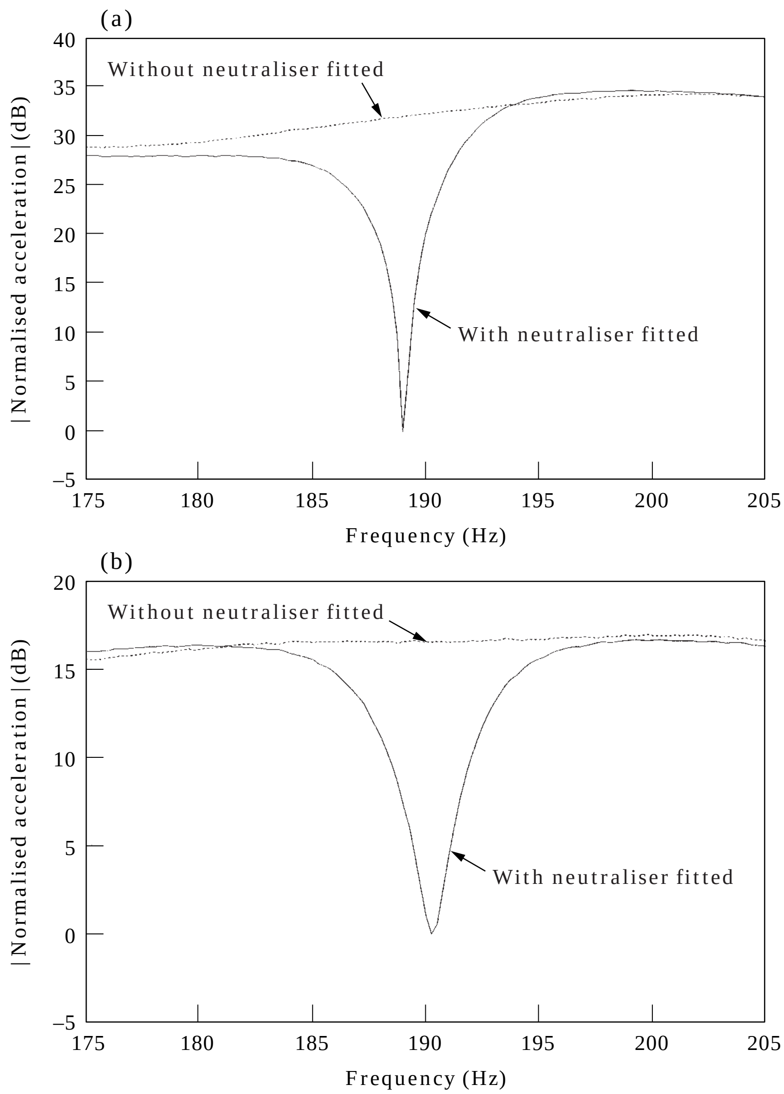

Figure 9. Experimental results—acceleration measurements on the beam. (a) Normalised acceleration at position 1 on the beam. (b) Normalised acceleration at position 3 on the beam.

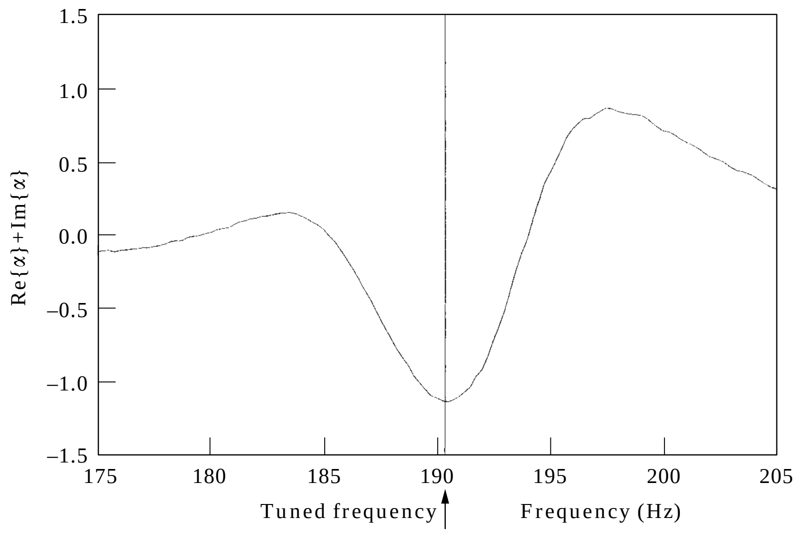

Figure 10. Experimental results—plot of the measured objective function versus frequency.

