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DAMPING PREDICATION OF SANDWICH STRUCTURES BY ORDER-REDUCTION-ITERATION APPROACH

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This paper presents an order-reduction-iteration approach for vibration analysis of viscoelastically damped sandwiches. The damping property of all kinds of viscoelastically damped sandwich structures can be predicted by the approach that consists of two steps, i.e., the first-order asymptotic solution of the non-linear real eigenequation and the order-reduction-iteration of the non-linear complex eigenequation. The experimental results for sandwich beams agree well with the numerical ones in this paper.

1. INTRODUCTION

The use of distributed viscoelastically damped materials in engineering has a long history as a means of controlling structural resonant vibration and acoustic radiation. One of the most effective methods of incorporating viscoelastic material in a built-up structure is in the form of constrained layers, i.e., sandwiches. Because of the frequency-variant coefficient property of the stiffness matrices of the sandwiches and their complicated geometrical property, it is difficult to get theoretical solutions about their eigenequations directly. By now there are four types of finite element analysis methods for analyzing viscoelastically damped sandwiches, i.e., (1) the direct frequency response method, (2) the complex eigenvalue method, (3) the modal strain energy method, and (4) the asymptotic method.

1.1. THE DIRECT FREQUENCY RESPONSE METHOD [1]

The direct frequency response method is based on the so-called correspondence principle of linear viscoelasticity. The equation of motion for a structure or structural element is derived in the way that considers the Young's modulus of all materials as real. The equation is then solved for the case of an applied load that varies sinusoidally in time. The Young's modulus of the viscoelastic material is then taken to be complex, and the ratio of the imaginary part of Young's modulus to the real part is called the material loss factor. This implies that stress and strain in the viscoelastic material can be out of phase and thus energy can be dissipated. The storage and loss modulus of the viscoelastic

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material are generally obtained by using an experimental method with material specimens having imposed sinusoidal stress or strain.

1.2. THE COMPLEX EIGENVALUE METHOD [2]

The complex eigenvalue method is similar to the direct frequency response method in that the complex Young's modulus is assumed. The following eigenequation for free vibration of the viscoelastically-damped system is then obtained:

$$(-\omega^2\mathbf{M} + \mathbf{K} + \mathrm{j}\beta\mathbf{K}_2)\mathbf{X} = \mathbf{0}, \tag{1}$$

where $\mathbf{X}$ is the complex amplitude vector and β is the material loss factor of the viscoelastic material. ω is the circular frequency, $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{K}_2$ are the mass matrix, the real part of the total stiffness matrix and the real part of the stiffness matrix formed by the viscoelastic material, respectively. If the stiffness matrices $\mathbf{K}$ and $\mathbf{K}_2$ are constant, equation (1) may be recast as an algebraic eigenvalue problem with complex eigenvalues and eigenvectors. The damping ratios of the complex modes are obtained from the complex eigenvalues.

1.3. THE MODAL STRAIN ENERGY METHOD [3]

In the modal strain energy method, it is assumed that the damped structure can be represented in term of real normal modes of the associated undamped system if appropriate damping terms are inserted into the uncoupled modal equation of motion. That is

$$\ddot{\alpha}_r + \eta^{(r)}\omega_r\dot{\alpha}_r + \omega_r^2\alpha_r = 0, \tag{2}$$

$$\mathbf{x} = \sum \boldsymbol{\phi}^{(r)}\alpha_r(t) \qquad r = 1, 2, 3, ..., \tag{3}$$

where α_r is the rth modal co-ordinate, ω_r is the natural circular frequency of the rth mode, $\boldsymbol{\phi}^{(r)}$ is the rth modal vector of the associated undamped system, $\eta^{(r)}$ is the loss factor of the rth mode, and $\mathbf{x}$ is the displacement vector. The modal loss factors are calculated by using the undamped modal vectors and all material loss factors. For sandwich structures, the material loss factors of the metal face sheets are very small compared with the viscoelastic core. In this situation, the rth modal loss factor is found from

$$\eta^{(r)} = \beta^{(r)}(V_v^{(r)}/V^{(r)}), \tag{4}$$

where $\beta^{(r)}$ is the material loss factor of the viscoelastic core evaluated at the rth resonant frequency and $V_v^{(r)}/V^{(r)}$ is the fraction of elastic strain energy attributable to the sandwich core when the structure deforms in the rth mode.

