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A PERTURBATION APPROACH TO ANALYZE THE VIBRATION OF STRUCTURES CONVEYING FLUID

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A method is presented to analyze real structures, all or part of which convey fluid. With well established equations of motion taken as a starting point, the real and imaginary parts of the natural frequencies are calculated by using first and second order perturbation theory. It is shown that whilst first order theory is adequate to predict the imaginary part, a second order calculation is required to predict the behaviour of the real frequency variation. Comparison is made with finite element formulations for a cantilever system and it is shown that the method is valid over ranges of practical interest. The method is easily applied to a general structure as a post processor using calculated modal properties.

1. INTRODUCTION

The equations governing the vibrational behaviour of pipes conveying fluid have been understood for 40 years. The flow modifies the behaviour of the pipework owing to the transport of momentum during the vibration cycle. Whilst the physical understanding of this situation is straightforward, only simple situations of beams subject to simple boundary conditions have been analyzed. Even in the case of a simple straight beam the analysis is somewhat tedious. Whilst there are many engineering structures in which flow is an important feature the effects of flow on vibrational properties are not included in standard analysis packages. Indeed to include such features would involve a significant overhead.

In this paper a perturbation technique is proposed to take account of the fluid flow effects. As a test of the theory, a comparison is given with the exact results for a straight pipe and it is shown that the analysis is valid for a range of flow rates that will cover most areas of interest. Whilst the simple situation is used to prove the argument, the method can be applied to a structure of arbitrary complexity, requiring only a prior calculation of the (zero flow) natural frequencies and mode shapes of the structure concerned.

In section 2, a discussion is given of the equation for a straight pipe, and some features of the vibrational behaviour. The nature of the general solution for a cantilever pipe is explained and the difficulties of the solution are summarized.

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Section 3 introduces a finite element formulation for a general structure to derive the changes on mode shape and natural frequency due to the presence of flow. Section 4 presents the derivation of the perturbation method and in section 5, a comparison is made of the perturbation results and the exact results for the case of a straight pipe. Finally, the treatment of a more general structure including components conveying fluid is described.

2. ANALYSIS FOR A STRAIGHT PIPE

The vibratational behaviour of a pipe containing stationary fluid may be analyzed by using the Euler beam theory upon assuming it has the appropriate aspect ratio. Allowance for the mass of the fluid within the pipe is made simply by mass loading. When flow is introduced, two extra terms are introduced into the equations as derived by Gregory and Païdoussis [1] and discussed by Païdoussis and co-workers [2, 3]. The two terms describe the transport of lateral momentum and centripetal effects. With the transverse vibration amplitude denoted by y, the free motion is described by the equation

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A v^2 \frac{\partial^2 y}{\partial x^2} + 2\rho A v \frac{\partial^2 y}{\partial x \partial t} + M\frac{\partial^2 y}{\partial t^2} = 0, \tag{1}$$

where the symbols are defined in the Appendix. Note that ρ is the fluid density, v is the fluid velocity and M is the combined mass per unit length of pipe and fluid. End conditions are applied to this equation in the usual manner. As explained by Blevins [4], a solution of equation (1) is sought of the form

$$y(x,\, t) = Y(x)\mathrm{e}^{\mathrm{i}\Omega t}. \tag{2}$$

This leads to a non-dimensional equation

$$\frac{\mathrm{d}^4\eta}{\mathrm{d}\xi^4} + V^2\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} + 2\mathrm{i}\beta^{1/2}V\omega\frac{\mathrm{d}\eta}{\mathrm{d}\xi} - \omega^2\eta = 0, \tag{3}$$

in which the following substitutions have been made:

$$\eta = Y/L, \quad \xi = x/L, \quad \beta^2 = \rho A/M,$$

$$\omega = \Omega L^2 (M/EI)^{1/2}, \quad V = vL(\rho A/EI)^{1/2}, \quad M = \rho A + m. \tag{4}$$

Equation (3) for the spatial dependence is considerably complicated by the presence of the flow terms. A solution is sought of the form $\eta = \eta_0 \mathrm{e}^{s\xi}$. Inserting this trial form gives an equation for s. As shown in reference [1], because the roots of equation (3) become more complicated due to the flow terms, imposition of the boundary conditions results in a set of highly non-linear equations for the parameter s, which leads to the natural frequencies.

