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ASYMPTOTIC BEHAVIOUR OF AN OSCILLATOR EXCITED BY DRY FRICTION FORCES

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A single-degree-of-freedom damped oscillator excited by dry friction forces of unknown exact characteristics is considered. Asymptotic behaviour of the oscillator has been studied by using the method of optimal Lyapunov functions. Optimal estimates of the limit region have been obtained for various bounding functions. Four illustrative examples of Lyapunov function optimization and parameter modification are given.

1. INTRODUCTION

Dry friction phenomena play an important role in many mechanical systems [1]. It is well known that dry friction forces always have non-linear characteristics and can cause very complex behaviour of a given system: e.g., non-linear excited (stick–slip) oscillations [2, 3] or a chaotic motion and bifurcations [1, 4, 5].

Since mechanical systems with dry friction are non-linear in principle, the analysis of properties of such systems is difficult. Even in the case of single-degree-of-freedom systems usually approximate and/or numerical methods are applied [4, 5]. Additional difficulties follow from the fact that friction characteristics in a given system usually are not known exactly and can vary in time: e.g., due to thermal and/or stick effects [1]. Therefore, qualitative methods of analysis of properties of mechanical systems with dry friction applicable when information about friction characteristics is incomplete seem to be an appropriate theoretical approach to such systems as an alternative to numerical methods.

The second method of Lyapunov is widely used for the stability analysis of smooth and non-smooth dynamical systems and systems with dry friction in particular. Both smooth [7] and non-smooth [8] Lyapunov functions can be applied.

In this paper the method of optimal quadratic Lyapunov functions [9–11] is applied to the problem of asymptotic behaviour of a single-degree-of-freedom damped oscillator excited by dry friction forces, namely the oscillator which can

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be described by the differential equation

$$\ddot{x}_1 + 2p\dot{x}_1 + qx_1 = f(t, \nu - \dot{x}_1), \tag{1}$$

where $p > 0$ is a damping coefficient, $q > 0$ is the stiffness, f is a function representing dry friction force and $v = v(t)$ is a background speed function (here the normalized case with the mass parameter $m = 1$ is assumed). Many practical systems can be described by equation (1): e.g., a pipe–soil underground system exposed to an earthquake excitation [6].

Vibrations of system (1) excited by Coulomb friction forces and a harmonic excitation $v(t)$ have been analyzed: e.g., in reference [6]. However, in practice one often has to deal with systems described by equation (1) when the friction force f is not known exactly. At most some information on qualitative properties of the friction force is available. Therefore, in this paper a more practical case is considered in which a bounding function $g = g(w)$ is known such that the condition

$$|f(t, w)| \leq g(w) \tag{2}$$

is satisfied for every relative speed $w = v(t) - \dot{x}_1$ and $t \geqslant t_0$. Moreover, the case of a general bounded excitation is considered: i.e., it is assumed that $v(t)$ is any function such that $|v(t)| \leqslant v_0$ for a given $v_0 > 0$ and for every $t \geqslant t_0$. Then one can treat the friction force as a disturbance of the linear part of system (1) and estimate the stability region of the system without the exact information upon friction characteristics.

The method of optimal Lyapunov functions [9–11] is applied to the stability analysis.

2. STABILITY OF EXCITED OSCILLATIONS

In this section a general conception is described of the method of optimal Lyapunov functions in application to linear systems under external excitations [9]. Concentration here is on a class of linear non-autonomous systems of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}z, \tag{3}$$

where $\mathbf{x} \in R^n$ is a state vector, $\mathbf{A}$ is a stable real $n \times n$ matrix, $\mathbf{B} \in R^n$ is a constant vector and z is a scalar exciting signal which is assumed to be dependent both on t and $\mathbf{x}$: i.e., $z = z(t, \mathbf{x})$, in general.

Since the matrix $\mathbf{A}$ is stable, there exists a positive definite matrix $\mathbf{S}$ such that the stability index

$$\gamma_0(\mathbf{S}) = -\sup_{\mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^{\mathrm{T}}\mathbf{S}\,\mathbf{A}\,\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{S}\,\mathbf{x}} \tag{4}$$

is positive [9–11]. Then the quadratic form $V_{\mathbf{S}}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{x}$ is a Lyapunov function of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ which is exponentially stable with respect to the norm $\|\cdot\|_{\mathbf{S}} = \sqrt{V_{\mathbf{S}}(\mathbf{x})}$.