1.4. THE ASYMPTOTIC METHOD [4]

The fourth method is the asymptotic method. The following equation is presented:

$$[(\mathbf{K} + \mathrm{j}\beta\mathbf{K}_2) - \omega^2(1 + \mathrm{j}\eta)\mathbf{M}]\boldsymbol{\phi} = \mathbf{0}. \tag{5}$$

In order to avoid solving the complex eigenvalue problem, the eigenvalues and the eigenvectors are expressed as the progression of the complex parameter $\mu = \mathrm{j}\beta$:

$$\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \mu\boldsymbol{\phi}_1 + \mu^2\boldsymbol{\phi}_2 + \mu^3\boldsymbol{\phi}_3 + \dots,$$

$$\omega^2 = \omega_0^2 + \mu^2\omega_2^2 + \mu^4\omega_4^2 + \dots,$$

$$\mathrm{j}\eta = \mu\eta_1 + \mu^3\eta_3 + \mu^5\eta_5 + \dots. \quad (6)$$

Substituting equation (6) into equation (5), one can gain the successive equations that the asymptotic solution must satisfy. After solving those equations, the factors $\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \dots, \omega_0, \omega_2, \omega_4, \dots, \eta_1, \eta_3, \eta_5, \dots,$ can be determined. The asymptotic method can improve the precision of resonant frequencies and loss factors can be performed easily.

In this paper, an order-reduction-iteration approach of finite element analysis for sandwich structures is developed to indicate and solve the frequency-variant coefficient property of the stiffness matrices efficiently.

2. THE ORDER-REDUCTION-ITERATION APPROACH

The order-reduction-iteration approach can get resonant frequencies and modal loss factors of sandwiches more precisely and more easily. It consists of two steps, i.e., the first-order asymptotic solution of the non-linear real eigenequation and the order-reduction-iteration of the complex eigenequation. The discrete (i.e., finite element) partial differential equation for free vibration of sandwiches (or any structure) is:

$$\mathbf{M}\ddot{\mathbf{X}} + (\mathbf{K} + \mathrm{j}\beta\mathbf{K}_2)\mathbf{X} = \mathbf{0}, \quad (7)$$

where the meanings of all parameters are the same as equation (1). $\mathbf{K}$, $\mathbf{K}_2$ and β are all the functions of the real storage module $G_R(\lambda)$ that varies with the frequency. Equation (7) can be converted into an eigenvalue problem by assuming

$$\mathbf{X} = \boldsymbol{\varphi}\mathrm{e}^{\mathrm{j}\sqrt{\lambda}t}, \quad (8)$$

where λ and $\boldsymbol{\varphi}$ are the complex eigenvalue and eigenvector. Substituting equation (8) into equation (7), one obtains eigenequation

$$(\mathbf{K} + \mathrm{j}\beta\mathbf{K}_2)\boldsymbol{\varphi} = \lambda\mathbf{M}\boldsymbol{\varphi}. \quad (9)$$

2.1. THE FIRST-ORDER ASYMPTOTIC SOLUTION OF THE NON-LINEAR REAL EIGENEQUATION

If the damping in equation (9) is overlooked, the real eigenequation is

$$\mathbf{K}\boldsymbol{\varphi} = \lambda\mathbf{M}\boldsymbol{\varphi}. \quad (10)$$

Assume λ_r and $\boldsymbol{\varphi}_r$ are the rth real eigenvalue and eigenvector when the storage module $G_R(\lambda)$ is equal to G_{R_0}. Here G_{R_0} is the selected initial storage module of the core material. $\mathbf{K}_0$ and $\mathbf{K}_{20}$ are the corresponding values to $G_R(\lambda)=G_{R_0}$. Then, $\bar{\lambda}_r$, $\bar{\boldsymbol{\varphi}}_r$, $\mathbf{K}_r$, and $\mathbf{K}_{2r}$ are the values when the storage module is equal to $G_R(\lambda_r)$ which corresponds to the frequency $\sqrt{\lambda_r}$. Considering that $\mathbf{K}_r$ and $\mathbf{K}_{2r}$ are linear functions of the module $G_R(\lambda)$, they can be expressed as the progressions of ΔG_R

$$\bar{\lambda} = \lambda_r + \frac{\partial \lambda_r}{\partial G_R}\Delta G_R + \frac{1}{2!}\frac{\partial^2 \lambda_r}{\partial G_R^2}\Delta G_R^2 + \ldots, \tag{11}$$