3. A FINITE ELEMENT STUDY

The dynamic equation (1) may be recast in terms of a finite element formulation. Chen and Fan [5] have given a formulation in terms of Timoshenko beam theory. In this paper the pipe motion is derived by using Euler theory. This is achieved by considering the virtual work done in increasing the deflection of the pipe by an amount δy, and then integrating to obtain

$$E_T = \int_0^L \delta y \left(EI \frac{\partial^4 y}{\partial x^4} + \rho A v^2 \frac{\partial^2 y}{\partial x^2} + 2\rho A v \frac{\partial^2 y}{\partial x \partial t} + M \frac{\partial^2 y}{\partial t^2} \right) \mathrm{d}x. \tag{5}$$

Here one observes that, by using standard Euler beam elements, the deflection may be expressed as

$$y(x) = \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix} \{\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3\}. \tag{6}$$

Note that the undetermined parameters α_n are related to the terminal conditions of each element via the relationship

$$\begin{Bmatrix} y_1 \\ \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_1 \\ y_2 \\ \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_2 \end{Bmatrix} = [C]\{\alpha\}. \tag{7}$$

Differentiation of the total energy yields a force equation given by

$$[K_{0e}]\{y\} + \rho A v^2 [K_{1e}]\{y\} + 2\rho A v [K_{2e}] \left\{ \frac{\partial y}{\partial t} \right\} + [M_{0e}] \left\{ \frac{\partial^2 y}{\partial t^2} \right\} = \{0\}, \tag{8}$$

which may be written in non-dimensional terms as

$$[k_{0e}]\{\eta\} + V^2 [k_{1e}]\{\eta\} + 2\beta^{1/2} V [k_{2e}] \left\{ \frac{\partial \eta}{\partial \tau} \right\} + [m_{0e}] \left\{ \frac{\partial^2 \eta}{\partial \tau^2} \right\} = \{0\}. \tag{8a}$$

In this equation, the terms in curly brackets represent corresponding nodal quantities, and $[k_{0e}]$, $[m_{0e}]$ are the usual element stiffness and mass matrices, respectively. Extra terms arise from the flow: in describing the first of these, the matrix is given by

$$[k_{1e}] = [C^{-1}]^{\mathrm{T}} \int_0^1 \begin{Bmatrix} 0 \\ 0 \\ 2 \\ 6\xi^2 \end{Bmatrix} \{1 \quad \xi \quad \xi^2 \quad \xi^3\} \,\mathrm{d}\xi [C^{-1}], \tag{9}$$

whilst the second term is given by

$$[k_{2e}] = [C^{-1}]^{\mathrm{T}} \int_0^1 \begin{Bmatrix} 0 \\ 1 \\ 2\xi \\ 3\xi^2 \end{Bmatrix} \{1 \quad \xi \quad \xi^2 \quad \xi^3\} \, \mathrm{d}\xi [C^{-1}]. \tag{10}$$

Note that neither of these matrices is necessarily symmetric. This reflects the physics of the fluid-conveying pipe, which in general will not be a conservative system. The completion of the multiplication and integration is straightforward giving

$$[k_{1e}] = \frac{1}{30l} \begin{bmatrix} -36 & -33l & 36 & -3l \\ 03l & -4l^2 & 3l & l^2 \\ 36 & 3l & -36 & 33l \\ -3l & l^2 & 3l & -4l^2 \end{bmatrix}$$

and

$$[k_{2e}] = \begin{bmatrix} -1/2 & l/10 & 1/2 & -l/10 \\ -l/10 & 0 & l/10 & -l^2/60 \\ -1/2 & -l/10 & 1/2 & l/10 \\ l/10 & l^2/60 & -l/10 & 0 \end{bmatrix}. \tag{11}$$

The four matrices are now assembled in the usual manner, to form four global matrices leading to an equation for the motion of the system:

$$[k_0]\{\eta\} + V^2[k_1]\{\eta\} + 2V\beta^{1/2}[k_2]\{\dot{\eta}\} + [m_0]\{\ddot{\eta}\} = \{0\}. \tag{12}$$

Here the dot superscript denotes a time derivative. It is convenient to re-express this equation in state space form:

$$\begin{bmatrix} k_0 + V^2 k_1 & 2V\beta^{1/2} k_2 \\ 0 & -k_0 \end{bmatrix} \begin{Bmatrix} \eta \\ \dot{\eta} \end{Bmatrix} + \begin{bmatrix} 0 & m_0 \\ k_0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\eta} \\ \ddot{\eta} \end{Bmatrix} = \{0\}. \tag{13}$$

