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MODELLING OF NON-LINEAR OSCILLATIONS OF ELASTIC STRUCTURES IN HEAVY FLUID LOADING CONDITIONS

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In a problem of structural acoustics with non-linear formulation of structural dynamics, a linearized compatibility condition at the fluid–structure interface is used along with the linear wave equation for the acoustic medium [1–7]. This approach is referred to as a light acoustic loading limit. Another aproach is to formulate a compatibility condition at the moving boundary and solve a non-linear wave equation in a volume. As it is shown in references [3–5], a solution for this problem predicts shock wave formation at a certain distance from a vibrating surface. In the present paper, one more model of interaction between an acoustic medium and a non-linear structure is suggested for heavy fluid loading conditions. In this model, propagation of acoustic waves is described by a linear wave equation, but the continuity condition is formulated at the moving boundary and the contact acoustic pressure acting at the vibrating non-linear structure is calculated by the Bernoulli integral with a quadratic velocity term retained. Two model problems of coupled structural acoustics are considered—oscillations of a fluid-loaded piston and oscillations of an infinitely long periodically supported elastic plate. A method of multiple scales is used for analysis of the local non-linear dynamics of the model systems, whilst matched asymptotic expansions are used to model the fluid's motion. Several specific effects of structural vibrations generated by the non-linearity of fluid–structure interaction, rather than by structural non-linearity are demonstrated.

1. INTRODUCTION

Various aspects of non-linear interaction between elastic structures and acoustic medium have been explored by many authors; see the comprehensive list of references in references [1–7]. In particular, special interest has been focused on perturbations in a linear acoustic field generated by structural non-linearities. To use a theory of linear acoustics, structural–acoustic coupling conditions have also been taken in a linear formulation as a light fluid loading limit. However, in

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the case of heavy fluid loading, non-linearities in the coupling formulation should be taken into account and influence both the acoustic field and the structural dynamics. Then the whole formulation of structural–acoustic coupling should be re-examined. The aims of this paper are to suggest a formulation of a problem of structural acoustics for non-linear heavy fluid loading of a non-linear elastic structure and to exemplify the theory by analytical solutions for two elementary model problems.

1.1. LINEAR FORMULATION OF THE PROBLEM

For clarity we begin with the statement of a linear problem of structural–acoustic coupling. This formulation has been proved to be able to predict reliably dynamics of fluid loaded elastic structures when the excitation frequency does not coincide with their eigenfrequencies. It contains two conditions posed at the fluid–structure interface S. The first one constitutes continuity of velocities of fluid and solid particles at the fluid–structure interface

$$\partial w_v / \partial t = v_v. \tag{1}$$

Here w_v is a displacement component normal to surface S, and v_v is the fluid particle velocity in the same direction. The positive normal is directed out of acoustic medium, positive directions of w_v and v_v coincide with each other. As a coupled formulation is explored, velocities of the structure are not prescribed. They are governed by a driving force acting at the structure (which is given) and by fluid loading (a contact acoustic pressure which should be found). The second condition at the fluid–structure interface formulates equations of structural dynamics containing a fluid-loading term (contact pressure)

$$\mathbf{Lw} = \mathbf{q} + p\nu, \tag{2}$$

where $\mathbf{L}$ is a matrix of the linear differential operator acting at a vector of structural displacements, $\mathbf{q}$ is a vector of driving forces, p is a contact acoustic pressure, and ν is a normal to the fluid–structure interface, outward to the acoustic medium.

Fields of fluid and solid variables are matched at the immobile interface (a surface of the undeformed structure) in linear problems of fluid–structure interaction. A field of velocities of fluid is assumed to be irrotational and velocity potential φ is induced as

$$\mathbf{v} = \mathbf{grad}\ \varphi. \tag{3}$$

Then the contact pressure becomes

$$p = -\rho_0 \partial\varphi / \partial t. \tag{4}$$

ρ_0 is the density of the undisturbed acoustic medium.

As equations (3 and 4) are used along with linear wave equation for dynamics of fluid one obtains the purely linear formulation of coupling. This formulation fits well with a linear theory of structural dynamics (2) and constitutes a theory of linear structural acoustics.

1.2. NON-LINEAR FORMULATION

Apparently, in the non-linear dynamics of fluid-loaded structures, it is necessary to consistently revise formulations of (i) structural response, (ii) fluid response and (iii) compatibility conditions. In this section, we briefly discuss the roles of these non-linearities in the formulation of a coupled problem.

It is well-known that essential non-linear phenomena of fluid–structure interaction are generated by the structure [1–7]. Non-linearities in structural response are typically classified into two groups—non-linearities in the constitutive law for a material of the structure and non-linear geometry. We concentrate our attention on the non-linear geometry (normally resulting in cubic non-linearity of governing equations for structural dynamics) and neglect other sources of structural non-linearities. As has been shown in references [1, 6], structural non-linearity controls the stationary dynamics of structures in conditions of light fluid loading. In effect, the assumption of light fluid loading implies that a zero-order problem is posed as linear vibrations of a structure which does not experience any acoustical "feedback". Then structural non-linearities and fluid–structure interaction effects (both, the linear and the non-linear ones) show up in problems of higher orders. Specifically, if the problem of the first order is considered, then this assumption in fact couples an enhanced theory of non-linear structural dynamics with a simplified linear theory of fluid–structure (structural–acoustic) interaction. This approach has been developed in references [6, 7].

Apparently, when heavy fluid loading is considered, strucural and acoustical problems become coupled in equal shares. Thus, a problem of each order should be treated as essentially coupled and non-linearities in fluid response and compatibility conditions should be brought to light. This simple observation makes it necessary to address the theory of non-linear acoustics [3–5, 8]. It is well known that as soon as the non-linear wave equation is used, then acoustic waves distort and eventually become shock waves at a certain distance from a source. Thus, a problem of non-linear vibrations of a heavy fluid-loaded structure cannot be posed in the framework of standard structural acoustics. Actually, these considerations have been treated in references [6, 7] as the motivation to use the light fluid loading limit in the analysis of wave propagation in an acoustic medium in contact with a structure which exhibits large amplitude oscillations.

The third source of non-linearity is associated with the formulation of compatibility conditions at the fluid–structure moving interface. In the coupled formulation of a problem, its position at any instant of time is not known and should be found simulteneously with a contact pressure. In fact, this non-linearity is closely linked with the non-linearity of the structural geometry (large deformations).

In the following section, some physical motivations are discussed for adopting a model of generation of linear acoustic waves by an oscillating surface when a contact acoustic pressure is calculated by a non-linear Bernoulli integral at the moving boundary.

1.3. MODEL OF GENERATION OF ACOUSTIC WAVES BY AN OSCILLATING SURFACE

In classic works [9–11] it is anticipated that the whole volume of the acoustic medium may be divided into two regions. The fluid motions in the vicinity of a vibrating body are predominantly similar to a flow of incompressible fluid. Apparently, the thickness of such a layer is very small and it is controlled by the amplitude of the motions of a vibrating structure. In the outer region the predominant type of fluid motion is propagation of acoustic waves. Hence, in the inner region energy transportation is associated with kinetic energy of flowing (in effect, incompressible) fluid, whereas in the outer region energy transportation is associated with propagation of acoustic waves. We should note that a similar concept has recently been explored to analyze sound generation by vorticity [12, 13].

To clarify the matter reference [11] is followed and the process of *radiation* of *small amplitude sound waves* by an oscillating body is considered. Then (i) local variability of density and sound speed is neglected and (ii) it is assumed that sufficiently far from a surface the fluid motion is acoustical—it radiates a sound wave and velocities are governed by the linear wave equation.

Two inequalities are now assumed in the whole volume occupied by a fluid: relative changes in the fluid's density are very small so that for the surplus pressure p one has

$$p \ll \rho_0 c_0^2, \tag{5}$$

and the velocity of the fluid particle, is much less than c_0, the speed of a sound wave,

$$v \ll c_0. \tag{6}$$

Let the velocity of an oscillating structure U also be small as compared with the speed of sound. As $U \sim A\omega$, (A, ω are the magnitude and the frequency of vibrations), then

$$\varepsilon = A\omega/c_0 \ll 1. \tag{7}$$

This small parameter in the literature is called the acoustic Mach number. If a length of the acoustic wave λ, radiated by the structure obeys an inequality $\lambda = c_0/\omega \gg l$ (l being the characteristic size of the structure), then in the vicinity of a vibrating structure it is possible to neglect the term $(1/c_0^2)\partial^2\varphi/\partial t^2$ in the wave equation for the velocity potential,

$$\Delta\varphi - (1/c_0^2)\partial^2\varphi/\partial t^2 = 0, \tag{8}$$

because it is of order φ/λ^2, while the first term is of order φ/l^2. Thus, near the vibrating structure the fluid appears incompressible [9–11] and the pressure p is determined by the Bernoulli integral

$$p/\rho_0 = -\partial\varphi/\partial t - \tfrac{1}{2}(\nabla\varphi)^2. \tag{9}$$

On the other hand, sufficiently far from a vibrating structure wave motion is governed by standard linear acoustics.

In the linear formulation of the structural–acoustic coupling problem amplitudes of oscillations of a structure are treated as negligibly small by definition. This automatically results in the elimination of an ‘‘inner’’ zone—its length tends to zero. In non-linear structural acoustics, compatibility conditions are posed at the moving boundary and it is essential to distinguish between these two zones. In fact, a contact pressure which acts at the vibrating structure and controls amplitude of vibrations is developed in the ‘‘inner’’ zone. The above considerations become of the most importance from the ‘‘structural’’ viewpoint, since structural dynamics becomes very sensitive to non-linearities at resonant excitation conditions.

A use of formula (9) in acoustics has been discussed in references [14–16]. In reference [14] it was shown for an uncoupled problem that temporal fluctuations of velocity of the flow result in fluctuations of presure in equation (9) not only because of explicit time-dependence of the first term, but also due to time-dependence of the second term. In reference [15] the uncoupled problem of pulsation of an axisymmetric cavity has been considered and it has been shown that propagation of waves should be described by a linear wave equation, while the pressure should be found in the form of a Bernoulli integral (9). Similar results have been obtained in reference [16]. The above examples do not deal with phenomena of non-linear structural dynamics, but they demonstrate regimes described by the linear wave equation and the non-linear formulation (9) for a pressure.