$$\bar{\boldsymbol{\varphi}}_r = \boldsymbol{\varphi}_r + \frac{\partial \boldsymbol{\varphi}_r}{\partial G_R}\Delta G_R + \frac{1}{2!}\frac{\partial^2 \boldsymbol{\varphi}_r}{\partial G_R^2}\Delta G_R^2 + \ldots, \tag{12}$$

$$\mathbf{K}_r = \mathbf{K}_0 + \frac{\partial \mathbf{K}}{\partial G_R}\Delta G_R, \tag{13}$$

$$\mathbf{K}_{2r} = \mathbf{K}_{20} + \frac{\partial \mathbf{K}}{\partial G_R}\Delta G_R, \tag{14}$$

where

$$\Delta G_R = G_R(\lambda_r) - G_{R_0}. \tag{15}$$

Substituting equations (11), (12), and (13) into equation (10), one obtains

$$\begin{aligned} &\left(\mathbf{K}_0 + \frac{\partial \mathbf{K}}{\partial G_R}\Delta G_R\right)\left(\boldsymbol{\varphi}_r + \frac{\partial \boldsymbol{\varphi}_r}{\partial G_R}\Delta G_R + \frac{1}{2!}\frac{\partial^2 \boldsymbol{\varphi}_r}{\partial G_R^2}\Delta G_R^2 + \ldots\right) \\ &- \left(\lambda_r + \frac{\partial \lambda_r}{\partial G_R}\Delta G_R + \frac{1}{2!}\frac{\partial^2 \lambda_r}{\partial G_R^2}\Delta G_R^2 + \ldots\right) \\ &\times \mathbf{M}\left(\boldsymbol{\varphi}_r + \frac{\partial \boldsymbol{\varphi}_r}{\partial G_R}\Delta G_R + \frac{1}{2!}\frac{\partial^2 \boldsymbol{\varphi}_r}{\partial G_R^2}\Delta G_R^2 + \ldots\right) = \mathbf{0}. \end{aligned} \tag{16}$$

From equation (16), the following equations are obtained:

$$(\mathbf{K}_0 - \lambda_r \mathbf{M})\boldsymbol{\varphi}_r = \mathbf{0}, \tag{17}$$

$$(\mathbf{K}_0 - \lambda_r \mathbf{M})\frac{\partial \boldsymbol{\varphi}_r}{\partial G_R} = \left(\frac{\partial \lambda_r}{\partial G_R}\mathbf{M} - \frac{\partial \mathbf{K}}{\partial G_R}\right)\boldsymbol{\varphi}_r. \tag{18}$$

Equation (17) is the real eigenvalue problem of the constant stiffness matrix $\mathbf{K}_0$. The rth eigenvalue λ_r and eigenvector $\boldsymbol{\varphi}_r$ can be calculated easily by means of the subspace iteration method or other methods.

The rth eigenvector $\bar{\boldsymbol{\varphi}}_r$ satisfies the following regularization equation

$$\bar{\boldsymbol{\varphi}}_r^{\mathrm{T}}\mathbf{M}\bar{\boldsymbol{\varphi}}_r = 1. \tag{19}$$

Substituting equation (12) into equation (19) leads to the following equation

$$\left(\boldsymbol{\varphi}_r + \frac{\partial \boldsymbol{\varphi}_r}{\partial G_R} \Delta G_R + \dots\right)^{\mathrm{T}} \mathbf{M} \left(\boldsymbol{\varphi}_r + \frac{\partial \boldsymbol{\varphi}_r}{\partial G_R} \Delta G_R + \dots\right) = 1. \tag{20}$$

From equation (20), one has

$$\boldsymbol{\varphi}_r^{\mathrm{T}} \mathbf{M} \boldsymbol{\varphi}_r = 1, \qquad 2\boldsymbol{\varphi}_r^{\mathrm{T}} \mathbf{M} \frac{\partial \boldsymbol{\varphi}_r}{\partial G_R} = 0. \tag{21, 22}$$

The real modal shape of regularization is solved from equation (21). Then if equation (18) is multiplied by $\boldsymbol{\varphi}_r^{\mathrm{T}}$ on the left, one has

$$\frac{\partial \lambda_r}{\partial G_R} = \boldsymbol{\varphi}_r^{\mathrm{T}} \frac{\partial \mathbf{K}}{\partial G_R} \boldsymbol{\varphi}_r. \tag{23}$$