Now if a vector $\{z\} = \begin{Bmatrix} \eta \\ \dot{\eta} \end{Bmatrix}$ is introduced, then this equation may be rewritten as

$$[A]\{z\} + [B]\{\dot{z}\} = \{0\}. \tag{14}$$

Then making the substitution $\{z\} = \{z_0\}\mathrm{e}^{-\lambda\tau}$ yields

$$[[A] - \lambda[B]]\{z_0\} = \{0\}, \tag{15}$$

which is a standard eigenvalue problem. In this equation

$$[A] = \begin{bmatrix} k_0 + V^2 k_1 & 2V\beta^{1/2} k_2 \\ 0 & -k_0 \end{bmatrix} \quad \text{and} \quad [B] = \begin{bmatrix} 0 & m_0 \\ k_0 & 0 \end{bmatrix}. \tag{16}$$

Table 1 shows the calculated first natural frequency for a cantilever beam modelled with eight Euler beam elements, as the non-dimensional velocity is increased from zero to seven, using three different values of β, 0·1, 0·2, and 0·5. For these calculations five elements have been used but as with any finite

TABLE 1

Effects of flow for first mode; natural frequencies as a function of V, β

V	$\beta = 0{\cdot}1$	$\beta = 0{\cdot}2$	$\beta = 0{\cdot}5$
0·1	3·5167 + 0·0633i	3·5160 + 0·0895i	3·5142 + 0·1414i
0·2	3·5185 + 0·1266i	3·5160 + 0·1790i	3·5085 + 0·2829i
0·5	3·5312 + 0·3174i	3·5156 + 0·4485i	3·4687 + 0·7076i
1	3·5786 + 0·6420i	3·5155 + 0·9051i	3·3237 + 1·4180i
2	3·8002 + 1·3493i	3·5353 + 1·8820i	2·6912 + 2·8595i
5	6·3420 + 7·3869i	4·2752 + 7·8864i	0 + 5·3187i
7	0 + 14·0122i	0 + 12·5046i	−11·077 + 4·049i

element approach, the convergence of results with respect to mesh refinement needs some study.

For a relatively high flow velocity, $v = 5$, and $\beta = 0{\cdot}2$ the effects of mesh discretization is illustrated in Table 2. The table shows the real and imaginary parts of the calculated frequency arising from the flow effects.

4. PERTURBATION ANALYSIS

It has been shown in the previous section how a finite element analysis may be used to analyze a structure containing pipes conveying fluid. However, in such a study there are four system matrices rather than simply stiffness and mass, and there is the further complication of the need to use a state space formulation in order to accommodate the convective term. This imposes heavy additional memory requirements. For a number of applications, such as heat exchangers, it is desirable to make some assessment of the flow effects without incurring the penalties of a full calculation as given in the preceding section. Provided that the flow rates are not too great, it is possible to describe the dynamic behaviour of a pipe in terms of the modal properties of the system with no flow.

To develop this approach, one can return to the case of a cantilever pipe. By using the FE study of the preceding section, the equation of motion may be written as

$$[k_0]\{\eta\} + V^2[k_1]\{\eta\} + 2\mathrm{i}\beta^{1/2}V\omega[k_2]\{\eta\} - \omega^2[m_0]\{\eta\} = \{0\}. \qquad (17)$$

One can now seek to determine the changes in eigenvalues and eigenvectors which are brought about by the flow of the fluid. In principle this could be achieved by expanding equation (17) in terms of the non-dimensional velocity V

TABLE 2

Model convergence; frequencies of first mode with v = 5, β = 0·2

Mode	2 elements	4 elements	8 elements	16 elements
1	4·53 + 7·98i	4·29 + 7·89i	4·27 + 7·89i	4·26 + 7·89i
2	14·67 + 1·40i	14·54 + 1·36i	14·48 + 1·33i	14·48 + 1·33i
3	68·76 + 5·43i	52·83 + 4·44i	52·19 + 4·30i	52·14 + 4·29i

and seeking the solution by matching powers of V. However, because the terms in equation (17) contain two powers of V this approach is inconvenient, and it is more appropriate to introduce a parameter λ which designates the extent of the perturbation; thus, in reality, the value $\lambda = 1$ will apply but it is helpful to regard this as a continuous variable. (A similar approach is used in reference [6].)

It is therefore valid to consider an equation

$$[k_0]\{\eta\} + \lambda(V^2[k_1]\{\eta\} + 2\mathrm{i}\beta^{1/2}V\omega[k_2]\{\eta\}) - \omega^2[m_0]\{\eta\} = \{0\}, \tag{18}$$

and to examine the way in which the eigen-solutions depend on the parameter λ.