In order to study stability properties of equation (3) one estimates the stability index [9, 10]

$$\gamma(\mathbf{S}, c) = -\sup_{\mathbf{x},\, t}\left[\frac{\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{x}} + \frac{\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{x}} z(\mathbf{x}, t)\right], \tag{5}$$

where the supremum is over $t \geqslant t_0$ and $\mathbf{x} \in \{\mathbf{y} : \|\mathbf{y}\|_{\mathbf{S}} = c\}$. It is known [9, 10] that the condition $\gamma(\mathbf{S}, c) > 0$ for a given $c > 0$ guarantees that every extern trajectory of the system reaching the surface of the ellipsoid $X(\mathbf{S}, c) = \{\mathbf{x} : \|\mathbf{x}\|_{\mathbf{S}} < c\}$ enters the ellipsoid. One says that system (3) is exponentially stable in $X(\mathbf{S}, c_1)$ with a limit region contained in $X(\mathbf{S}, c_2)$ if index (5) is positive for every $c \in (c_2, c_1)$ and negative for $c < c_2$.

In order to estimate index (5) without the exact knowledge of the perturbation z one assumes that a stationary bound for $z(\mathbf{x}, t)$ is known: i.e., there is a positive function $g = g(\mathbf{x})$ such that

$$\mathop{\forall}_{t \geqslant t_0} \mathop{\forall}_{\mathbf{x}} |z(\mathbf{x}, t)| \leqslant g(\mathbf{x}). \tag{6}$$

If one additionally assumes that $g(\mathbf{x})$ is a polynomial function (i.e., $g(\mathbf{x}) = g_0 + g_1(\mathbf{x}) + g_2(\mathbf{x}) + \cdots + g_k(\mathbf{x})$, where $g_i(\mathbf{x})$; $i = 1, \ldots, k$ is a homogeneous function of order i and g_0 is a positive real), one obtains the estimate

$$\begin{aligned}\gamma(\mathbf{S}, c) &\geqslant -\sup_{\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{x}=c^2}\left[\frac{\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{x}} + \frac{|\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{x}|}{\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{x}} g(\mathbf{x})\right] \geqslant \gamma^{\sim}(\mathbf{S}, c) \\ &= -\sup_{\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{x}=1,}\left[\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{x} + |\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{x}|\left(h_1(\mathbf{x}) + ch_2(\mathbf{x}) + \ldots + c^{k-1}h_k(\mathbf{x}) + \frac{g_0}{c}\right)\right],\end{aligned} \tag{7}$$

where $h_i(\mathbf{x}) = |g_i(\mathbf{x})|$, $i = 1, \ldots, k$.

It can be proved [10] that $\gamma^{\sim}(\mathbf{S}, c)$ given by expression (7) is a continuous function of c for $c > 0$. Moreover, as one can easily find out, $\gamma^{\sim}(\mathbf{S}, c) \to -\infty$ as $c \to 0$. Similarly, if $k > 1$, then $\gamma^{\sim}(\mathbf{S}, c) \to -\infty$ as $c \to +\infty$. Thus one can hope only that the index is positive in a certain range (c_2, c_1), if the perturbation z is small enough.

To prove the above hypothesis one can make a further estimate

$$\gamma^{\sim}(\mathbf{S}, c) \geqslant \gamma_0(S) - \left(\varepsilon_1 + c\,\varepsilon_2 + \cdots + c^{k-1}\varepsilon_k + \frac{g_0}{c}||\mathbf{B}||_{\mathbf{S}}\right), \tag{8}$$

where $\gamma_0(\mathbf{S})$ is given by equation (4) and

$$\varepsilon_i = \varepsilon_i(\mathbf{S}) = \sup_{\mathbf{x}^{\mathrm{T}}\mathbf{S}\,\mathbf{x}=1}[|\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{x}||g_i(\mathbf{x})|], \quad i = 1, \ldots, k. \tag{9}$$

Now, it is easy to see that $\gamma^{\sim}(\mathbf{S}, c)$ is positive in a certain range of c, if $\gamma_0(\mathbf{S}) > \varepsilon_1$ and g_0 is sufficiently small.

3. ASYMPTOTIC BEHAVIOUR OF THE OSCILLATOR WITH DRY FRICTION

One can now apply the method of Lyapunov functions described in the previous section to the problem of asymptotic behaviour of the motion of oscillator (1). First one can rewrite equation (1) into the matrix form (3) where $\mathbf{x} = [x_1, x_2 = \dot{x}_1]^{\mathrm{T}}$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -q & -2p \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, z(t, \mathbf{x}) = f(t, \nu(t) - \mathbf{B}^{\mathrm{T}}\mathbf{x}.) \tag{10}$$

One can perform stability analysis separately for certain bounding functions which seem to be important in practice and are convenient for theoretical analysis, simultaneously. Namely, one can consider three cases; constant bounding, linear increasing bounding and quadratic bounding functions.