Therefore, the first-order asymptotic real eigensolution of equation (10) is written as

$$\bar{\lambda}_r = \lambda_r + \frac{\partial \lambda_r}{\partial G_R} \Delta G_R, \qquad \frac{\partial \lambda_r}{\partial G_R} = \boldsymbol{\varphi}_r^{\mathrm{T}} \frac{\partial \mathbf{K}}{\partial G_R} \boldsymbol{\varphi}_r,$$

$$\mathbf{K}_r = \mathbf{K}_0 + \frac{\partial \mathbf{K}}{\partial G_R} \Delta G_R, \qquad \mathbf{K}_{2r} = \mathbf{K}_{20} + \frac{\partial \mathbf{K}}{\partial G_R} \Delta G_R \tag{24}$$

2.2. THE ORDER-REDUCTION-ITERATION OF THE COMPLEX EIGENEQUATION

If the estimated rth eigenvalue in equation (9) is determined by using equation (24), the exact rth eigenvalue can be calculated by the direct iteration method. But generally speaking, the order of equation (9) is too big to be performed in practice. Here the Lanczos algorithm [5] is used to form m order orthogonal complex base vectors. Then equation (9) can be reduced from an n order eigenequation to an m order eigenequation at the base vectors formed. Because $m \ll n$, the order-reduced eigenequation can be easily calculated by the direct iteration method and the computation efficiency can be improved greatly when $m = \max(2r + 4, r + 16)$ [6].

If $\mathbf{K}_r$ and $\mathbf{K}_{2r}$ are the stiffness matrix of the structure and the stiffness matrix formed by the viscoelastic core, respectively, when λ is equal to the rth first-order asymptotic eigenvalue $\bar{\lambda}_r$, the following iteration equations are defined by the Lanczos algorithm:

$$\boldsymbol{\gamma}_i = \beta_{i+1} \mathbf{q}_{i+1} = \mathbf{K}_r^{-1} \mathbf{M} \mathbf{q}_i - \alpha_i \mathbf{q}_i - \beta_i \mathbf{q}_{i-1}, \qquad i = 1, \dots m. \tag{25}$$

Here α_i and β_i are the orthogonal factors, $\boldsymbol{\gamma}_i$ and $\mathbf{q}_i$ are all Lanczos vectors; $\mathbf{q}_i$ is the regularization Lanczos vector, β_{i+1} is the regularization factor, and $\mathbf{q}_0 = \mathbf{0}$. Assuming

$$\mathbf{Q}_m = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \dots \quad \mathbf{q}_m], \tag{26}$$

$\mathbf{Q}_m$ satisfies the following orthogonality condition equation:

$$\mathbf{Q}_m^{\mathrm{T}}\mathbf{M}\mathbf{Q}_m = \mathbf{I}. \tag{27}$$

From equation (27), the factors α_i, β_i, and β_{i+1} can be written as

$$\alpha_i = (\mathbf{K}_r^{-1}\mathbf{M}\mathbf{q}_i)^{\mathrm{T}}\mathbf{M}\mathbf{q}_i, \tag{28}$$

$$\beta_i = (\mathbf{K}_r^{-1}\mathbf{M}\mathbf{q}_i)^{\mathrm{T}}\mathbf{M}\mathbf{q}_{i-1}, \tag{29}$$

$$\beta_{i+1} = (\boldsymbol{\gamma}_i^{\mathrm{T}}\mathbf{M}\boldsymbol{\gamma}_i)^{1/2}. \tag{30}$$

Substituting equations (13) and (14) into equation (9) leads to the following equation:

$$\left[\mathbf{K}_0 + \frac{\partial \mathbf{K}}{\partial G_R}\Delta G_R + \mathrm{j}\beta(\mathrm{Re}(\lambda))\left(\mathbf{K}_{20} + \frac{\partial \mathbf{K}}{\partial G_R}\Delta G_R\right)\right]\boldsymbol{\varphi} = \lambda\mathbf{M}\boldsymbol{\varphi}, \tag{31}$$

where Re(●) denotes the real part of a complex quantity.