2. NON-LINEAR DYNAMICS OF A FLUID LOADED PISTON

In this section a model 1-D problem of non-linear dynamics of a structure in heavy fluid loading conditions is considered. The aim of this analysis is two-fold. First, since it presents serious mathematical difficulties to carry out asymptotic matching of solutions of fluid dynamics problem in the ‘‘inner’’ and the ‘‘outer’’ zones for a general 3-D case, it is relevant to exemplify this matching by considering a relatively simple 1-D case. This outlines the contents of section 2.1. Second, the simple model problem of non-linear fluid–structure interaction solved in section 2.2 is of some practical value for analysis of motion of elastic membranes designed to control pressure jumps in pipe lines.

2.1. ANALYSIS OF HYDRODYNAMICS FOR 1-D PROBLEM

Non-linear formulation of the problem in physical variables (a pressure p and a velocity v) is

$$\frac{1}{\rho_0 c_0^2}\left(\frac{\partial p}{\partial t}+\frac{\partial p}{\partial x}v\right)+\frac{\partial v}{\partial x}=0, \quad \frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}v+\frac{1}{\rho_0}\frac{\partial p}{\partial x}=0. \qquad \text{(10a, b)}$$

The boundary condition is posed at the moving surface $x = w(t)$, x is the Eulerian co-ordinate

$$v|_{x=w(t)} = U = \mathrm{d}w/\mathrm{d}t.$$

Our aim is to construct an asymptotic solution for equations (10) in the "inner" zone and match it with the solution for a linear problem in the "outer" zone.

2.1.1. *Formulation in the "inner" zone*

Non-dimensional variables are introduced as $\bar{t} = \omega t$, $\bar{v} = v/(A\omega)$, $\bar{w} = w/A$, $\overline{U}/(A\omega)$, $\bar{p} = p/(\rho_0 c_0^2 \varepsilon)$. The last formula provides the same order for non-dimensional functions $\bar{v}$ and $\bar{p}$. This is necessary for further asymptotic analysis.

In the "inner" zone spatial variability is scaled by the magnitude of the vibration amplitude A, and it is meaningful to induce non-dimensional spatial co-ordinate $\bar{z}$ as $\bar{z} = (x - w)/A$. In the new variables the equations of hydrodynamics are (upper symbols are omitted)

$$\varepsilon\left(\frac{\partial p}{\partial t} - (v - U)\frac{\partial p}{\partial z}\right) + \frac{\partial v}{\partial z} = 0, \quad \varepsilon\left(\frac{\partial v}{\partial t} - (v - U)\frac{\partial v}{\partial z}\right) + \frac{\partial p}{\partial z} = 0, \quad v|_{z=0} = U(t). \tag{11}$$

A solution of equations (11) is sought in the form of asymptotic expansions in the small parameter ε. Then the following zero order problem is formulated:

$$\partial p_0/\partial z = 0, \quad \partial v_0/\partial z = 0, \quad v_0|_{z=0} = U(t).$$

A problem of the first order becomes

$$\partial p_0/\partial t + \partial v_1/\partial z = 0, \quad \partial v_0/\partial t + \partial p_1/\partial z = 0, \quad v_1|_{z=0} = 0$$

and a problem of the second order is

$$\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_0}{\partial z} + \frac{\partial p_2}{\partial z} = 0, \quad \frac{\partial p_1}{\partial t} + v_1\frac{\partial p_0}{\partial z} + \frac{\partial v_2}{\partial z} = 0, \quad v_2|_{z=0} = 0.$$

The solution for the velocity and pressure in the "inner" zone is

$$v(t, z) = U - \varepsilon z\frac{\mathrm{d}f_1}{\mathrm{d}t} + \varepsilon^2\left(\frac{z^2}{2}\frac{\mathrm{d}^2 U}{\mathrm{d}t^2} - z\frac{\mathrm{d}f_2}{\mathrm{d}t}\right), \tag{12}$$

$$p(t, z) = f_1(t) + \varepsilon\left(-z\frac{\mathrm{d}U}{\mathrm{d}t} + f_2(t)\right) + \varepsilon^2\frac{z^2}{2}\frac{\mathrm{d}^2 f_1}{\mathrm{d}t^2}. \tag{13}$$

It is valid for small z and ε.

Equations (12) and (13) contain undetermined functions $f_1(t)$, $f_2(t)$ because one has the degeneration of the equations (11) as $\varepsilon \to 0$. To find these functions one should address the problem in the "outer" zone.

2.1.2. *Formulation in the "outer" zone*

As has been discussed in section 1.3, at a certain distance from a vibrating piston fluid's motion becomes of acoustical nature. Thus, instead of having to know all about the fluid dynamics in the "inner" zone, one can assume the hydrodynamics in the "outer" zone are purely linear. The spatial scale in the "outer" zone is defined by the acoustic wavelength and one can introduce the new non-dimensional spatial variable $\bar{x} = x\omega/c_0$ $(\bar{x} = \bar{z}\varepsilon)$.

In non-dimensional variables the linearized equations of hydrodynamics are

$$\partial p/\partial t + \partial v/\partial x = 0, \quad \partial v/\partial t + \partial p/\partial x = 0. \tag{14}$$

By substitution $\beta = x - \varepsilon w$ equations (14) can be re-written as

$$\partial p/\partial t - \varepsilon U\, \partial p/\partial \beta + \partial v/\partial \beta = 0, \quad \partial v/\partial t - \varepsilon U\, \partial v/\partial \beta + \partial p/\partial \beta = 0. \tag{15}$$

One can now find a solution of equations (15) in the form of an asymptotic expansion in the small parameter ε and retain only the first two terms (similar to those in the 'inner" zone). The substitution of these series into equations (15) gives the following zero order problem:

$$\partial p_0/\partial t + \partial v_0/\partial \beta = 0, \quad \partial v_0/\partial t + \partial p_0/\partial \beta = 0. \tag{16}$$

A solution of this problem is $v_0(\beta, t) = p_0(\beta, t) = F_1(t - \beta)$. Here F_1 is an arbitrary function. The problem of the first order becomes

$$\partial p_1/\partial t + \partial v_1/\partial \beta - U\, \partial p_0/\partial \beta = 0, \quad \partial v_1/\partial t + \partial p_1/\partial \beta - U\, \partial v_0/\partial \beta = 0. \tag{17}$$

As pressure is excluded from equations (17), one obtains an inhomogeneous wave equation for the velocity v_1:

$$\frac{\partial^2 v_1}{\partial \beta^2} - \frac{\partial^2 v_1}{\partial t^2} = -2U \frac{\partial^2 v_0}{\partial \beta\, \partial t} - \frac{\mathrm{d}U}{\mathrm{d}t}\frac{\partial v_0}{\partial \beta}. \tag{18}$$

The particular solution of this inhomogeneous wave equation is $-\{\mathrm{d}F_1(t-\beta)/\mathrm{d}t\}w(t)$, and the general solution of the homogeneous equation is $F_2(t-\beta)$ (here F_2 is an arbitrary function). Thus, the general solution of the first order problem is

$$v_1(t, \beta) = F_2(t - \beta) - \frac{\mathrm{d}F_1(t-\beta)}{\mathrm{d}t} w(t). \tag{19}$$

Finally, one has thus obtained the following formula for the velocity in the "outer" zone:

$$v(t, \beta) = v_0 + \varepsilon v_1 = F_1(t - \beta) + \varepsilon\left[F_2(t - \beta) - \frac{\mathrm{d}F_1(t-\beta)}{\mathrm{d}t} w(t)\right]. \tag{20}$$

Equation (20) is valid for large β and small ε and it contains undetermined functions $f_1(t)$, $F_2(t)$ which should be defined by matching of the solutions for the "inner" and the "outer" zones.

2.1.3. *The matching of asymptotic expansions*

In following the van Dyke method [17] of matching asymptotic expansions for the velocity (12) and (20) it is necessary to rewrite the "outer" representation

(20) by using the "inner" spatial co-ordinate z and to expand the result in power seria on small parameter $z\varepsilon$:

$$v(t, \beta) = F_1(t - z\varepsilon) + \varepsilon\left[F_2(t - z\varepsilon) - \frac{dF_1(t - z\varepsilon)}{dt} w(t)\right] \sim F_1(t)$$
$$+ \varepsilon\left(-z\frac{dF_1(t)}{dt} - F_2(t) - w\frac{dF_1(t)}{dt}\right)$$
$$+ \varepsilon^2\left(\frac{z^2}{2}\frac{d^2F_1(t)}{dt^2} - z\frac{dF_2(t)}{dt} + zw(t)\frac{d^2F_1(t)}{dt^2}\right). \quad (21)$$

Comparing expression (21) with equation (12), one concludes that

$$f_1(t) = F_1(t) = U, \quad F_2(t) = w\, dU/dt, \quad f_2(t) + \tfrac{1}{2}U^2,$$

and from equation (13) one obtains

$$p(t, z) = U + \varepsilon(-z\, dU/dt + \tfrac{1}{2}U^2). \quad (22)$$

In particular, on the surface of the piston one has

$$p(t, 0) = U + \varepsilon\tfrac{1}{2}U^2. \quad (23)$$

From the standpoint of structural dynamics, formula (23) is sufficient to perform analysis of non-linear oscillations of a fluid-loaded piston, as it gives a relation between the velocity of piston at any instant of time and a contact pressure acting at the piston (fluid loading).

We have matched asymptotic expansions for velocity in "inner" and "outer" zones up to terms of order ε. Matching of pressure is achieved only up to terms of order 1. Physically this means that the pressure in the "outer" zone is calculated as if there were no "inner" zone at all. To construct a unique representation for pressure one should use the non-linear equation (10b), to obtain

$$\frac{p}{\rho_0} = -\int \frac{\partial v}{\partial t}\, dx - \frac{1}{2}v^2, \quad (24)$$

and formula (20) for velocity. Note, that in reference [18, #244] it is suggested to retain the second term in equation (24) for analysis of the pressure field in the whole volume. Substituting equation (24) into equation (10a), yield, to terms of order ε the following equation in dimensional variables:

$$-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\int v\, dx - \frac{2}{c_0^2}v\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0.$$

The second term is relevant to the kinetic energy of the fluid particles in acoustic waves and in the "outer" zone this term is omitted.