It is reasonable to assume that for a given (modest) value of V, each eigenvalue and eigenvector can be expressed as a power series of the parameter λ. Thus

$$\{\eta_n\} = \{\eta_{0n}\} + \lambda\{\eta_{1n}\} + \lambda^2\{\eta_{2n}\} + \cdots, \tag{19}$$

where the coefficients in this equation have yet to be determined. Similarly, the new eigenvalues are given by

$$\omega_n = \omega_{0n} + \lambda\omega_{1n} + \lambda^2\omega_{2n}\cdots. \tag{20}$$

Inserting these expansions into equation (18) leads to

$$\begin{aligned}
&[k_0]\{\eta_{0n} + \lambda\eta_{1n} + \lambda^2\eta_{2n} + \cdots\} + \lambda V^2[k_1]\{\eta_{0n} + \lambda\eta_{1n} + \lambda^2\eta_{2n} + \cdots\} \\
&\quad + 2\mathrm{i}\lambda\beta^{1/2}V(\omega_{0n} + \lambda\omega_{1n} + \lambda^2\omega_{2n} + \cdots)[k_2]\{\eta_{0n} + \lambda\eta_{1n} + \lambda^2\eta_{2n} + \cdots\} \\
&\quad - (\omega_{0n} + \lambda\omega_{1n} + \lambda^2\omega_{2n} + \cdots)^2[m_0]\{\eta_{0n} + \lambda\eta_{1n} + \lambda^2\eta_{2n} + \cdots\} = \{0\}.
\end{aligned} \tag{21}$$

This equation requires that the coefficient of each power of λ is zero and hence yields a set of equations relating the terms in equations (19) and (20). Note that the first equation, involving the coefficients of λ^0 is simply a restatement of the original, no-flow equation of motion. The first three equations of the series are used in the current study and these are stated as follows: for λ^0,

$$[k_0]\{\eta_{0n}\} - \omega_{0n}^2[m_0]\{\eta_{0n}\} = \{0\}; \tag{22}$$

for λ^1,

$$\begin{aligned}
&[k_0]\{\eta_{1n}\} + V_2[k_1]\{\eta_{0n}\} + \mathrm{i}2\omega_{0n}\beta^{1/2}V_0[k_2]\{\eta_{0n}\} \\
&\quad - \omega_{0n}^2[m_0]\{\eta_{1n}\} - 2\omega_{0n}\omega_{1n}[m_0]\{\eta_{0n}\} = \{0\};
\end{aligned} \tag{23}$$

for λ^2,

$$\begin{aligned}
&[k_0]\{\eta_{2n}\} + V^2[k_1]\{\eta_{1n}\} + \mathrm{i}2\omega_{0n}\beta^{1/2}V[k_2]\{\eta_{1n}\} \\
&\quad + \mathrm{i}2\omega_1\beta^{1/2}V[k_2]\{\eta_{0n}\} - \omega_0^2[m_0]\{\eta_{2n}\} - \omega_{1n}^2[m_0]\{\eta_{0n}\} \\
&\quad - 2\omega_{0n}\omega_{1n}[m_0]\{\eta_{1n}\} - 2\omega_{0n}\omega_{2n}[m_0]\{\eta_{0n}\} = \{0\}.
\end{aligned} \tag{24}$$

Each equation of this set becomes progressively more complicated. However, the technique is widely used in physics [6] and engineering [7]. Equation (22) is simply a re-statement of the problem with no flow, and this is conveniently solved by using standard methods; hence, the frequencies ω_{0n} and the corresponding mode shapes $\{\eta_{0n}\}$ are known and furthermore, these mode shapes form a complete ortho-normal set. The first order changes $\{\eta_{1n}\}$ are expressed in term of these known modes:

$$\{\eta_{1n}\} = \sum a_{nm}\{\eta_{0m}\}. \tag{25}$$

Inserting this expression into equation (23) gives

$$[k_0 - \omega_{0n}^2 m_0]\{\sum a_{nm}\eta_{0m}\} + [k_1]\{\eta_{0n}\} + \mathrm{i}2\beta^{1/2}V\omega_{0n}[k_2]\{\eta_{0n}\} - 2\omega_{0n}\omega_{1n}[m]\{\eta_{0n}\} = \{0\}. \tag{26}$$

Now, upon taking $n = m$, the first term on the left side of equation (26) becomes zero. The equation is now pre-multiplied by $\{\eta_{0n}\}^{\mathrm{T}}$ and using the orthogonality relationships one obtains

$$V^2\{\eta_{0n}\}^{\mathrm{T}}[k_1]\{\eta_{0n}\} + \mathrm{i}\omega_{0n}2\beta^{1/2}V\{\eta_{0n}\}^{\mathrm{T}}[k_2]\{\eta_{0n}\} - 2\omega_{0n}\omega_{1n} = \{0\}, \tag{27}$$

giving the first order frequency changes

$$\omega_{1n} = \frac{V^2\{\eta_{0n}\}^{\mathrm{T}}[k_1]\{\eta_{0n}\} + \mathrm{i}\omega_{0n}2\beta^{1/2}V\{\eta_{0n}\}^{\mathrm{T}}[k_2]\{\eta_{0n}\}}{2\omega_{0n}}. \tag{28}$$