If

$$\boldsymbol{\varphi} = \mathbf{Q}_m\bar{\boldsymbol{\varphi}}, \tag{32}$$

where $\bar{\boldsymbol{\varphi}}$ is the eigenvector of the order-reduced system, substituting equation (32) into equation (31) and multiplying equation (31) by $\mathbf{Q}_m^{\mathrm{T}}$ on the left, one has

$$\left[\bar{\mathbf{K}}_0 + \frac{\partial \bar{\mathbf{K}}}{\partial G_R}\Delta G_R + \mathrm{j}\beta(\mathrm{Re}(\lambda))\left(\bar{\mathbf{K}}_{20} + \frac{\partial \bar{\mathbf{K}}}{\partial G_R}\Delta G_R\right)\right]\bar{\boldsymbol{\varphi}} = \lambda\bar{\boldsymbol{\varphi}}, \tag{33}$$

where

$$\bar{\mathbf{K}}_0 = \mathbf{Q}_m^{\mathrm{T}}\mathbf{K}_0\mathbf{Q}_m, \tag{34}$$

$$\bar{\mathbf{K}}_{20} = \mathbf{Q}_m^{\mathrm{T}}\mathbf{K}_{20}\mathbf{Q}_m, \tag{35}$$

$$\frac{\partial \bar{\mathbf{K}}}{\partial G_R} = \mathbf{Q}_m^{\mathrm{T}}\frac{\partial \mathbf{K}}{\partial G_R}\mathbf{Q}_m. \tag{36}$$

Equation (33) is an mth order non-linear standard eigenequation. The iteration solution is introduced as follows [7]:

$$\left[\bar{\mathbf{K}}_0 + \frac{\partial \bar{\mathbf{K}}}{\partial G_R}\Delta G_R^{(i)} + \mathrm{j}\beta(\mathrm{Re}(\lambda_r^{(i)}))\left(\bar{\mathbf{K}}_{20} + \frac{\partial \bar{\mathbf{K}}}{\partial G_R}\Delta G_R^{(i)}\right)\right]\bar{\boldsymbol{\varphi}}^{(i+1)} = \lambda^{(i+1)}\bar{\boldsymbol{\varphi}}^{(i+1)}, \tag{37}$$

$$\Delta G_R^{(i)} = G_R(\mathrm{Re}(\lambda_r^{(i)})) - G_{R_0}, \tag{38}$$

where $\lambda_r^{(1)} = \bar{\lambda}_r$.

If

$$|\lambda_r^{(i+1)} - \lambda_r^{(i)}| < \varepsilon, \tag{39}$$

the iteration can be ended, where ε is the absolute error limit given.

The approximate solution of equation (9) in the condition of ε error is written as

$$\lambda_r = \lim_{i\to\infty} \lambda_r^{(i)}, \tag{40}$$

$$\boldsymbol{\varphi}_r = \mathbf{Q}_m \lim_{i\to\infty} \bar{\boldsymbol{\varphi}}_r^{(i)}. \tag{41}$$

According to reference [8], one has

$$\lambda_r = \omega_r^2(1 + j\eta_r). \tag{42}$$

Thus

$$\omega_r = \sqrt{\mathrm{Re}(\lambda_r)}, \qquad \eta_r = \frac{\mathrm{Im}(\lambda_r)}{\mathrm{Re}(\lambda_r)}, \tag{43, 44}$$

where ω_r, η_r are the rth circular frequency and modal loss factor, and Re(●), Im(●) denote the real part and the imaginary part of a complex quantity.

3. NUMERICAL RESULTS AND COMPARISON

Two simple test cases were run to assess the accuracy of the order-reduction-iteration approach above. The first one is a continuous sandwich beam shown in Figure 1. The geometrical dimension is shown in the figure, too. The material properties of the matrix layer, constrained layer, and the viscoelastic core are revealed in the Appendix. It is clamped on the left.

Table 1 lists the first three-order results computed by the order-reduction-iteration approach (ORIA) and measured experimental ones of the natural frequencies and modal loss factors of this case. It may be seen that the difference is quite small in this simple case.

The second example is more complicated than the first one. Here a discontinuous sandwich beam is shown in Figure 2. A detailed discussion is

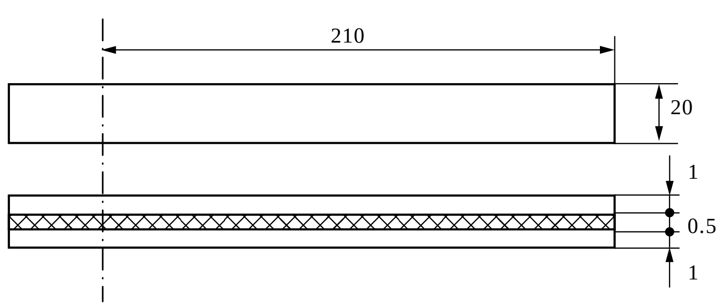

Figure 1. Continuous cantilever sandwich beam tested (unit: mm)