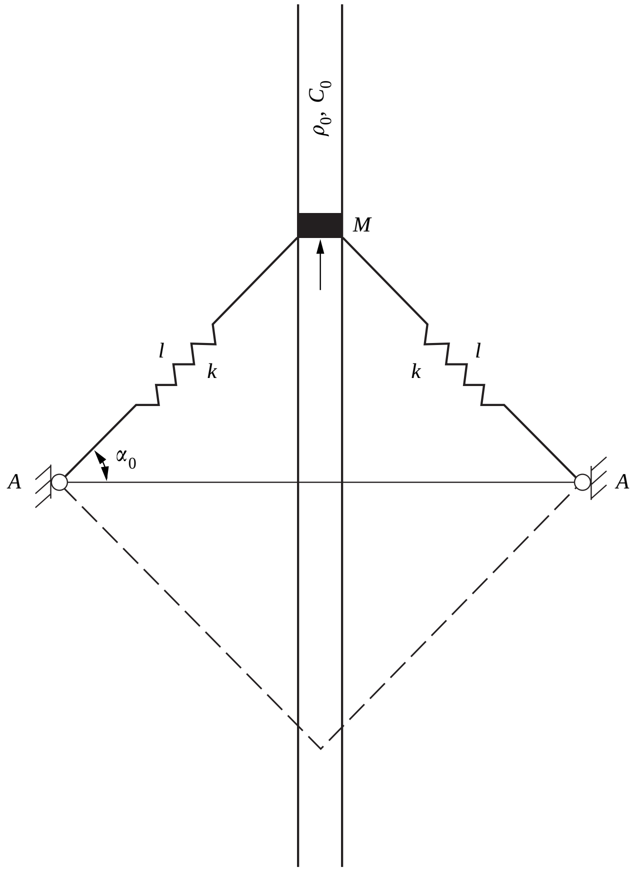

Figure 1. A model non-linear structure loaded by an acoustic medium.

2.2. COUPLED PROBLEM FOR A PISTON ATTACHED TO A MODEL NON-LINEAR STRUCTURE

Consider a structure modelling dynamics of a switch used in the automatic control of pressure jumps in pipe lines (see Figure 1). It consists of two weightless linear springs of stiffness k. The length of each spring in unloaded state is l. There is a concentrated mass m placed in the hinge connecting the springs. The angle between the horizontal axis and each spring in the unloaded position is α_0. The concentrated mass is attached to a piston in contact with an acoustic medium of density ρ_0 and sound speed c_0. It is assumed that a piston may move only along the x axis so that one has a one d.o.f. model characterized by the existence of three equilibrium configuration (two stable and one unstable) in the absence of external loading. The non-linear dynamics of this system with no contact with a fluid has been considered in reference [19].

The equation of motion of this system driven by a force F is

$$M\ddot{w} = F - 2k(l\sin\alpha_0 + w)\left[1 - \frac{1}{2\cos\alpha_0}\left(1 + \left(tg\alpha_0 + \frac{w}{l\cos\alpha_0}\right)^2\right)^{-0\cdot 5}\right] - p. \quad (25)$$

Here w is the displacement of mass along the z axis which coincides with the axis of the tube, p is a contact pressure acting at the piston from the fluid. In this paper we restrict ourselves by considering non-linear dynamics of a comparatively shallow system with no snap-through motion. Thus,

$$tg\alpha_0 \ll 1, \quad \frac{w}{l\cos\alpha_0} \ll 1,$$

so that equation (25) may be expanded in a power series about the vicintity of the upper unloaded position $w = 0$ (see Figure 1) as

$$M\ddot{w} = F - kl\alpha_0^3\left[2\frac{w}{l\alpha_0} + 3\left(\frac{w}{l\alpha_0}\right)^2 + \left(\frac{w}{l\alpha_0}\right)^3\right] - p. \tag{26}$$

A contact pressure is given by equation (24). Introducing the following non-dimensional parameters

$$\xi = \frac{w}{l\alpha_0}, \quad \omega_0^2 = \frac{2k\alpha_0^2}{M}, \quad \tau = \omega_0 t, \quad \gamma = \frac{\rho_0 l\alpha_0}{M}, \quad \mu = \frac{\rho_0 c_0}{M\omega_0}, \quad f(\tau) = \frac{f(\omega_0 t)}{2kl\alpha_0^3},$$

one obtains the equation

$$\ddot{\xi} + \xi + \tfrac{3}{2}\xi^2 + \tfrac{1}{2}\xi^3 + \mu\dot{\xi} - \tfrac{1}{2}\gamma\dot{\xi}^2 = f(\tau). \tag{27}$$

This is Duffing's equation for a driven non-linear oscillator. If a linear formulation of fluid dynamics is used then this equation does not contain terms quadratic in the velocity (those underlined). A new parameter γ (the parameter of the fluid's non-linearity) has a simple physical interpretation. It is the ratio of fluid mass contained in a channel between two stable equilibrium positions of the piston (see Figure 1) to the mass of the piston. In principle it may vary in a wide range and is not necessarily small. The parameter μ is relevant to acoustical damping in the linear formulation of the problem and may be presented as $\mu = \gamma\, c_0/\omega_0 l\alpha_0$. The factor $c_0/\omega_0 l\alpha_0$ may also vary in a wide range depending upon the stiffness k of the spring.

To illustrate the effects generated by the additional non-linear term one analyzes by the multiple scales method weak excitation an primary resonance ($\omega = \omega_0$) and hard excitation at $\omega = \omega_0/2$.

A detailed formulation of the multiple scales method is available, for example, in references [20, 21]; we adhere to notations adopted in reference [22].

2.2.1. *Weak excitation at primary resonance*

First, equation (27) is analyzed, which is re-written as

$$\ddot{\xi} + \xi = \varepsilon_0(f - \tfrac{3}{2}\xi^2 - \tfrac{1}{2}\xi^3 - \mu\dot{\xi} + \tfrac{1}{2}\gamma\dot{\xi}^2), \tag{28}$$

with ε_0 selected as a bookkeeper for asymptotically small terms.

A solution for the problem of zero-order $\ddot{\xi}_0 + \xi_0 = 0$ is simply

$$\xi_0 = A(\tau_1)\exp i\tau + cc. \tag{29}$$

Here τ_1 is a slow time which controls the modulation of amplitudes. We induce a detuning parameter $\nu\tau = \sigma\tau_1 + \tau$ and present the driving force as

$$f = \tfrac{1}{2} f_0(\exp(\mathrm{i}\nu\tau) + cc). \tag{30}$$

Then the following equation describes the non-linear dynamics to ε_0^1 order:

$$\begin{aligned}\ddot{\xi}_1 + \xi_1 = {} & \tfrac{1}{2} f_0 \exp \mathrm{i}\tau \exp \mathrm{i}\sigma\tau_1 + \tfrac{1}{2}\gamma(-A^2 \exp 2\mathrm{i}\tau + A\bar{A}) - \mu \mathrm{i} A \exp\ \mathrm{i}\tau \\ & - \tfrac{3}{2}(A^2 \exp 2\mathrm{i}\tau + A\bar{A}) - \tfrac{1}{2}(A^3 \exp 3\mathrm{i}\tau + 3A^2\bar{A} \exp\ \mathrm{i}\tau) \\ & - 2\mathrm{i}(\partial A/\partial\tau_1) \exp\ \mathrm{i}\tau + cc. \end{aligned} \tag{31}$$

To remove secular terms from the solution for this equation it is necessary to satisfy the condition

$$\tfrac{1}{2} f_0 \exp\ \mathrm{i}\sigma\tau_1 - \mathrm{i}\mu A - \tfrac{3}{2}A^2\bar{A} - 2\mathrm{i}\ \partial A/\partial\tau_1 = 0.$$

Hereafter differentiation by τ_1 will be denoted as ($'$).

Solving this modulation equation it is assumed that $A = \frac{1}{2} a \exp(\mathrm{i}\varphi)$, where a is amplitude, and φ is a phase of stationary solution; after some standard transformations one obtains two equations,

$$\tfrac{1}{2} f_0 \sin(\sigma\tau_1 - \varphi) = a' + \tfrac{1}{2}\mu a, \tag{32a}$$

$$-\tfrac{1}{2} f_0 \cos(\sigma\tau_1 - \varphi) = a\varphi' - \tfrac{3}{16}a^3, \tag{32b}$$

in "slow time". It is more convenient to induce a new variable $\psi = \sigma\tau_1 - \varphi$, so that $\varphi' = \sigma - \psi'$. Stationarity conditions $a' = \varphi' = 0$ give a solution for amplitude and phase shift:

$$\left(\frac{3a^3}{16} - a\sigma\right)^2 + \frac{1}{4}\mu^2 a^2 = \frac{1}{4} f_0^2, \quad tg\psi = \frac{\mu}{2}\left(\frac{3a^2}{16} - \sigma\right)^{-1}. \tag{33a, b}$$

One can see that at the primary resonance both the amplitude and the phase shift are controlled only by linear damping and a term quadratic in the velocity does not enter the modulation equations. However, a complete solution for the ε_0^1 problem does contain terms generated by fluid-induced quadratic non-linearity:

$$\xi = a\cos(\nu\tau - \psi) + \frac{\gamma - 3}{4}a^2 + \frac{\gamma + 3}{12}a^2\cos(2\nu\tau - 2\psi) + \frac{a^3}{64}\cos(3\nu\tau - 3\psi). \tag{34}$$

Examination of this formula reveals that the contribution of fluid-induced non-linearity is small as long as

$$\gamma = \rho_0 l \alpha_0 / M \ll 1. \tag{35}$$

This means a light acoustic medium in contact with a heavy piston. In effect, structural non-linearity and linear acoustic damping govern dynamics of a piston. Therefore, this case is not discussed any further and proceed to hard subharmonic excitation of a piston.