3.1. CONSTANT BOUNDING

Suppose that the friction force modulus can be bounded by a constant $g(w) = g_0 = a > 0$. Then stability index (5) for oscillator (1) can be estimated as

$$\begin{aligned}
\gamma(\mathbf{S}, c) &= -\sup_{\mathbf{x}^{\mathrm{T}}\mathbf{S}\,\mathbf{x}=1}\left[\mathbf{x}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{x} + \frac{a}{c}|\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{x}|\right] \geq \gamma^{\sim}(\mathbf{S}, c) \\
&= -\frac{\mathbf{C}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{C}}{||\mathbf{C}||_{\mathrm{S}}^2} - \frac{a}{c}||\mathbf{B}||_{\mathrm{S}} - \sup_{\xi \in \langle 0, 1\rangle}\left[\xi^2\left(\frac{\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{B}}{||\mathbf{B}||_{\mathrm{S}}^2} - \frac{\mathbf{C}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{C}}{||\mathbf{C}||_{\mathrm{S}}^2}\right)\right. \\
&\qquad \left. + \frac{|\mathbf{B}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{S} + \mathbf{S}\mathbf{A})\mathbf{C}|}{||\mathbf{B}||_{\mathrm{S}}||\mathbf{C}||_{\mathrm{S}}}\xi\sqrt{1-\xi^2}\right] = -\frac{1}{2}\left(\frac{\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{B}}{||\mathbf{B}||_{\mathrm{S}}^2} + \frac{\mathbf{C}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{C}}{||\mathbf{C}||_{\mathrm{S}}^2}\right) \\
&\qquad - \frac{a}{c}||\mathbf{B}||_{\mathrm{S}} - \frac{1}{2}\sqrt{\left(\frac{\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{B}}{||\mathbf{B}||_{\mathrm{S}}^2} - \frac{\mathbf{C}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{C}}{||\mathbf{C}||_{\mathrm{S}}^2}\right)^2 + \frac{|\mathbf{B}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{S} + \mathbf{S}\mathbf{A})\mathbf{C}|^2}{||\mathbf{B}||_{\mathrm{S}}||\mathbf{C}||_{\mathrm{S}}}} \\
&= \gamma_0(\mathbf{S}) - \frac{a}{c}||\mathbf{B}||_{\mathrm{S}},
\end{aligned} \tag{11}$$

where $\mathbf{A}$, $\mathbf{B}$ are given by expression (10) and $\mathbf{C} \neq 0$ is a vector $\mathbf{S}$-orthogonal to $\mathbf{B}$ i.e., such that $\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{C} = 0$. It is easy to derive from the stability condition $\gamma^{\sim}(\mathbf{S}) > 0$ the following estimate for the radius of the limit ellipsoid:

$$c_2 \leq c_2^{\sim}(\mathbf{S}) = \frac{a}{\gamma_0(\mathbf{S})}||\mathbf{B}||_{\mathrm{S}}. \tag{12}$$

3.2. LINEAR BOUNDING

Suppose that the friction force modulus can be bounded by a linear function of the relative speed $g(w) = a|w| + b$, where a, b are positive reals. Since the bounding function $g(w) = g(v(t) - x_2) = g(v(t) - \mathbf{B}^{\mathrm{T}}\mathbf{x}) \leqslant g(v_0) + a|\mathbf{B}^{\mathrm{T}}\mathbf{x}|$, the stability index (5) of oscillator (1) can be estimated as follows:

$$\gamma(\mathrm{S},c) \geqslant -\sup_{\mathbf{x}^{\mathrm{T}}\mathbf{S}\,\mathbf{x}=1} [\mathbf{x}^{\mathrm{T}}\mathbf{SAx} + |\mathbf{B}^{\mathrm{T}}\mathbf{Sx}|(a|\mathbf{B}^{\mathrm{T}}x| + (av_0+b)/c]$$

$$\geqslant \gamma^{\sim}(\mathbf{S},c) = \gamma_1(\mathbf{S}) - \frac{g_1(v_0)||\mathbf{B}||_{\mathbf{S}}}{c} = -\frac{\mathbf{C}^{\mathrm{T}}\mathbf{SAC}}{||\mathbf{C}||_{\mathbf{S}}^2} - (av_0+b)||\mathbf{B}||_{\mathbf{S}}/c$$