It is straightforward to insert this expression back into equation (26), assume $n \neq m$ and deduce an equation for a_{nm} giving the changes in mode shape. The form of this expression is made physically clearer by noting that the changes in frequency squared may be written as

$$\Delta\omega_n^2 = 2\omega_{0n}\omega_{1n} = V^2\{\eta_{0n}\}^{\mathrm{T}}[k_1]\{\eta_{0n}\} + \mathrm{i}\omega_{0n}2\beta^{1/2}V\{\eta_{0n}\}^{\mathrm{T}}[k_2]\{\eta_{0n}\}, \tag{29}$$

which becomes simply

$$\Delta\omega_n^2 = 2\omega_{0n}\omega_{1n} = V^2 I_{1n} + \mathrm{i}\omega_{0n}2\beta^{1/2}VI_{2n}. \tag{30}$$

It is seen from equation (28) that to calculate the first order frequency changes two quantities need to be calculated for each mode, namely

$$I_{1n} = \{\eta_{0n}\}^{\mathrm{T}}[k_1]\{\eta_{0n}\} \quad \text{and} \quad I_{2n} = \{\eta_{0n}\}^{\mathrm{T}}[k_2]\{\eta_{0n}\}. \tag{31}$$

In order to test this calculation, these integrals were calculated by using five elements, and the values of these terms for a non-dimensionalized system are shown for the first three modes in Table 3. Figure 1 shows the changes in the real and imaginary parts. It is clear that the shapes of the curves have been well predicted in general terms: the real part shows a quadratic dependence of frequency change with velocity whilst the imaginary part shows a linear dependence. Furthermore, the gradient of the imaginary part dependence is accurately given by the perturbation calculation by using the values in Table 3. However, the calculation does miss some vital physics in the variation of the real

TABLE 3

Modal products; calculated by using five elements

	1	2	3
$I_{1n} = \{y_{0n}\}^T[k_1]\{y_{0n}\}$	0·8582	−13·2940	−45·9036
$I_{2n} = \{y_{0n}\}^T[k_2]\{y_{0n}\}$	2·001	2·0039	2·0039

part of the frequency: in particular no dependence on the parameter β is predicted whereas results of the non-linear calculation show a strong dependence. This illustrates the need to continue the calculation to second order. It is thought that the physical basis for this requirement is that the introduction of the convective term brings a qualitative change to the nature of the system modal behaviour.

Before continuing the calculation, note that a reconsideration of equation (22) for the case $n \neq m$ leads to

$$a_{nm} = \frac{\{\eta_{0n}\}^T[V^2k_1 + i2\omega_{0n}\beta^{1/2}Vk_2]\{\eta_{0m}\}}{\omega_{0n}^2 = \omega_{0m}^2}. \tag{32}$$

Equation (29) gives the first order perturbation solutions to the problem of a general pipe. Figure 1 shows the exact solutions for three values of β for the first mode. It is clear that all three cases show the predicted dependence for relatively small values of flow and furthermore the linear plots of the imaginary part have the gradient predicted by the perturbation study.

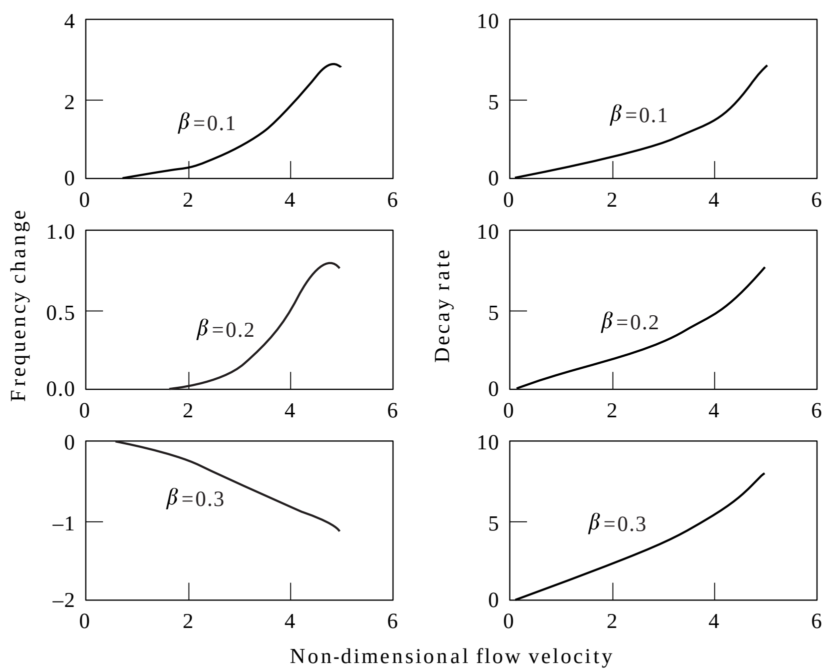

Figure 1. Real and imaginary parts—mode 1.