2.2.2. *Hard excitation at $\omega = \omega_0/2$*

In the case of hard excitation equation (27) is re-written as

$$\ddot{\xi} + \xi = f + \varepsilon_0(-\tfrac{3}{2}\xi^2 - \tfrac{1}{2}\xi^3 - \mu\dot{\xi} + \tfrac{1}{2}\gamma\dot{\xi}^2). \tag{36}$$

A detuning parameter is induced and the driving force is presented as

$$f = \frac{1}{2}q\left(\exp\left(\frac{\tau}{2} + \frac{\sigma\tau_1}{2}\right)\mathrm{i} + cc\right).$$

A solution for the problem of zero-order becomes

$$\xi_0 = A(\tau_1)\exp\ \mathrm{i}\tau + \frac{2}{3}q\exp\left[\mathrm{i}\left(\frac{\tau}{2} + \frac{1}{2}\sigma\tau_1\right)\right] + cc. \tag{37}$$

Then the following equation describes the non-linear dynamics of ε_0^1 order:

$$\begin{aligned}\ddot{\xi}_1 + \xi_1 = &-2A\mathrm{i}\exp\ \mathrm{i}\tau + \frac{1}{2}\mathrm{i}q\sigma\exp\left[\mathrm{i}\left(\frac{\tau}{2} + \frac{1}{2}\sigma\tau_1\right)\right] - \frac{1}{18}\gamma q^2\exp(\mathrm{i}\tau + \mathrm{i}\sigma\tau_1)\\ &- \mu A\mathrm{i}\exp\ \mathrm{i}\tau - \frac{2}{3}q^2\exp(\mathrm{i}\tau + \mathrm{i}\sigma\tau_1)\\ &- \frac{3}{2}A^2\bar{A}\exp\ \mathrm{i}\tau - \frac{4}{3}Aq^2\exp\ \mathrm{i}\tau + \cdots + cc.\end{aligned} \tag{38}$$

To remove secular terms from the solution of this equation it is necessary to solve the modulation equation

$$-2A'\mathrm{i} - \tfrac{1}{18}\gamma q^2\exp\sigma\tau_1\mathrm{i} - \mathrm{i}\mu A - \tfrac{2}{3}q^2\exp\sigma\tau_1\mathrm{i} - \tfrac{3}{2}A^2\bar{A} - \tfrac{4}{3}Aq^2\exp\sigma\tau_1\mathrm{i} = 0. \tag{39}$$

As before one searches for a solution of equation (39) in the form

$$A = \tfrac{1}{2}a\exp(\mathrm{i}\varphi).$$

After some standard transformations one obtains, for the stationary regime $a' = \varphi' = 0$,

$$(a\sigma - \tfrac{2}{3}q^2a - \tfrac{3}{16}a^3)^2 + (\mu a/2)^2 = q^4(\tfrac{1}{18}\gamma + \tfrac{2}{3})^2, \tag{40a}$$

$$tg\varphi = -\tfrac{1}{2}\mu/(\sigma - \tfrac{2}{3}q^2 - \tfrac{3}{16}a^2), \tag{40b}$$

and it follows from equation (40a) that the amplitude of the vibrations at the resonance frequency ω_0 is controlled by the fluid's non-linearity, while a phase shift is governed by simple linear acoustic damping.

To plot a dependence of amplitude a upon frequency ω the dimensionless parameter are induced $\Omega = \omega/\omega_0$ and the detuning parameter then becomes $\sigma = \Omega - \frac{1}{2}$. Then equation (40a) may be rewritten as

$$\Omega = \frac{1}{2} + \frac{3a^2}{16} + \frac{2q^2}{3} \pm \left[\frac{q^4}{a^2}\left(\frac{\gamma}{18} + \frac{2}{3}\right)^2 - \frac{\mu^2}{4}\right]^{0\cdot 5}. \tag{41}$$

One should note that in a case of linear structural dynamics, formulas (40a, b) are reduced at $\sigma = 0$ to

$$a^2 = \frac{q^4 \left(\frac{1}{18}\gamma\right)^2}{\mu^2 + \left(\frac{4}{3}q^2\right)^2}, \quad \mathrm{tg}\varphi = \frac{3}{4}\frac{\mu}{q^2}, \quad \Omega = \frac{1}{2} \pm \sqrt{\frac{\gamma^2 q^4}{324a^2} - \frac{\mu^2}{4}}, \qquad (42a, b, 43)$$

so that oscillations at the frequency ω_0 are generated only by the fluid's non-linearity.

As there are three non-dimensional parameters q, μ, γ in the modulation equations, it is appropriate to analyze the roles of each one of them. In Figure 2, non-dimensional frequency response curves are plotted for $q = 0{\cdot}1$ and $\mu = 0{\cdot}1$. The curve 1 is relevant to $\gamma = 1$ (mass of fluid in volume between two stable equilibrium positions is equal to mass of piston), the curve 2 is plotted for heavy fluid ($\gamma = 10$). There is a pronounced non-linear effect of "hardening" type. This effect is controlled by the parameter γ, as well as by the loading parameter q, as is seen also from Figure 3. Graphs in this figure are plotted for $\mu = 0{\cdot}1$, $\gamma = 1$ and $q = 0{\cdot}3$ (curve 1) and $q = 0{\cdot}5$ (curve 2). It is clear that non-linearity in the dynamics of the model system becomes much stronger with an increase of the amplitude of excitation. The role of the linear damping μ is opposite: as it grows, the non-linearity is weakened (see Figure 4). A set of parameters is: $q = 0{\cdot}3$, $\gamma = 1$ and $\mu = 0{\cdot}1$ (curve 1), $\mu = 0{\cdot}2$ (curve 2).

In Figure 5, three sets of curves are presented to show a dependence of phase angle upon excitation frequency parameter. Specifically, in Figure 5(a), it is plotted for $q = 0{\cdot}3$, $\mu = 0{\cdot}2$ and $\gamma = 1$. A dependence of amplitude upon frequency parameter in this case is given by curve 2 in Figure 4. In this case non-linearity is weak and phase is simply changed from $\pi/2$ to $-\pi/2$ as a resonance

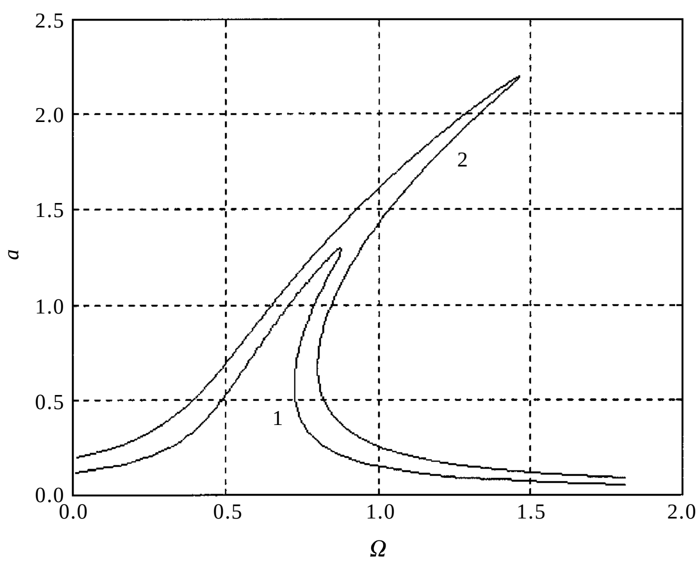

Figure 2. Non-linear forced amplitude response of a fluid-loaded piston at hard subharmonical excitation for $q = 0{\cdot}1$, $\mu = 0{\cdot}1$ and $\gamma = 1$ (curve 1), $\gamma = 10$ (curve 2).

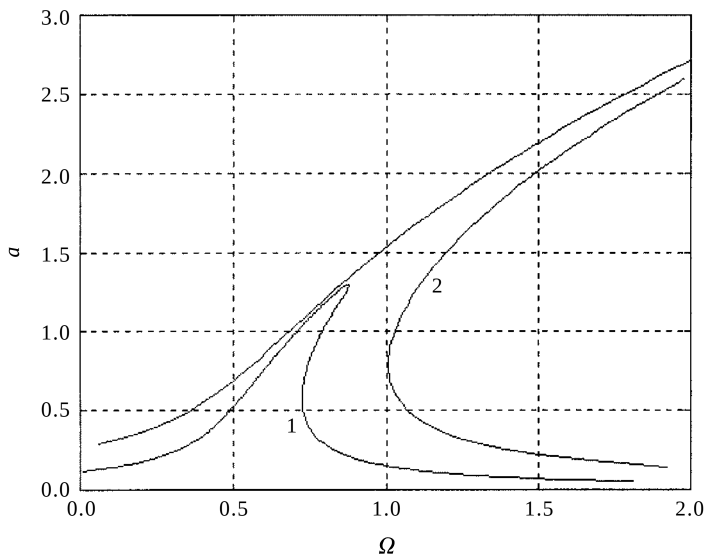

Figure 3. Non-linear forced amplitude response of a fluid-loaded piston at hard subharmonical excitation for $\gamma = 1$, $\mu = 0{\cdot}1$ and $q = 0{\cdot}3$ (curve 1), $q = 0{\cdot}5$ (curve 2).

peak is passed. Graphs in Figure 5(b) are plotted for $q = 0{\cdot}3$, $\mu = 0{\cdot}1$ and $\gamma = 1$. A dependence of amplitude upon frequency parameter in this case is given by curve 1 in Figure 3. In this case, there are three co-existing solutions in a zone $0{\cdot}67 < \Omega < 0{\cdot}91$. Curve 1 in Figure 4(b) presents a phase of stable solution given by the left part of the curve 1 in Figure 3. The upper branch of curve 2 (to the left from the dashed straight vertical line) in Figure 5(b) presents a phase of a post-resonance stable branch of curve 1 in Figure 3. The lower branch of the