$$-\sup_{\xi\in\langle 0,\,1\rangle}\left[\xi^2\left(\frac{\mathbf{B}^{\mathrm{T}}\mathbf{SAB}}{||\mathbf{B}||_{\mathbf{S}}^2} - \frac{\mathbf{C}^{\mathrm{T}}\mathbf{SAC}}{||\mathbf{C}||_{\mathbf{S}}^2} + a\mathbf{B}^{\mathrm{T}}\mathbf{B}\right)\right.$$

$$\left.+\left(\frac{|\mathbf{B}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{S}+\mathbf{SA})\mathbf{C}}{||\mathbf{B}||_{\mathbf{S}}||\mathbf{C}||_{\mathbf{S}}} + a||\mathbf{B}||_{\mathbf{S}}\frac{|\mathbf{B}^{\mathrm{T}}\mathbf{C}|}{||\mathbf{C}||_{\mathbf{S}}}\right)\xi\sqrt{1-\xi^2}\right]$$

$$= -\frac{1}{2}\left(\frac{\mathbf{B}^{\mathrm{T}}\mathbf{SAB}}{||\mathbf{B}||_{\mathbf{S}}^2} + \frac{\mathbf{C}^{\mathrm{T}}\mathbf{SAC}}{||\mathbf{C}||_{\mathbf{S}}^2} + a\mathbf{B}^{\mathrm{T}}\mathbf{B}\right) - \frac{av_0+b}{c}||\mathbf{B}||_{\mathbf{S}}$$

$$-\frac{1}{2}\sqrt{\left(\frac{\mathbf{B}^{\mathrm{T}}\mathbf{SAB}}{||\mathbf{B}||_{\mathbf{S}}^2} - \frac{\mathbf{C}^{\mathrm{T}}\mathbf{SAC}}{||\mathbf{C}||_{\mathbf{S}}^2} + a\mathbf{B}^{\mathrm{T}}\mathbf{B}\right)^2 + \left(\frac{|\mathbf{B}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{S}+\mathbf{SA})\mathbf{C}|}{||\mathbf{B}||_{\mathbf{S}}||\mathbf{C}||_{\mathbf{S}}} + a||\mathbf{B}||_{\mathbf{S}}\frac{|\mathbf{B}^{\mathrm{T}}\mathbf{C}|}{||\mathbf{C}||_{\mathbf{S}}}\right)^2} \tag{13}$$

As in the previous case, it is very easy to derive from the stability condition $\gamma^{\sim}(\mathbf{S}) > 0$ the following estimate for the radius of the limit ellipsoid:

$$c_2 \leqslant c_2^{\sim}(\mathbf{S},v_0) = \frac{(av_0+b)}{\gamma_1(\mathbf{S},a)}||\mathbf{B}||_{\mathbf{S}}. \tag{14}$$

This is applicable if $\gamma_1(\mathbf{S},a) > 0$.

3.3. QUADRATIC BOUNDING

Suppose that the friction force modulus can be bounded by a quadratic function of the relative speed $g(w) = aw^2 + b|w| + d$, where a, d are positive reals and $b \geqslant 0$. Then, since $g(w) = g(v(t) - \mathbf{B}^{\mathrm{T}}\mathbf{x}) \leqslant g(v_0) + (2av_0 + b)\,|\mathbf{B}^{\mathrm{T}}\mathbf{x}| + a(\mathbf{B}^{\mathrm{T}}\mathbf{x})^2$, the stability index (5) of oscillator (1) can be estimated as follows

$$\gamma(\mathbf{S},c) \geqslant -\sup_{\mathbf{x}^{\mathrm{T}}\mathbf{Sx}=1}\left[\mathbf{x}^{\mathrm{T}}\mathbf{SAx} + (2av_0+b)|\mathbf{B}^{\mathrm{T}}\mathbf{S}\,\mathbf{x}||\mathbf{B}^{\mathrm{T}}\mathbf{x}|\right.$$

$$\left.+ac|\mathbf{B}^{\mathrm{T}}\mathbf{Sx}|(\mathbf{B}^{\mathrm{T}}\mathbf{x})^2 + \frac{(av_0^2+bv_0+d)}{c}|\mathbf{B}^{\mathrm{T}}\mathbf{Sx}|\right]$$