However, the simple theory presented so far fails to predict the variation of the real frequency with flow for different ranges of β. Note that correct predictions are given for the rather artificial case of $\beta = 0$ but the first order calculation gives no dependence of the real part on β. To overcome this shortcoming, second order terms must be calculated by considering equation (24). The complete discussion of this analysis involves some rather tedious algebra so it is here simplified to retain the main points of interest. By following a similar treatment as in the first order case,

$$2\omega_{0n}\omega_{2n} = \{\eta_{0n}\}V^2[k_1]\{a_{nm}\eta_{0m}\} + \mathrm{i}\omega_{0n}\{\eta_{0n}\}2\beta^{1/2}V[k_2]\{a_{nm}\eta_{0m}\} + \mathrm{i}2\beta^{1/2}\omega_{1n}\{\eta_{0n}\}V[k_2]\eta_{0m}\} - \omega_1^2 - 2\omega_{0n}\omega_{1n}\{\eta_{0n}\}[m_0]\{a_{nm}\eta_{0m}\}. \quad (33)$$

Since a_{nm} is small in the present case, the three terms involving this coefficient are neglected in order to simplify the solutions. It is easily shown that in the present case for the lowest mode the neglected terms are a factor of 20 below the other expressions in the equations. The simplified form of equation (33) becomes

$$2\omega_{0n}\omega_{2n} \approx \mathrm{i}\omega_{1n}2\beta^{1/2}V\{\eta_{0n}\}[k_2]\{\eta_{0n}\} - \omega_{1n}^2. \quad (34)$$

Inserting expressions (30) into this equation gives

$$2\omega_{0n}\omega_{2n} \approx \mathrm{i}\left(\frac{V^2\{\eta_{0n}\}^{\mathrm{T}}[k_1]\{\eta_{0n}\} + \mathrm{i}\omega_{0n}2\beta^{1/2}V\{\eta_{0n}\}[k_2]\{\eta_{0n}\}}{2\omega_{0n}}\right)2\beta^{1/2}V\{\eta_{0n}\}[k_2]\{\eta_{0n}\} - \left(\frac{V^2\{\eta_{0n}\}^{\mathrm{T}}[k_1]\{\eta_{0n}\} + \mathrm{i}\omega_{0n}2\beta^{1/2}V\{\eta_{0n}\}[k_2]\{\beta_{0n}\}}{2\omega_{0n}}\right)^2. \quad (35)$$

Taking the real part of equation (35) and using the relationship of equation (30) one obtains

$$2\omega_{0n}\omega_{2n} = -2\beta V^2 I_{2n}^2 - \mathrm{Re}\left(\frac{V^2 I_{1n} + \mathrm{i}2\omega_0\beta^{1/2}VI_{2n}}{2\omega_0}\right)^2, \quad (36)$$

which becomes

$$2\omega_{0n}\omega_{2n} = -2\beta V^2 I_{2n}^2 + \beta V^2 I_{2n}^2 - V^4 I_{1n}^2/4\omega_0^2. \quad (37)$$

The last term on the right side may be neglected for small values of velocity; more specifically the term is unimportant provided that

$$V \ll 2\omega_{0n}\sqrt{1 - \beta I_{2n}^2/I_{1n}}. \quad (38)$$

The total change in natural frequency may be written as

$$\varDelta\omega = \omega_1 + \omega_2. \quad (39)$$

Taking now just the real part one has

$$\Delta\omega_n^2 = 2\omega_{0n}\omega_{1n} = V^2 I_{1n}(1 - \beta I_{2n}^2/I_{1n}). \tag{40}$$

Hence, expressions have been established for the perturbation solutions for both natural frequency and mode shapes.