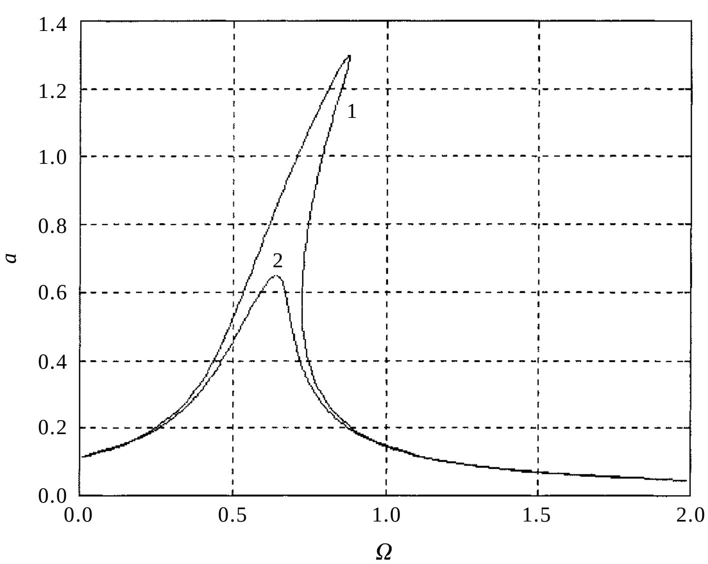

Figure 4. Non-linear forced amplitude response of a fluid-loaded piston at hard subharmonical excitation for $\gamma = 1$, $q = 0{\cdot}3$ and $\mu = 0{\cdot}1$ (curve 1), $\mu = 0{\cdot}2$ (curve 2).

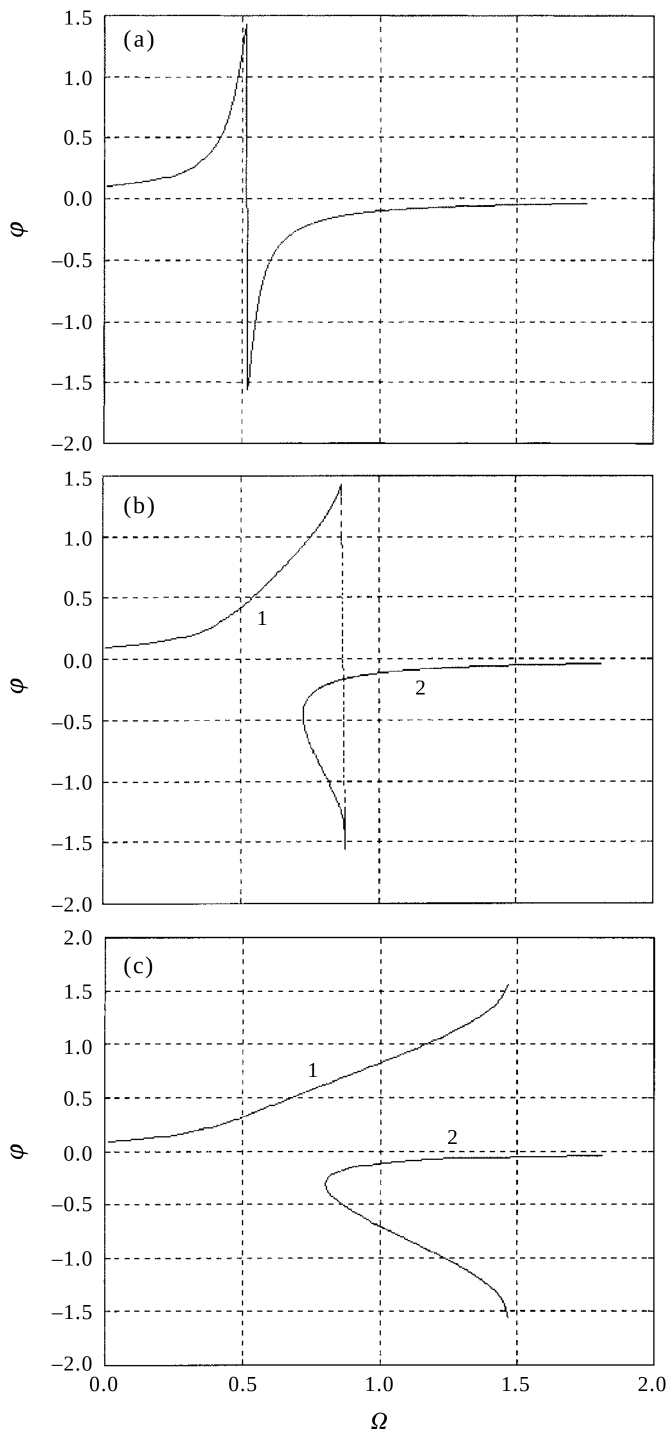

Figure 5. Phase angle versus excitation frequency parameter for (a) $q = 0{\cdot}3$, $\mu = 0{\cdot}2$, $\gamma = 1$; (b) $q = 0{\cdot}3$, $\mu = 0{\cdot}1$, $\gamma = 1$; (c) $q = 0{\cdot}3$, $\mu = 0{\cdot}1$, $\gamma = 10$.

curve 2 is relevant to an unstable solution. Graphs shown in Figure 5(c) are similar to those in Figure 5(b) but they are plotted for a system with stronger non-linearity: $q = 0{\cdot}3$, $\mu = 0{\cdot}1$ and $\gamma = 10$.

Finally, in Figure 6 the forced amplitude response of a linear structure (curve 1) obtained from formula (43) is plotted against the forced amplitude response of a non-linear structure (curve 2, formula (41)) for fluid parameters $\mu = 0{\cdot}1$, $\gamma = 10$ and excitation force $q = 0{\cdot}3$. Structural non-linearity enhances resonance behaviour of a system, as compared with that of a linear structure. However, it should be noted that a model of linear structure loaded by a linear fluid would give no response at all in subharmonic excitation conditions.

We do not dwell upon the analysis of characteristic curves for other combinations of parameters μ and γ because the model of a fluid-loaded piston is chosen here mostly to clarify with reasonably simple mathematics the physics of non-linear structural-acoustic coupling. In principle, any combination of these two parameters can be adjusted to some model of a structure with non-linear geometry, shown in Figure 1 (say, very soft spring + very heavy fluid or hard spring + heavy fluid, etc.) However, non-linear phenomena exposed in the above analysis are expected to show up in much more complicated, but realistic problems. We also do not discuss the possible transitions to chaos that are characteristic in the periodically forced Duffing's equation since this aspect lies beyond the framework of this paper.

In conclusion to this section we should note that the 1-D problem for fluid motions generated by a piston does not permit one to look at modal interaction which in other situations may result in pumping of energy from non-radiating modes to radiating ones. This aspect of the problem is discussed for another model problem.

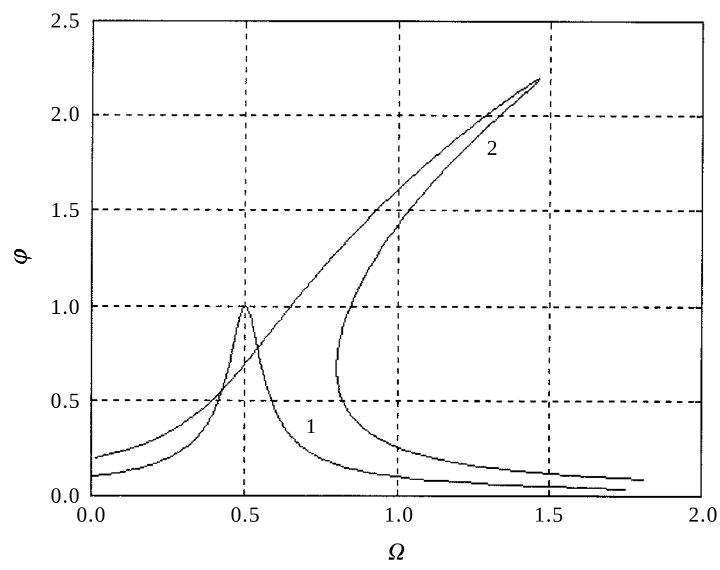

Figure 6. Non-linear forced amplitude response of a fluid-loaded piston at hard subharmonical excitation for $\gamma = 10$, $q = 0{\cdot}3$ and $\mu = 0{\cdot}1$. Curve 1, linear structure; curve 2, non-linear structure.

3. NON-LINEAR VIBRATIONS OF A FLUID-LOADED PERIODICALLY SUPPORTED INFINITE PLATE

An infinitely long plate which is periodically supported by immobile hinges is considered. The plate is acoustically loaded and a driving force is applied to provoke skew-symmetric motions. In effect, any span of a plate may be considered individually together with fluid in an attached channel-like domain if conditions at the vertical boundaries of a fluid domain are of soft baffle $\phi = 0$. This problem in a linear formulation has elementary analytical solution [23] that gives a clear insight into two distinct mechanisms of structural–acoustic coupling (pure radiation damping and pure accession to inertia). In reference [4] this problem has been analysed in a general non-linear formulation with an emphasis put on shock waves formation. Here we begin with specification of a model suggested in section 1 for the problem in hand. Then two typical cases in non-linear dynamics are considered in detail.

3.1. FORMULATION OF THE PROBLEM

The co-ordinates of a plate at rest are specified as

$$y(s) = s, \quad z(s) = 0. \tag{44}$$

A driving force is lateral and it produces predominantly normal displacements, so that the co-ordinates of the vibrating plate are

$$y(s, t) = s, \quad z(s, t) = w(s, t). \tag{45}$$

Then compatibility condition (velocities continuity condition) at the moving fluid–structure interface may be formulated as

$$\left(\frac{\partial\phi}{\partial z} - \frac{\partial w}{\partial s}\frac{\partial\phi}{\partial s}\right)\bigg|_{z=w(s,\,t)} = \frac{\partial w}{\partial t}. \tag{46}$$

As suggested in reference [2], one may expand equation (46) as

$$\frac{\partial\phi}{\partial z}\bigg|_{z=w(s,\,t)} = \frac{\partial\phi}{\partial z}\bigg|_{z=0} + \frac{\partial^2\phi}{\partial z^2}\bigg|_{z=0} w + \dots, \tag{47}$$

provided that $w(s, t)$ is "not too large". Then boundary conditions are formulated at an immobile interface, but the second order derivative of the velocity potential enters this condition. This indicates an approximate nature of posing the problem because the second order derivative ϕ_{zz} is the leading order term in the wave equation.