$$\geqslant -\frac{\mathbf{C}^{\mathrm{T}}\mathbf{SAC}}{||\mathbf{C}||_{\mathbf{S}}^{2}} - ac||\mathbf{B}||_{\mathbf{S}} \sup_{\xi\in\langle 0,\,1\rangle}\left[\xi\left(\frac{\mathbf{B}^{\mathrm{T}}\mathbf{B}}{||\mathbf{B}||_{\mathbf{S}}}\xi + \frac{\mathbf{B}^{\mathrm{T}}\mathbf{C}}{||\mathbf{C}||_{\mathbf{S}}}\sqrt{1-\xi^{2}}\right)^{2}\right]$$

$$-\frac{g(v_0)}{c}||\mathbf{B}||_{\mathbf{S}} - \sup_{\xi\in\langle 0,\,1\rangle}\left[\xi^{2}\left(\frac{\mathbf{B}^{\mathrm{T}}\mathbf{SAB}}{||\mathbf{B}||_{\mathbf{S}}^{2}} - \frac{\mathbf{C}^{\mathrm{T}}\mathbf{SAC}}{||\mathbf{C}||_{\mathbf{S}}^{2}} + (2av_0+b)\mathbf{B}^{\mathrm{T}}\mathbf{B}\right)\right.$$

$$\left. + \left(\frac{|\mathbf{B}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{S}+\mathbf{SA})\mathbf{C}|}{||\mathbf{B}||_{\mathbf{S}}||\mathbf{C}||_{\mathbf{S}}} + (2av_0+b)||\mathbf{B}||_{\mathbf{S}}\frac{|\mathbf{B}^{\mathrm{T}}\mathbf{C}|}{||\mathbf{C}||_{\mathbf{S}}}\right)\xi\sqrt{1-\xi^{2}}\right]$$

$$\geqslant \gamma^{\sim}(\mathbf{S}, c) = -\frac{1}{2}\left[\frac{\mathbf{B}^{\mathrm{T}}\mathbf{SAB}}{||\mathbf{B}||_{\mathbf{S}}^{2}} + \frac{\mathbf{C}^{\mathrm{T}}\mathbf{SAC}}{||\mathbf{C}||_{\mathbf{S}}^{2}} + (2av_0+b)\mathbf{B}^{\mathrm{T}}\mathbf{B}\right]$$

$$-\frac{1}{2}\left[\left(\frac{\mathbf{B}^{\mathrm{T}}\mathbf{SAB}}{||\mathbf{B}||_{\mathbf{S}}^{2}} - \frac{\mathbf{C}^{\mathrm{T}}\mathbf{SAC}}{||\mathbf{C}||_{\mathbf{S}}^{2}} + (2av_0+b)\mathbf{B}^{\mathrm{T}}\mathbf{B}\right)^{2}\right.$$

$$\left. + \left(\frac{|\mathbf{B}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{S}+\mathbf{SA})\mathbf{C}|}{||\mathbf{B}||_{\mathbf{S}}||\mathbf{C}||_{\mathbf{S}}} + (2av_0+b)||\mathbf{B}||_{\mathbf{S}}\frac{|\mathbf{B}^{\mathrm{T}}\mathbf{C}|}{||\mathbf{C}||_{\mathbf{S}}}\right)^{2}\right]^{1/2}$$

$$- ac||\mathbf{B}||_{\mathbf{S}}\left[\left(\frac{\mathbf{B}^{\mathrm{T}}\mathbf{B}}{||\mathbf{B}||_{\mathbf{S}}}\right)^{2} + \left(\frac{\mathbf{B}^{\mathrm{T}}\mathbf{C}}{||\mathbf{C}||_{\mathbf{S}}}\right)^{2}\right]^{1/2}$$

$$-\frac{(av_0^2+bv_0+d)}{c}||\mathbf{B}||_{\mathbf{S}} = \gamma_2(\mathbf{S}, 2av_0+b) - ac\delta_2 - \frac{g(v_0)||\mathbf{B}||_{\mathbf{S}}}{c}.$$

The stability condition $\gamma^{\sim}(\mathbf{S}) > 0$ leads in this case to a quadratic inequality which has solutions if and only if the following condition of stability is satisfied:

$$\gamma_2^2(\mathbf{S}, 2av_0+b) \geqslant 4a\delta_2(\mathbf{S})\, g(v_0)||\mathbf{B}||_{\mathbf{S}}. \tag{16}$$

Then one obtains the critical values, $c_1 > c_2$,

$$c_{1,\,2} = \frac{\gamma_2(\mathrm{S}) \pm \sqrt{\gamma_2^2(\mathbf{S}) - 4a\delta_2(\mathbf{S})g(v_0)||\mathbf{B}||_{\mathbf{S}}}}{2a\delta_2(\mathbf{S})} \tag{17}$$

such that the approximate index $\gamma^{\sim}(\mathbf{S}, c)$ is positive for every $c \in (c_2, c_1)$. Thus, estimate (15) ensures that every trajectory of the system, starting from the ellipsoid $X(\mathbf{S}, c_1)$, converge exponentially to the limit ellipsoid $X(\mathbf{S}, c_2)$.