give an objective function similar to that described in section 3. The result is plotted in Figure 10 where it can be seen that there is indeed a minimum in this function at the frequency when the neutraliser optimally attenuates the incident propagating wave. This figure should be compared with Figure 6.

5. CONCLUSIONS

This paper has described an analytical and experimental investigation into the use of a tunable vibration neutraliser to control flexural waves on a beam. Expressions have been derived which describe the frequency at which the neutraliser should be tuned in order to suppress an incident propagating flexural wave. Although there is not a simple closed-form solution for the tuned frequency, relatively simple expressions do exist for the tuned frequency provided that the mass of the neutraliser is either small or large compared to the mass of one wavelength of the beam at the tuned frequency. The attenuation of the incident propagating wave has been shown to be proportional to the ratio of the neutraliser mass to the mass of one wavelength of the beam at the tuned frequency, and inversely proportional to damping in the neutraliser. A simple expression relating these quantities has been derived. As well as influencing the attenuation of the incident wave on the tuned frequency, the damping also controls the bandwidth. It has been shown that the bandwidth of the neutraliser with a small mass ratio is simply equal to the loss factor, but when the mass ratio is large then the bandwidth increases by one-third. Finally a simple control algorithm has been proposed that uses two signals from accelerometers positioned on the beam spaced a small distance apart, either side of the neutraliser. The control algorithm is a two-tier control system consisting of an open-loop part which tunes the neutraliser so that its natural frequency is coincident with the forcing frequency, and a closed-loop control algorithm that uses the signals from the accelerometers positioned on the beam at the base of the neutraliser. The closed-loop control algorithm fine-tunes the neutraliser to suppress the incident propagating wave. The main advantage of this system is that it is potentially simple and compact.

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APPENDIX: THE CONTROL OF FLEXURAL WAVES ON AN INFINITE BEAM USING A DISCONTINUITY

In this Appendix the effects of mass and spring-like discontinuities on flexural wave propagation on an infinite Euler–Bernoulli beam are discussed. Consider an infinite beam with a discontinuity of dynamic stiffness K_d, as shown in Figure 1. The ratios of the transmitted and reflected waves to the incident wave are given by [10]:

$$\frac{A_t}{A_i} = \frac{4-\varepsilon}{4-\varepsilon-\mathrm{j}\varepsilon}, \quad \frac{A_r}{A_i} = \frac{-\varepsilon}{\varepsilon+\mathrm{j}(4-\varepsilon)}, \tag{A1a, b}$$

where

$$\varepsilon = \frac{K_d}{EIk_f^3}. \tag{A2}$$

If the dynamic stiffnesses of simple jumped parameter elements are substituted into equation (A2), one can determine the relationship between ε and frequency for the various elements: for a mass $\varepsilon \propto -\omega^{1/2}$, for a viscous damper $\varepsilon \propto \mathrm{j}\,\omega^{-1/2}$ and for a stiffness $\varepsilon \propto \omega^{-3/2}$. Examining equation (A1a) one can see that for the transmitted wave to be set to zero requires $\varepsilon = 4$, which means that ε must be real and positive. It is clear that only a stiffness element can meet this criterion.