An alternative way to formulate the continuity condition is to use deformable co-ordinates in a way used in a piston problem: s, $\beta = z - w(s, t)$ and then to formulate velocities continuity as

$$\left(\frac{\partial\phi}{\partial\beta} - \frac{\partial w}{\partial s}\frac{\partial\phi}{\partial s}\right)\bigg|_{\beta=0} = \frac{\partial w}{\partial t}. \tag{48}$$

The formulation of the contact acoustic pressure in equations of structural vibrations in deformable co-ordinates (s, β) becomes

$$p = -\rho_0\left[\frac{\partial\phi}{\partial t} - \frac{\partial w}{\partial t}\frac{\partial\phi}{\partial\beta} + \frac{1}{2}\left(\frac{\partial\phi}{\partial s}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial\beta}\right)^2\right].$$

As one substitutes here equation (48) for $\mathrm{d}w/\mathrm{d}t$ one obtains

$$p|_{\beta=0} = -\rho_0\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\frac{\partial\phi}{\partial s}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial\beta}\right)^2\right], \tag{49}$$

which in fact expresses quadratic non-linearity of compatibility conditions at the moving boundary.

Similar to the case of a fluid-loaded piston one can assume amplitudes of vibrations of a structure to be large enough to pose continuity conditions at the moving boundary, but at any point of a fluid volume velocities are much less than the sound speed (6) and acoustic pressure is much less than the bulk modulus of the fluid (5). One can transform the linear wave equation to deformable co-ordinates s, β as

$$\frac{\partial^2\phi}{\partial\beta^2} + \frac{\partial^2\phi}{\partial s^2} - 2\frac{\partial w}{\partial s}\frac{\partial^2\phi}{\partial s\partial\beta} - \frac{\partial^2 w}{\partial s^2}\frac{\partial\phi}{\partial\beta} - \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} + \frac{2}{c_0^2}\frac{\partial w}{\partial t}\frac{\partial^2\phi}{\partial\beta\partial t} + \frac{1}{c_0^2}\frac{\partial^2 w}{\partial t^2}\frac{\partial\phi}{\partial\beta} = 0, \tag{50}$$

and use a theory (48, 49, 50) coupled with an elementary theory of non-linear dynamics of a plate (a beam):

$$\frac{h^2}{12}w^{(4)} + \frac{p(1-\nu^2)}{Eh} + \frac{\rho(1-\nu^2)}{E}\ddot{w} - \frac{1-\nu^2}{2l}w''\int_0^l (w')^2\,\mathrm{d}x - \frac{q(1-\nu^2)}{Eh} = 0. \tag{51}$$

Here h is the thickness of the plate, ρ is its material density, E is its Young's modulus, ν is Poisson's ratio. A driving force is denoted by q and w is the displacement of the plate.

3.2. WEAK EXCITATION AT PRIMARY RESONANCE

At the primary resonance, weak excitation of a plate under heavy fluid loading is assumed, which is described by a set of equations

$$\frac{h^2}{12}w^{(4)} + \frac{p(1-\nu^2)}{Eh} + \frac{\rho(1-\nu^2)}{E}\ddot{w} - \frac{1-\nu^2}{2l}\varepsilon w''\int_0^l w'^2\,\mathrm{d}x - \varepsilon\frac{q(1-\nu^2)}{Eh} = 0, \tag{52a}$$

$$\Phi_{\beta\beta} + \Phi_{ss} - \frac{1}{c_0^2}\Phi_{tt} - \varepsilon\left[w''\Phi_\beta + 2w'\Phi_{ss} - \frac{2}{c_0^2}\dot{w}\Phi_{\beta t} - \frac{1}{c_0^2}\ddot{w}\Phi_\beta\right] = 0, \tag{52b}$$

$$\Phi_\beta - \varepsilon w'\Phi_s = \dot{w}, \tag{52c}$$

$$p = -\rho_0[\Phi_t - \varepsilon\dot{w}\Phi_\beta + \varepsilon\tfrac{1}{2}\Phi_s^2 + \tfrac{1}{2}\varepsilon\Phi_\beta^2]. \tag{52d}$$

Here ε is a bookkeeper of asymptotically small terms.

A linear problem of zero order becomes

$$\frac{h^2}{12} w_0^{(4)} + \frac{p_0(1-\nu^2)}{Eh} + \frac{\rho(1-\nu^2)}{E} \ddot{w}_0 = 0, \tag{53a}$$

$$(\Phi_0)_{\beta\beta} + (\Phi_0)_{ss} - (1/c_0^2)(\Phi_0)_{tt} = 0, \tag{53b}$$

$$(\Phi_0)_\beta|_{\beta=0} = \dot{w}_0, \quad p_0 = -\rho_0(\Phi_0)_t. \tag{53c, d}$$

A solution can be presented as

$$\Phi_0(s, \beta, t) = \tilde{\varphi}_0 \exp(-\mathrm{i}\omega_0 t)\sin(\pi s/l) + cc, \tag{54a}$$

$$w_0(s, t) = C\exp(-\mathrm{i}\omega_0 t)\sin(\pi s/l) + cc, \tag{54b}$$

and this brings one to the well-known formulation of structural response:

$$\left[-\omega^2 + \frac{h^2}{12}\left(\frac{\pi}{l}\right)^4 \frac{E}{\rho(1-\nu^2)} - \frac{\rho_0}{\rho}\frac{l}{h}\frac{\omega^2}{\sqrt{\pi^2 - (\omega l/c_0)^2}}\right] C = 0. \tag{55}$$

A solution of resonance type may exist only if the following inequality applies:

$$\omega_0 l/c_0 < \pi, \tag{56}$$

which in turn defines a non-acoustical behaviour of a fluid—the second term in brackets is purely real and has a physical meaning of added mass. One can easily find the eigenfrequency of a fluid-loaded plate from the equation

$$\left(\frac{\omega_0 l}{c_0}\right)^2 \left(1 + \frac{\rho_0}{\rho}\frac{l}{h}\frac{1}{\sqrt{\pi^2 - (\omega_0 l/c_0)^2}}\right) = \frac{\pi^4}{12}\left(\frac{h}{l}\right)^2 \left(\frac{c}{c_0}\right)^2 \frac{1}{(1-\nu^2)}, \quad c^2 = \frac{E}{\rho} \tag{57}$$

In the case when a sign in the inequality (56) is inverted there is no added mass effect and the fluid produces only radiation damping (the second term in the brackets becomes pure imaginary). In such a case a resonance solution in the zero-order approximation does not exist and to be able to use a multiple scales method it becomes necessary to revise the assumption regarding heavy fluid loading of the structure. This case is left for discussion later and examination is continued of the case when the fluid produces added mass so that a problem of order ε^0 has a solution

$$w_0 = [C_1(T_1)\exp(-\mathrm{i}\omega_0 t) + C_2(T_1)\exp(\mathrm{i}\omega_0 t)]\sin(\pi s/l), \quad C_2 = \bar{C}_1, \tag{58}$$

where ω_0 is an eigenfrequency of vibrations of a plate with an added mass, defined by equation (57). A formulation of the velocity potential is

$$\Phi_0 = \frac{\mathrm{i}\omega l}{\kappa}[C_1(T_1)\exp(-\mathrm{i}\omega_0 t) - C_2(T_1)\exp(\mathrm{i}\omega_0 t)]\exp\left(\frac{-\kappa}{l}\beta\right)\sin\frac{\pi s}{l},$$

$$\kappa = \sqrt{\pi^2 - \left(\frac{\omega_0 l}{c_0}\right)^2}, \qquad (59)$$

and the motions of the fluid's particles are essentially localized near the vibrating structure. Note that all formalism of the multiple scales method that have been used in section 2 are valid for analysis of this problem and T_1 is presented as "slow time", $T_1 = \varepsilon t$.

Now one can address the problem of order ε^1 which is formulated by

$$\frac{h^2}{12}w_1^{(4)} + \frac{p_1(1-\nu^2)}{Eh} + \frac{\rho(1-\nu^2)}{E}\ddot{w}_1$$
$$= -\frac{\rho(1-\nu^2)}{E}2(w_0)_{T_1} + \frac{1-\nu^2}{2l}w_0''\int_0^l (w_0')^2\,\mathrm{d}x + \frac{q(1-\nu^2)}{Eh}, \qquad (60\text{a})$$

$$(\Phi_1)_{\beta\beta} + (\Phi_1)_{ss} - \frac{1}{c_0^2}(\Phi_1)_{tt}$$
$$= -\frac{2}{c_0^2}(\Phi_0)_{T_1} - \frac{2}{c_0^2}\dot{w}_0(\Phi_0)_{\beta t} - \frac{1}{c_0^2}\ddot{w}_0(\Phi_0)_\beta + w_0 + 2w_0(\Phi_0)_{\beta s}, \qquad (60\text{b})$$

$$p_1 = -\rho\left[\frac{\partial\Phi_1}{\partial t} + \frac{\partial\Phi_0}{\partial T_1} - \dot{w}_0\frac{\partial\Phi_0}{\partial\beta} + \frac{1}{2}\left(\frac{\partial\Phi_0}{\partial\beta}\right)^2 + \frac{1}{2}\left(\frac{\partial\Phi_0}{\partial s}\right)^2\right], \qquad (60\text{c})$$

$$-(\Phi_1)_\beta + \dot{w}_1 = -w_0'(\Phi_0)_s - (w_0)_{T_1}. \qquad (60\text{d})$$

Modulations of the amplitudes C_1, C_2 in equations (58) and (59) should be determined from conditions of cancelling of secular terms in equation (60a). A driving force is specified as $q = \frac{1}{2}Q\sin(\pi s/l)[\exp(-\mathrm{i}\Omega t) + \exp(\mathrm{i}\Omega t)]$. One can search for solutions for velocity potential and displacement in the two-terms expansions

$$w_1 = w_{11}\sin\frac{\pi s}{l} + w_{12}\sin\frac{2\pi s}{l}, \quad \Phi_1 = \Phi_{11}\sin\frac{\pi s}{l} + \Phi_{12}\sin\frac{2\pi s}{l}. \qquad (61\text{a, b})$$