4. LYAPUNOV FUNCTION OPTIMIZATION AND PARAMETERS MODIFICATION

In the previous section useful estimates were obtained of the stability index and the limit ellipsoid of oscillator (1) with various assumed boundings for dry friction characteristics. One can now show how optimal estimates of the limit region of the system can be obtained and how to choose system parameters in order to achieve optimal stability properties.

It is assumed that one has an estimate $\gamma^{\sim}(\mathbf{S}, c)$ of the stability index of system (1) and the corresponding estimate $c_2^{\sim} \geqslant c_2$ of the radius of the limit ellipsoid derived from the stability condition $\gamma^{\sim}(\mathbf{S}, c) > 0$. It is clear that the radius $c_2^{\sim}$ depends, in general, on the choice of Lyapunov function as well as on certain parameters $\mathbf{p}$ of the system: i.e., $c_2^{\sim} = c_2^{\sim}(\mathbf{S},\mathbf{p})$.

If system parameters are fixed (i.e., $\mathbf{p} = \mathbf{p}_0 = \text{const.}$), then one usually wants to find an optimal estimate of the limit region of the system. To do that one has to determine a quality function $Q = Q(\mathbf{S}, \mathbf{p}_0)$ (e.g., a measure of the ellipsoid $X(\mathbf{S}, c_2^{\sim}(\mathbf{S}, \mathbf{p}_0))$ and perform an optimization with respect to $\mathbf{S}$ belonging to a class of positive definite matrices $n \times n$. In the result one obtains an optimal Lyapunov function $\mathbf{x}^{\mathrm{T}}\hat{\mathbf{S}}\mathbf{x}$ and the corresponding optimal estimate $\hat{X}_2 = X(\hat{\mathbf{S}}, c_2(\hat{\mathbf{S}}, \mathbf{p}_0)$ of the limit region.

In order to provide illustrative examples one can consider system (1) with $p \in (0, 1)$ and $q = 1$, and apply a one-parameter class of Lyapunov functions

$$V_S(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{S}(s)\mathbf{x} = \mathbf{x}^{\mathrm{T}}\begin{bmatrix} 2s^2 - 2ps + 1, & s \\ s, & 1 \end{bmatrix}\mathbf{x}, \quad s \in R, \tag{18}$$

which proves to be especially convenient in calculations [9, 10]. It is easy to see that $\mathbf{S}(s)$ is positive definite for all $s \in R$ and $p \in (0, 1)$. Thus, one can perform Lyapunov function optimization with respect to the parameter $s \in R$.

Example 1. Perform Lyapunov function optimization in the class of positive definite quadratic forms (18) for system (1) with constant bounding function $g(w) = a > 0$. Assume the radius $c_2(\mathbf{S})$ as the minimized quality function.

Solution. Applying equation (18) to estimates (11) one obtains the following estimate of the stability index [5]:

$$\gamma^{\sim}(\mathbf{S}, c) = \left\{ \begin{array}{ll} 2p - s - \dfrac{a}{c}, & \text{if} \quad s \geqslant p \quad \text{or} \quad s < p \quad \text{and} \quad c \leqslant \dfrac{a}{4(p-s)} \\ s - \dfrac{a^2}{8c^2(p-s)}, & \qquad\qquad \text{if} \quad c < p \quad \text{and} \quad c > \dfrac{a}{4(p-s)} \end{array} \right\} \tag{19}$$

An estimate of the radius c_2 of the limit ellipsoid $X(\mathbf{S}, c_2)$ can be easily derived from formula (19). For example, taking $s = p$ one obtains $c_2 = c_2(\mathbf{S}) = a/p$. One can also perform Lyapunov function optimization with respect to the parameter $s \in R$ under a suitable quality index. It is easy to deduce from equation (19) that the minimal radius c_2 is achieved for $s = \hat{s} = 2p/3$ and $\hat{c}_2 = c_2(\mathbf{S}(\hat{s}), p) = 3a/4p$. Thus, all trajectories of the system converge exponentially to the ellipsoid $\{(x_1, x_2) \in R^2 : (1 - 8p^2/9)x_1^2 + 4px_1x_2/3 + x_2^2 \leqslant 9a^2/16p^2\}$.