5. COMPARISON WITH "EXACT" RESULTS

The second order results for the real frequency change show very good agreement with the values from the "exact" calculations. For mode 1, shown in Figure 2, the frequency changes agree well up to fluid velocities of order two. Furthermore, in this case the flow effects increase the natural frequencies. At higher values of β the frequencies are reduced and, as shown in Figure 4, for $\beta = 0{\cdot}3$, the perturbation solution is accurate over a wider range of flow. From equation (40) it can be seen that the "critical" value of β is given by

$$\beta_{crit} = I_{1n}/I_{2n}^2, \tag{41}$$

and using the figures from Table 2 yields the value $\beta_{crit} \approx 0{\cdot}215$. In view of this result it is not surprising that the quadratic perturbation result becomes erroneous for $u > 2$ with $\beta = 0{\cdot}2$, as shown in Figure 2. In that case the quartic term of equation (37) becomes important. As a check on the validity of this analysis apply equation (38) to this case. The valid range for the quadratic solution is given (using values from Table 3) by $V < 7^* \sqrt{1 - 0{\cdot}2^*4/0{\cdot}85} \approx 1{\cdot}7$, which agrees well with the curve shown in Figure 2. In the higher modes the dependence with β is much less, with no change of sign in the frequency shift.

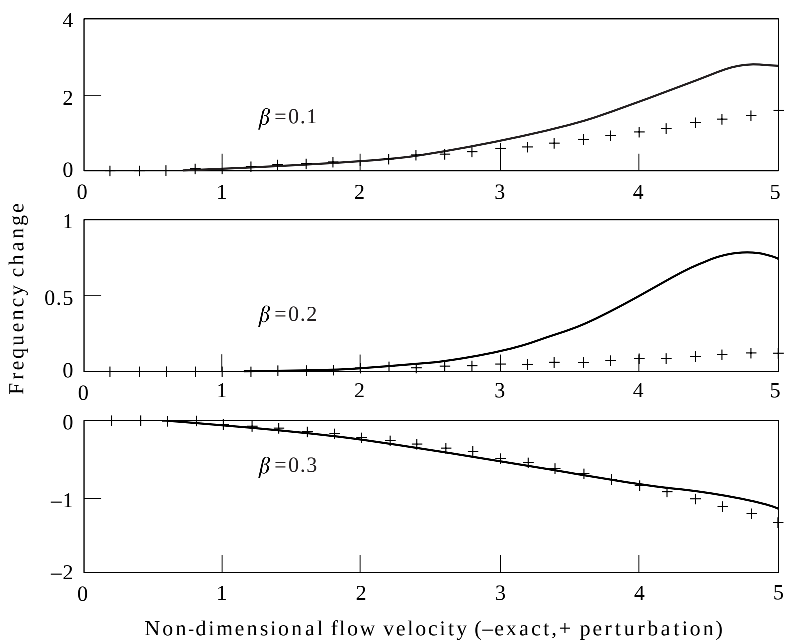

Figure 2. Comparison of "exact" and perturbation solutions—mode 1.

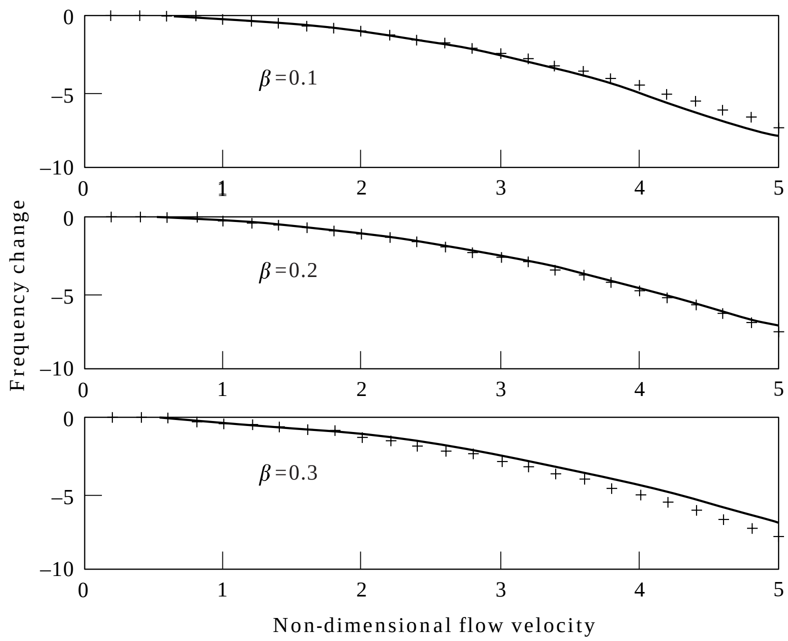

Figure 3. Comparison of "exact" and perturbation solutions—mode 2.

It can be concluded that the simple perturbation solutions presented give an accurate view of the dynamics of the system provided that the flow velocities are modest. The frequency changes, however, may be significant in these cases.