Orthogonalization of the boundary condition (60d) to selected trial functions gives a set of two equations:

$$(\Phi_{11})_{\beta\beta} = \dot{w}_{11} + \frac{\mathrm{d}C_1}{\mathrm{d}T_1}\exp(-\mathrm{i}\omega_0 t) + \frac{\mathrm{d}C_2}{\mathrm{d}T_1}\exp(\mathrm{i}\omega_0 t)$$
$$+ \frac{4\pi\omega_0}{l\kappa}[C_1^2\exp(-2\mathrm{i}\omega_0 t) - C_2^2\exp(2\mathrm{i}\omega_0 t)], \qquad (62\text{a})$$

$$(\Phi_{12})_\beta = \dot{w}_{12} \tag{62b}$$

Respectively, wave equation (60d) transforms to

$$(\Phi_1)_{\beta\beta} - \frac{\pi^2}{l^2}\Phi_{11} - \frac{1}{c_0^2}(\Phi_1)_{tt}$$
$$= \left\{\frac{2}{c_0^2}\frac{\partial\dot{\Phi}_0}{\partial T_1} - \frac{8i\omega^3 l}{\pi c_0^2}[C_1^2\exp(-2i\omega_0 t) - C_2^2\exp(2i\omega_0 t)]\right\}\exp(-\kappa\beta), \tag{63a}$$

$$(\Phi_{11})_{\beta\beta} - \frac{4\pi^2}{l^2}\Phi_2 - \frac{1}{c_0^2}(\Phi_{12})_{tt} = 0. \tag{63b}$$

One can see that the second terms in equations (61) are not of interest since there is no effect of modal coupling in equations (62) and (63).

Our attention is specified only at stationary response and put derivatives on ‘‘slow time’’ to zero:

$$dC_1/dT_1 = 0, \quad dC_2/dT_1 = 0.$$

Then, due to linearity of the problem (62a, 63a) a velocity potential is presented as

$$\Phi_{11} = \Phi_{110} + \Phi_{111} + \Phi_{112}, \tag{64}$$

Each component of equation (64) is a solution of the following problems:

$$(\Phi_{112})_{\beta\beta} - \frac{\pi^2}{l^2}\Phi_{112} - \frac{1}{c_0^2}(\Phi_{112})_{tt}$$
$$= -\frac{8\omega^3 i}{\pi c_0^2}[C_1^2\exp(-2i\omega_0 t) - C_2^2\exp(2i\omega_0 t)]\exp(-\kappa\beta), \tag{65a}$$

$$(\Phi_{112})_\beta = 0, \tag{65b}$$

$$(\Phi_{111})_{\beta\beta} - \frac{\pi^2}{l^2}\Phi_{111} - \frac{1}{c_0^2}(\Phi_{111})_{tt} = 0, \tag{66a}$$

$$(\Phi_{111})_\beta = \frac{4\pi\omega i}{l\kappa}[C_1^2\exp(-2i\omega_0 t) - C_2^2\exp(2i\omega_0 t)], \tag{66b}$$

$$(\Phi_{110})_{\beta\beta} - \frac{\pi^2}{l^2}\Phi_{110} - \frac{1}{c_0^2}(\Phi_{110})_{tt} = 0, \quad (\Phi_{110})_\beta = \dot{w}_{11}. \tag{67a, b}$$

As has been already discussed, inequality (56) is held. Now one can see that a qualitative type of solution for problems (65) and (66) is governed by the sign of the expression

$$\pi^2 - (2\omega_0 l/c_0)^2. \tag{68}$$

If it is positive, then similarity to a solution for a zero-order problem will have localised motions of an acoustic medium. However, it is also possible that formula (68) is negative, as for example, one chooses the following set of parameters: $\rho_0/\rho = 1{\cdot}65e - 0{\cdot}4$, $c_0/c = 0{\cdot}0736$, $h/l = 0{\cdot}05$, here c is the sound in the plate's material. A natural frequency of a fluid-loaded plate, $\omega_0 l/c_0 = 1{\cdot}9354$, satisfies both inequalities. This combination of dimensionless parameters is relevant, for instance, to a steel plate ($\rho = 7{\cdot}8$ g/cm^3, $c = 4500$ m/s) vibrating in air $\rho_0 = 0{\cdot}00129$ g/cm^3, $c_0 = 331$ m/s. The length of the span is 2 cm, while the thickness of the plate is 0·1 cm. Then the first natural frequency becomes 5098 Hz, and as vibration at this frequency is not associated with propagation of waves from a structure, at the doubled frequency 10 196 Hz there is sound radiation from the plate.

Then some simple algebra gives

$$\Phi_{12} = -\frac{8\omega C_1^2}{3\pi}\left[\mathrm{i}\exp\left(-\frac{\kappa\beta}{l}\right) + \frac{\kappa}{\kappa_2}\exp\left(\frac{\mathrm{i}\kappa_2\beta}{l}\right)\right]\exp(-2\mathrm{i}\omega_0 t)$$
$$+\frac{8\omega C_2^2}{3\pi}\left[\mathrm{i}\exp\left(-\frac{\kappa\beta}{l}\right) - \frac{\kappa}{\kappa_2}\exp\left(-\frac{\mathrm{i}\kappa_2\beta}{l}\right)\right]\exp(2\mathrm{i}\omega_0 t), \tag{69a}$$

$$\Phi_{111} = \frac{4\pi\omega}{3\kappa\kappa_2}\left[C_1^2\exp\left(\frac{\mathrm{i}\kappa_2\beta}{l}\right)\exp(-2\mathrm{i}\omega_0 t) + C_2^2\exp\left(-\frac{\mathrm{i}\kappa_2\beta}{l}\right)\exp(2\mathrm{i}\omega_0 t)\right], \tag{69b}$$

where $\kappa_2 = \sqrt{(2\omega_0 l/c_0)^2 - \pi^2}$, while a solution for a problem (67) does not contain amplitudes C_1, C_2 which must be found.

In turn, a problem of structural dynamics is defined as

$$\frac{h^2}{12}w_1^{(4)} + \frac{\rho(1-\nu^2)}{E}\ddot{w}_1 - \frac{\rho_0(1-\nu^2)}{Eh}(\Phi_{110})_t = \frac{1-\nu^2}{2l}w''_0\int_0^l (w'_0)^2\,\mathrm{d}x$$
$$+\frac{q(1-\nu^2)}{Eh} + \frac{\rho_0(1-\nu^2)}{Eh}\left(w_0(\Phi_0)_\beta - \frac{1}{2}(\Phi_0)_\beta^2\right) \tag{70}$$

and after Galerkin's orthogonalization one arrives at the following modulation equation

$$\tfrac{3}{16}(\pi/l)^4 C_1^2 C_2 = Q/Eh.$$

Its solution,

$$C_1 = C_2 = \sqrt[3]{\frac{16Ql^4}{3\pi^4 Eh}} = a, \tag{71}$$

controls amplitudes of displacements at frequency ω_0. Formula (71) is valid for a stationary regime at detuning parameter equal to zero. It does not present any difficulty to derive modulation equations for a general case when the detuning parameter is induced in a standard way (see section 2) and derivatives on slow time are retained in solution, but the detailed analysis of such a case is beyond the scope of this paper.

We should emphasis that a potential, induced by non-linearity of an acoustic medium is expressed as

$$\Phi_{111} + \Phi_{112} = -\frac{16\omega a^2}{3\pi}\exp\left(-\frac{\beta\kappa}{l}\right)\sin 2\omega t + \frac{8\omega a^2}{3\pi}\frac{2(\omega l/c_0) - \pi^2}{\sqrt{\pi^2 - (\omega l/c_0)^2}\sqrt{(2\omega l/c_0)^2 - \pi^2}}\cos\left(\frac{\kappa_2\beta}{l} - 2\omega t\right), \quad (72)$$

and while the first term has the same type as a solution for a zero-order problem, i.e., exponentially decaying wave, the second term describes qualitatively a new effect—generation of a propagating wave. It is presented in deformable co-ordinates, but it may be transformed to Eulerian co-ordinates as β tends to infinity:

$$\tilde{\Phi} = 8\omega a^2 \frac{2(\omega l/c_0)^2 - \pi^2}{3\pi\sqrt{\pi^2 - (\omega l/c_0)^2}\sqrt{(2\omega l/c_0)^2 - \pi^2}}\cos\left(\frac{\kappa_2 z}{l} - 2\omega t\right). \quad (73)$$

It is remarkable that sound radiation appears in the case which from the standpoint of the linear formulation of acoustics is classified as trivial. Of course, the intensity of sound radiation is very weak, but it is still likely to be registered experimentally.

When the sign in inequality (56) is inverted, characteristic equation (57) has no real roots. Thus, in heavy fluid loading conditions the problem in the zeroth order cannot be posed as weak excitation at a primary resonance. This is not a case in light fluid loading, see reference [6], when no non-linearities in structural–acoustic interaction enter a problem of the first order.

It should be emphasized that in the model problem of vibrations of an infinite periodically supported plate in contact with an acoustic medium, the latter produces either purely real added mass or purely imaginary radiation losses. As soon as one addresses vibration of, say, a baffled plate, then one gets both the added mass effect and the effect of radiation damping. As is well-known (see, for example reference [24]), accession to inertia in a heavy fluid loaded plate may be very large. Then a resonance frequency must be found with fluid added mass taken into account and a non-linear problem of a weak excitation at primary resonance may be posed with effect of added mass included in the zeroth order approximation while radiation damping and non-linear terms are accounted for in the problem of the first order.