It is clear that the ellipsoid obtained can be a conservative estimate of the limit region of system trajectories in certain special cases (e.g., for high frequency harmonic excitation). However, if one wants to estimate safe stability bounds in the case of uncertain information on friction forces and excitations, then one should consider the most disadvantageous scenario of the background excitation. It is usually assumed that the dry friction force is opposite to speed $\dot{x}_1$ (i.e.,

$f = -a\ \mathrm{sign}(\dot{x}_1)$ for the Coulomb model). In this case, however, one can imagine a background motion controlled in such a way that the friction force is always in agreement with the speed direction: i.e., $f = a\ \mathrm{sign}(\dot{x}_1)$. Then the work done by friction forces is continuously pumped into the system. Therefore, one can expect, in such a case, the maximal limit region of system trajectories. More precisely, a limit cycle can be expected as the result of frictional pumping and viscous damping in the system. It is illustrated in Figure 1 for the case of oscillator (1) with $p = 0{\cdot}5$, $q = 1$, $a = 0{\cdot}8$.

It is significant that external trajectories enter the limit ellipsoid and converge to the limit cycle although they are not exponentially stable inside the ellipsoid.

Example 2. Find the minimal radius of the limit ellipsoid for system (1) with linear bounding function $g(w) = a|w| + b$ and parameters $a = p = 0{\cdot}5$. perform Lyapunov function optimization in the class of functions defined by equation (18).

Solution. Applying expression (18) to formula (14), one obtains the following formula for the approximate index:

$$\gamma^{\sim}(\mathbf{S}, c) = p - \frac{a}{2} - \frac{1}{2}\sqrt{(2s + a - 2p)^2 + \frac{a^2 s^2}{s^2 - 2ps - 1}} - \frac{av_0 + b}{c}. \tag{20}$$

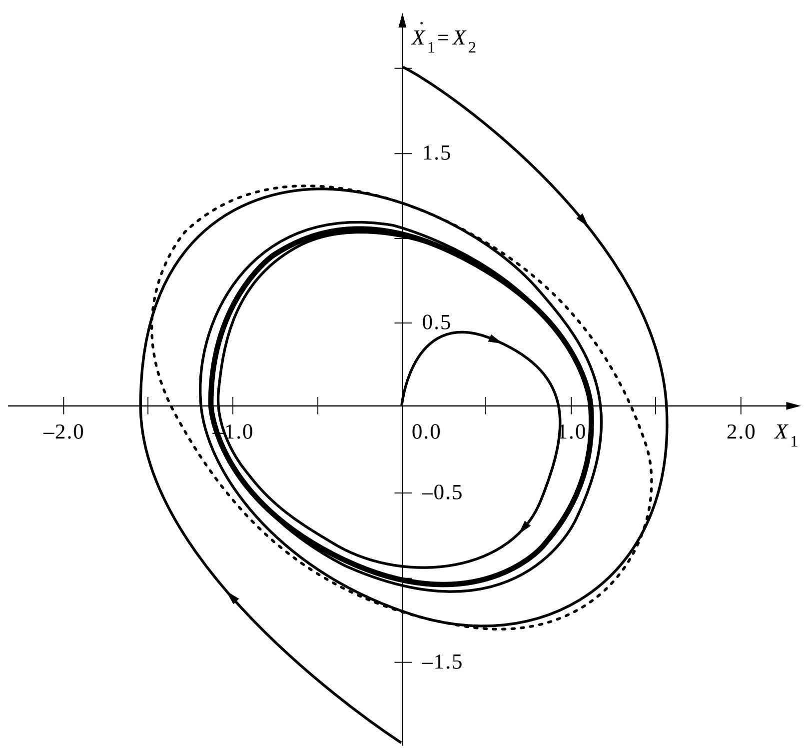

Figure 1. Phase portrait of oscillator (1) with $p = 0{\cdot}5$ and $q = 1$ excited by a Coulomb friction force of amplitude $a = 0{\cdot}8$ (the dotted ellipsoid $7x_1^2 + 6x_1x_2 + 9x_2^2 = 12{\cdot}96$ represents the bounds of the optimal Lyapunov estimate of the limit region).