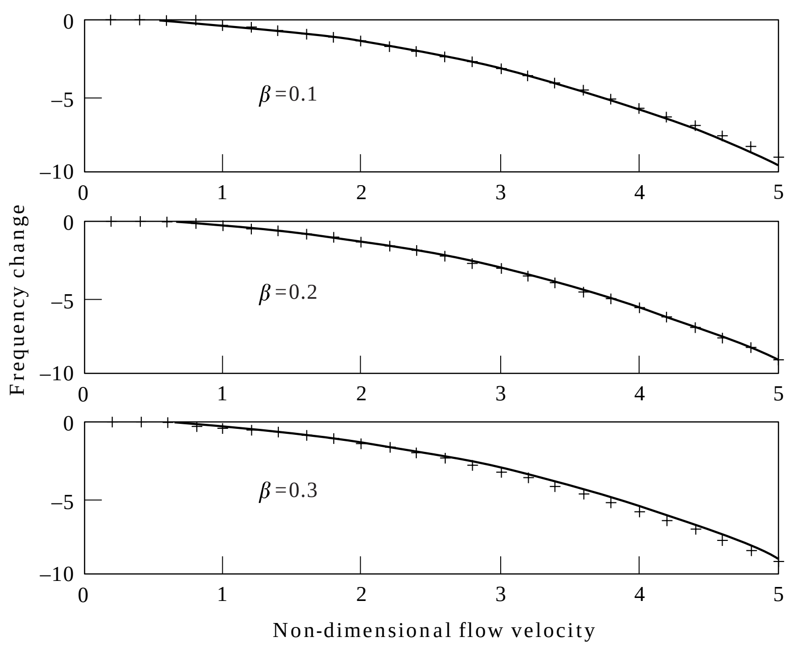

Figure 4. Comparison of "exact" and perturbation solutions—mode 3.

6. DISCUSSION

The dynamics of pipes conveying fluids have been studied extensively in the literature, but most of the study has been focused on the problems of stability. At flow velocities below the instability values, forces arise which modify the dynamic properties of the system. The methods presented in section 4 have been shown to give a good description of the influence of flow at modest velocities and it should be noted that the calculations were carried out by using only the normalized mode shapes of the structure under study. This implies that a simple post processing routine may be applied to calculate the change to the dynamic properties of an arbitrarily complex structure such as a heat exchanger. Note also that for steel components, the range of non-dimensionalized flow corresponds to very fast flows. For example, in a typical pipe span of length 5 m/s having a diameter of 0·2 m and wall thickness 0·01 m, a non-dimensional velocity of 1 corresponds to an actual velocity of 46 m/s.

The analysis of this paper has been presented in terms of non-dimensional parameters, but clearly the results may be transformed into dimensioned terms and this may be more appropriate for a general case. Note that in applying the flow corrections to a general structure, the additional matrices may be assembled by using the element forms given in section 3. However, only those parts of the structure which convey fluid will have non-zero terms. Thus, the approach is particularly convenient to apply.

7. CONCLUSIONS

By using a finite element formulation of Euler beam theory for comparison, it has been shown that a second order perturbation solution gives an accurate assessment of the influence of flow. This solution technique helps to clarify a number of trends in the dynamics of a pipe conveying fluid. The method is easily applied as a post processor using computed (zero flow) natural frequencies and mode shapes. This provides a useful approach to a number of practical calculations.

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APPENDIX: NOTATION

a_{nm}	modal mixing coefficients—first order
b_{nm}	modal mixing coefficients—second order
A	cross-sectional area of flow
C	element transformation matrix
E	Young's modulus for pipe
E_T	pseudo energy of system
i	$\sqrt{-1}$
I	second moment of area of the pipe
I_{nm}	cross integral of mode shapes and k_m stiffness terms
K_0	stiffness matrix for system
K_1	matrix expressing centripital term
K_2	convective matrix
M_0	mass matrix
k_0	stiffness matrix for system (non-dimensional)
k_1	matrix expressing centripital term (non-dimensional)
k_2	convective matrix (non-dimensional)
l	element length
L	total pipe length
m_0	mass matrix (non-dimensional)
M	fluid mass per unit length
s	spatial exponent
t	time
v	fluid velocity
V	non-dimensional fluid velocity
x	distance along pipe
y_{mn}	mth correction to nth mode shape

Greek symbols

β	mass ratio of fluid to pipe and fluid
ω_{mn}	mth correction to nth natural frequency
λ	perturbation order parameter
η	analytic non-dimensional mode shape
ξ	normalized position along pipe
τ	normalized time
Ω	natural frequency

Subscript

e element