3.3. HARD EXCITATION AT $\Omega = \omega_0/2$

Heavy fluid loading of a plate is still considered and resonant vibrations are assumed to be non-acoustical in nature: i.e., there are purely real eigenfrequencies defined by equation (57). Then hard excitation of a plate at the frequency of $\omega_0/2$ is described by the set of equations (52). However, the last term in equation (52) now does not contain the multiplier ε and becomes simply $q(1-\nu^2)/Eh$. Hence, the problem of zero order becomes

$$\frac{h^2}{12} w_0^{(4)} + \frac{p_0(1-\nu^2)}{Eh} + \frac{\rho(1-\nu^2)}{E} \ddot{w}_0 = \frac{q(1-\nu^2)}{Eh}, \tag{74a}$$

$$(\Phi_0)_{\beta\beta} + (\Phi_0)_{ss} - (1/c_0^2)(\Phi_0)_{tt} = 0, \tag{74b}$$

$$(\Phi_0)_\beta|_{\beta=0} = \dot{w}_0, \quad p_0 = -\rho_0(\Phi_0)_t, \tag{74c, d}$$

and a driving force can be chosen as

$$q = \tfrac{1}{2} Q \sin(\pi s/l)[\exp(-\mathrm{i}\Omega t) + \exp(\mathrm{i}\Omega t)], \quad \Omega = (\omega_0/2) + (\varepsilon\sigma/2).$$

A solution for the linear problem of zero order becomes

$$w_0 = \{A_q[\exp(-\mathrm{i}\Omega t) + \exp(\mathrm{i}\Omega t)] + [C\exp(-\mathrm{i}\omega_0 t) + \bar{C}\exp(\mathrm{i}\omega_0 t)]\}\sin(\pi s/l), \tag{75}$$

where

$$A_q = \frac{Q(1-\nu^2)}{Eh}\left\{\frac{h^2}{12}\left(\frac{\pi}{l}\right)^4 - \frac{\rho\Omega^2(1-\nu^2)}{E}\left(1 + \frac{\rho_0}{\rho}\frac{l}{h\kappa_1}\right)\right\}, \quad \kappa_1 = \sqrt{\pi^2 - \left(\frac{\Omega l}{c_0}\right)^2}, \tag{76}$$

and C is an amplitude of the resonance wave that should be found from the modulation equation. Respectively, the velocity potential is

$$\Phi_0 = \frac{\mathrm{i}\Phi l}{\kappa_1} A_q[\exp(-\mathrm{i}\Omega t) - \exp(\mathrm{i}\Omega t)]\sin\frac{\pi x}{l}\exp\left(-\frac{\kappa_1\beta}{l}\right)$$
$$+ \frac{\mathrm{i}\omega_0 l}{\kappa}[C\exp(-\mathrm{i}\omega_0 t) - \bar{C}\exp(\mathrm{i}_0 t)]\sin(\pi x/l)\exp(-\kappa\beta/l). \tag{77}$$

Similarly as in the previous case of weak excitation one can search for a stationary solution at $\sigma = 0$ and put derivatives of C on slow time to zero.

Then the problem of order ε^1 becomes

$$\frac{h^2}{12} w_1^{(4)} - \frac{\rho_0(1-\nu^2)}{Eh}(\Phi_1)_t + \frac{\rho(1-\nu^2)}{E}\ddot{w}_1$$
$$= \frac{1-\nu^2}{2l} w_0'' \int_0^l (w_0')^2 \, \mathrm{d}x + \frac{\rho_0(1-\nu^2)}{Eh}\left[-\dot{w}_0(\Phi_0)_\beta + \frac{1}{2}(\Phi_0)_s^2 + \frac{1}{2}(\Phi_0)_\beta^2\right], \tag{78a}$$

$$(\Phi_1)_{\beta\beta} + (\Phi_1)_{ss} - \frac{1}{c_0^2}(\Phi_1)_{tt} = -\frac{2}{c_0^2}\dot{w}_0(\Phi_0)_{\beta t} - \frac{1}{c_0^2}\ddot{w}_0(\Phi_0)_\beta + (w_0)_{ss}(\Phi_0)_\beta + 2w_0(\Phi_0)_{\beta s},$$

$$-(\Phi_1)_\beta + (w_1)_t = -(w_0)_s(\Phi_0)_s. \tag{78b}$$

One can now search for a solution of the problem (78) as $\Phi_1 = \Phi_{11} + \Phi_{12} + \Phi_{13}$. Then equations (78) are split into a set of three problems:

$$(\Phi_{11})_{\beta\beta} + (\Phi_{11})_{ss} - \frac{1}{c_0^2}(\Phi_{11})_{tt} = 0, \quad (\Phi_{11})_\beta = -w_0(\Phi_0)_s, \tag{79a, b}$$

$$(\Phi_{12})_{\beta\beta} + (\Phi_{12})_{ss} - \frac{1}{c_0^2}(\Phi_{12})_{tt} = -\frac{2}{c_0^2}\dot{w}_0(\Phi_0)_{\beta t} - \frac{1}{c_0^2}\ddot{w}_0(\Phi_0)_\beta + (w_0)_{ss}(\Phi_0)_\beta + 2w_0(\Phi_0)_{\beta s} \quad (\Phi_{12})_\beta = 0, \tag{80a, b}$$

$$(\Phi_{13})_{\beta\beta} + (\Phi_{13})_{ss} - \frac{1}{c_0^2}(\Phi_{13})_{tt} = 0, \quad (\Phi_{13})_\beta = \dot{w}_1. \tag{81a, b}$$

A dependence of all functions herein upon the axial co-ordinate s is taken in a simple approximation $\sin(\pi s/l)$. Then the solution for a linear in w_1, Φ_1 problem (78a), (81) should not contain secular terms generated by the inhomogeneous part of equation (78b), so that it is necessary to find relevant components of potentials Φ_{11} and Φ_{12}. Solutions of the problems (79) and (80) that produce secular terms in equation (78a) are

$$\Phi_{11} = -\frac{4\pi}{3}\frac{\mathrm{i}\Omega}{\kappa\kappa_1}\sin\frac{\pi s}{l}\exp\left(-\frac{\beta\kappa}{l}\right)A_q^2(\exp(-2\mathrm{i}\Omega t) - \exp(2\mathrm{i}\Omega t)), \tag{82a}$$

$$\Phi_{12} = \frac{8\mathrm{i}\Omega}{3\pi}\left[\frac{\kappa_1}{\kappa}\exp\left(-\frac{\beta\kappa}{l}\right) - \exp\left(-\frac{\beta\kappa_1}{l}\right)\right]\sin\frac{\pi s}{l}A_q^2(\exp(-2\mathrm{i}\Omega t) - \exp(2\mathrm{i}\Omega t)). \tag{82b}$$

As one substitutes velocity potentials (77) and (82) into equation (78a) of the structural dynamics, performs orthogolalization to the selected trial function $\sin(\pi s/l)$ and collects secular terms, one arrives at the following modulation equation (one puts $\sigma = 0$ and $X = a\exp(\mathrm{i}\varphi)$):

$$\left(\frac{a}{l}\right)^3 + 2\left(\frac{A_q}{l}\right)^2\frac{a}{l} + \frac{16}{3\pi^4}\frac{\rho_0}{\rho}\left(\frac{c_0}{c}\right)^2\left(\frac{\Omega l}{c_0}\right)^2\frac{l}{h}\left(\frac{A_q}{l}\right)^2\left\{\frac{2\pi}{3\kappa\kappa_1} + \frac{1}{\pi} - \frac{4\kappa_2}{3\pi\kappa} + \frac{\pi}{6\kappa_1^2}\right\} = 0, \tag{83}$$

This equation gives a control of the amplitude of vibrations at the resonance frequency solely by the non-linearity in the acoustic medium. One should note that, unlike the former case (section 3.2), the acoustic medium does not exhibit its properties of compressibility and in effect behaves simply as an added mass. However, from the standpoint of the structural dynamics the effect defined by equation (83) is of practical importance.

In particular, for the set of parameters discussed in section 3.2, excitation by a non-dimensional driving force of $Q = 1{\cdot}0 \times 10^{-6}$ produces a forced response of $a/l = 6{\cdot}33 \times 10^{-4}$ at the resonance frequency. One may easily see, that for most cases of practical relevance (say, steel plate in water, composite plate in air, etc.)

the magnitude of vibration at the resonance frequency is much smaller than at the driving frequency $\omega_0/2$. However, as one neglects the non-linearity in the structural–acoustic coupling there is no excitation of resonant vibrations at all. It is assumed that in some specific exploitation conditions even minor disturbances at a resonance frequency may be extremely dangerous and, hence, should be avoided.

4. CONCLUSIONS

An investigation has been done of the non-linear formulation of a problem of structural–acoustic interaction in the framework of the potential theory of hydrodynamics of a compressible fluid. The motivation for re-examining of this problem is a necessity for consistently describing structural non-linearities and non-linearity of compatibility conditions for both the light fluid loading and the heavy fluid loading of a structure.

A model has been suggested for a heavy fluid loading of a structure exhibiting non-linear oscillations. It has been shown that at the surface of a structure one has to take into account the quadratic (velocity head) term in the Bernoulli integral for a pressure. However, this does not result in formation of shock waves in a far field zone because the velocity potential is governed by a linear wave equation.

Two examples have been explored in detail by a use of multiple scales method. Attention has been focused on qualitative effects that may be described by a simple approximation. At the primary resonance acoustical non-linearity does not control structural dynamics, but produces an acoustic wave of a doubled frequency. In the case of linear structural dynamics the non-linearity of the contact pressure is proved to be the only source of effects at subharmonic resonance.

Two phenomena have been discovered in the framework of the suggested theory for non-linear oscillations of an infinite plate under heavy fluid loading conditions. The first one is generation of a sound wave at the weak excitation of a heavily fluid-loaded plate at its primary resonance. A field in the acoustic medium at the resonance frequency is described by a decaying exponent; i.e., it is not of a propagating nature. However, it has been shown that for a specific range of parameters non-linearity in fluid–structure interaction produces a propagating wave of doubled frequency. Another effect concerns structural dynamics at the frequency of $\omega/2$. In these excitation conditions the fluid's behaviour remains non-acoustical, but resonant oscillations of the plate are controlled solely by the non-linearity of the structural–acoustic coupling.

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