The variable ε is a non-dimensional quantity, and for a *mass-like* discontinuity relates the mass m of the discontinuity to the mass in one wavelength of the beam. Substituting for $K_d = -\omega^2 m$ into equation (A2) and setting $\mu = -\varepsilon$ gives:

$$\mu = 2\pi\frac{m}{\rho A\lambda} = \frac{2\pi m}{\text{Mass of one wavelength of the beam}}. \tag{A3}$$

For a mass-like discontinuity, equations (A1a, b) can therefore be written as:

$$\frac{A_t}{A_i} = \frac{4+\mu}{4+\mu+\mathrm{j}\mu}, \quad \frac{A_r}{A_i} = \frac{\mu}{-\mu+\mathrm{j}(4+\mu)}. \tag{A4a, b}$$

The moduli of these equations are plotted in Figure A1(a). The solid lines give the acutal values computed using equations (A4a, b) and the dotted lines give the low and high frequency asymptotic values. It can be seen that the maximum reduction of an incident flexural wave using a mass discontinuity is only 3 dB, as reported by Mead [10]. The low frequency asymptote (small μ) for the reflection ratio is simply given by $-\mathrm{j}\mu/4$. By setting the modulus of this to $1/\sqrt{2}$, one finds that the point at which the asymptotes cross is given by $\mu = 2\sqrt{2}$.

As discussed above it is possible to completely suppress an incident flexural wave using a spring-like discontinuity. In practice, springs have some internal damping so it is of interest to look at the effects of a discontinuity consisting of a hysteretically damped spring of spring constant k, whose dynamic stiffness is given by $K_d = k(1 + \mathrm{j}\eta)$. In this case the non-dimensional parameter ε consists of real and imaginary parts and can be written as $\varepsilon(1 + \mathrm{j}\eta)$. The transmitted and reflected wave ratios are thus given by:

$$\frac{A_t}{A_i} = \frac{4-\varepsilon-\mathrm{j}\eta\varepsilon}{4-\varepsilon(1-\eta)-\mathrm{j}\varepsilon(1-\eta)}, \quad \frac{A_r}{A_i} = \frac{-\varepsilon(1+\mathrm{j}\eta)}{\varepsilon(1+\eta)+\mathrm{j}(4-\varepsilon)}. \tag{A5a, b}$$

For small damping, when $\eta \ll 1$, the transmitted wave is a minimum when $\varepsilon \approx 4$. In this case the transmitted and reflected wave ratios are given by:

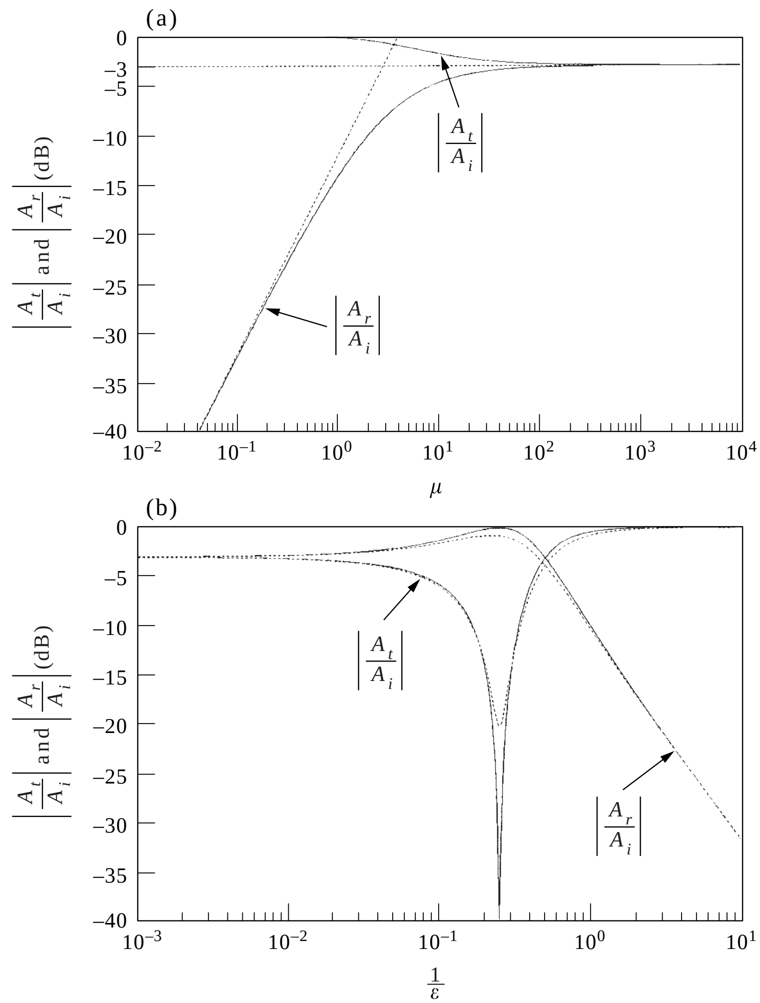

Figure A1. Effects of discontinuities on a propagating flexural wave on an infinite Euler–Bernoulli beam. (a) Transmission and reflection ratios of flexural waves on a beam with a mass discontinuity. (b) Transmission and reflection ratios of flexural waves on a beam with a hysteretically damped spring discontinuity. - - - - - -, $\eta = 0{\cdot}1$; —, $\eta = 0{\cdot}01$.

$$\frac{A_t}{A_i} \approx \frac{\eta}{1+\eta} \approx \eta, \quad \frac{A_r}{A_i} \approx \frac{-1}{1+\eta}. \tag{A6a, b}$$

The frequency at which the transmitted wave is a minimum can be determined setting $\varepsilon = 4$, in equation (A2) to give:

$$\omega_t = \sqrt{\frac{k}{\frac{2}{\pi}\rho A \lambda_t}}, \tag{A7}$$

where the subscript t denotes the tuned frequency, which is the frequency at which the amplitude of the transmitted wave is a minimum. Equation (A7) can be interpreted as a natural frequency where the equivalent mass of $2/\pi$ of a flexural wavelength of the beam is resonating on the stiffness of the discontinuity.

To find the bandwidth of the device (defined as the normalised frequency range over which the modulus of the transmitted wave is within 3 dB of the minimum), equate the square of the modulus of equation (A5a) with twice the square of equation (A6a). Solving this for ε, and ignoring higher orders of η gives:

$$\varepsilon_{1,2} = 4 \pm 4\eta. \tag{A8}$$

Subtracting ε_1 from ε_2 and noting that ε is related to frequency by $\varepsilon = 4(\omega_t/\omega)^{3/2}$ gives:

$$\left(\frac{\omega_t}{\omega_1}\right)^{3/2} - \left(\frac{\omega_t}{\omega_2}\right)^{3/2} = 2\eta. \tag{A9}$$

Using the approximation $a^x = 1 + x\ln(a)$ and noting that $a^x - b^x = x(\ln(a) - \ln(b)) = x\ln(a/b)$, equation (A9) can be written as:

$$\frac{\omega_2}{\omega_1} = e^{4/3\eta}. \tag{A10}$$

Subtracting 1 from both sides of equation (A10), and assuming that damping is small such that $\eta \ll 1$ and $\omega_1 \approx \omega_t$, then gives:

$$\frac{\Delta\omega}{\omega_t} = \frac{4}{3}\eta, \tag{A11}$$

which is the bandwidth of a hysteretically damped spring discontinuity on a beam.

The reflection and transmission coefficients for a hysteretically damped spring discontinuity are plotted as a function of $1/\varepsilon$ in Figure A1(b). ($1/\varepsilon$ is used as the independent variable rather than ε so that the frequency increases from left to right on the graph). It can be seen that the reduction of an incident flexural wave using a hysteretically damped spring discontinuity is 3 dB at low frequencies (ε large). The attenuation reaches a maximum when $\varepsilon = 4$, as discussed above, when the attenuation is then given by approximately $1/\eta$. This dip in the transmission has a bandwidth of $4\eta/3$. At high frequencies the reflection ratio asymptotically approaches $j\varepsilon/4$. It should be noted that the damping has only a marginal effect on the transmitted and reflected waves at frequencies away from the tuned frequency defined in equation (A7).