This is applicable for $s \in (0, 2p)$. Hence, one has the following estimate for the radius of the limit ellipsoid:

$$c_2^{\sim} = \frac{av_0 + b}{p - \dfrac{a}{2} - \dfrac{1}{2}\left[(2s - 2p + a)^2 + \dfrac{a^2 s^2}{s^2 - 2ps + 1}\right]^{1/2}}. \tag{21}$$

It is easy to see that the minimal value of the radius $c_2^{\sim} \cong 2{\cdot}306 * (0{\cdot}5v_0 + b)$ is achieved for $s = \hat{s} \cong 0{\cdot}2311$.

Example 3. Find an estimate of the maximal upper bound v_0 for the speed v which ensures exponential convergence of all trajectories of the system to the limit ellipsoid $\{(x_1, x_2) \in R^2 : x_1^2 + 1{\cdot}6x_1x_2 + x_2^2 \leqslant 1/4\}$. Perform calculations for $a = 0{\cdot}5$, $b = 0$, $c = 0{\cdot}5$, $d = 0{\cdot}1$, $p = s = 0{\cdot}8$ and for the Lyapunov function $\mathbf{S}(s)$ belonging to class (18).

Solution. Applying expression (18) to estimates (15) one obtains the following formula for the approximate stability index:

$$\gamma^{\sim}(\mathbf{S}, c) = p - av_0 - \frac{b}{2} - \frac{av_0^2 + bv_0 + d}{c} - ac\left[\frac{2s^2 - 2sp + 1}{s^2 - 2sp + 1}\right]^{1/2}$$
$$- \frac{1}{2}\sqrt{(2s - 2p + 2av_0 + b)^2 + \frac{(2av_0 + b)^2 s^2}{s^2 - 2ps + 1}} \tag{22}$$

Putting the assumed values of parameters into equation (22) one obtains $\gamma^{\sim}(\mathbf{S}, c) = (11/60) - (8/6)\ v_0 - v_0^2$. Hence, one finally concludes that exponential stability of the system to the assumed limit ellipsoid is ensured for $v_0 < v^* \cong 0{\cdot}1256$.

If a mechanical system is at the design stage and some system parameters $\mathbf{p}$ are not fixed one usually wants to choose them in an optimal way in order to achieve the best stability properties of the system. In such a case one can perform a two-stage optimization. At the first stage one performs a Lyapunov function optimization (e.g., the minimization of a quality function $Q(\mathbf{S}, \mathbf{p})$ with respect to $\mathbf{S}$) for any parameters $\mathbf{p}$. In the result one obtains an optimal matrix $\hat{\mathbf{S}}$, usually dependent on the parameters: i.e., $\hat{\mathbf{S}} = \hat{\mathbf{S}}(\mathbf{p})$.

Then, at the second stage, one can perform further parameter optimization (e.g., the minimization of the quality function $Q(\hat{\mathbf{S}}(\mathbf{p}), \mathbf{p})$ with respect to $\mathbf{p}$ belonging to a set of admissible values of parameters). In the result one obtains an optimal vector of parameters $\hat{\mathbf{p}}$ for which the quality function achieves its minimum: i.e., system (1) with optimal parameters $\hat{\mathbf{p}}$ achieves the limit region of the minimal measure.

Example 4. Find an optimal damping parameter p for oscillator (1) with constant bounding on dry friction forces such that the system achieves a limit ellipsoid of a minimal area.

Solution. One has just performed Lyapunov function optimization for the system (see example 1) in class (18) of Lyapunov functions. In the result one has obtained the optimal parameter $\hat{s} = 2p/3$ and the corresponding optimal radius

$\hat{c}_2 = 3a/4p$. Now one can perform further optimization: namely, one can minimize the area of the limit ellipsoid $X(\mathbf{S}(\hat{s}), \hat{c})_2$. It is easy to see that the area is equal to

$$\mathbb{P} = \frac{27\pi a^2}{16p^2\sqrt{9-8p^2}}. \tag{23}$$

Hence, the optimal damping $\hat{p} = \sqrt{3}/2 \cong 0{\cdot}866$ and the minimal area is $\mathbb{P}_{MIN} = 3\sqrt{3}\pi a^2/4$.

5. CONCLUSIONS

The Lyapunov approach presented has proved to be an effective method for the analysis of qualitative properties of vibrating systems of one degree of freedom with dry friction. Lyapunov function optimization enables one to obtain optimal estimates of safe stability limits without exact information on friction characteristics and background excitation. This is essential in many practical problems when the exact system identification is not possible.

By using the described method it is also possible to perform an optimal parametric modification with respect to stability properties of a given system. it can be utilized in the designing of optimal mechanical systems excited by dry friction forces.

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