TABLE 1

Natural frequencies and modal loss factors of the continuous sandwich beam in Figure 1

	Natural frequency (Hz)		Modal loss factor	
Modal number	ORIA	Experiment	ORIA	Experiment
1	38·4	37·2	0·337	0·348
2	185·6	182·5	0·410	0·432
3	472·6	470·3	0·420	0·407

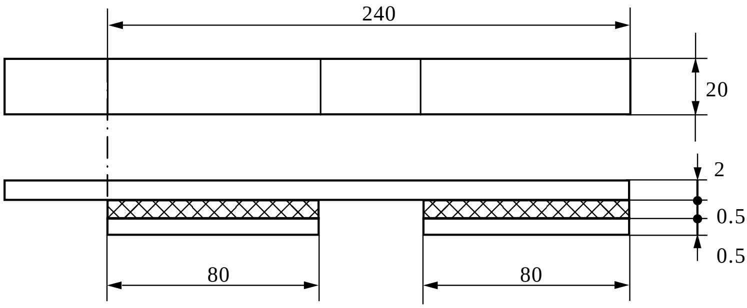

Figure 2. Discontinuous cantilever sandwich beam tested (unit: mm).

available in reference [6]. The material properties are the same as in the first case. It is also clamped on the left side. But the eigenequation is similar to equation (9). Table 2 lists the numerical and experimental results of this example. There are also very good agreements obtained between them. Therefore, it is concluded that the order-reduction-iteration approach has almost the same high accuracy in analysis of both uniformly and non-uniformly distributed damping layer structures. It is also seen that the uniform damping layer structures have more efficient damping than distributed ones.

In the first-order asymptotic analysis, G_{R_0} must be given by anticipation. In order to evaluate the effect of G_{R_0} on the computed results, Table 3 lists the numerical results of the first case at different G_{R_0}. It seems that G_{R_0} has very little effect on the results. Hence, the approach is very useful in engineering application.

TABLE 2

Natural frequencies and modal loss factors of the discontinuous sandwich beam in Figure 2

	Natural frequency (Hz)		Modal loss factor	
Modal number	ORIA	Experiment	ORIA	Experiment
1	28·6	27·9	0·125	0·130
2	186·1	183·5	0·0945	0·0952
3	499·6	495·4	0·0959	0·0980

TABLE 3

Natural frequencies and modal loss factors of the continuous sandwich beam in Figure 1 at different G_{R_0}

	Natural frequency (Hz)			Modal loss factor		
G_{R_0}(N/m^2)	f_1	f_2	f_3	η_1	η_2	η_3
1×10^7	38·438	185·602	472·348	0·3366795	0·4099854	0·4201920
$1{\cdot}5 \times 10^7$	38·431	185·599	472·351	0·3366293	0·4099673	0·4202000
$2{\cdot}5 \times 10^7$	38·432	185·597	472·347	0·3366345	0·4099467	0·4201904

4. CONCLUSION

Methods for finite element analysis of structures with layer viscoelastic damping have been reviewed. None of these methods is yet in widespread use for day-to-day design work. The order-reduction-iteration approach appears to be the most promising method for large-scale applications.

The order-reduction-iteration approach simplifies the frequency-variant coefficient non-linear complex eigenequation by combining first-order real asymptotic analysis with order-reduced iteration. It has the advantage of simplicity, practicality, and being implemented with a modern finite element code that is widely accessible and familiar to a large number of engineers. The numerical results indicate that the predicated G_{R_0} has very little effect on the results. The method can treat not only uniformly but also non-uniformly distributed sandwiches. It provides information of the direct usefulness in designing a viscoelastically-damped treatment. The method has been demonstrated for three-layer cantilever sandwich beams. Very good agreement has been obtained with experimental solutions for natural frequencies and modal loss factors.

Additional work is still required to make the order-reduction-iteration approach a practical design tool. Further comparison with experiments will also be required under more complicated conditions.

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APPENDIX: MATERIAL PROPERTIES

The material of the matrix layer and the constrained layer are all steel. The viscoelastic core is the damping rubber. Their material properties are: density $\rho_1 = \rho_3 = 7831$ kg/m^3, $\rho_2 = 990$ kg/m^3; Poisson ratio $\mu_1 = \mu_3 = 0{\cdot}3$, $\mu_2 = 0{\cdot}45$; modulus $E_1 = E_3 = 2{\cdot}1 \times 10^{11}$ N/m^2. The subscripts 1, 2, and 3 denote the matrix layer, the damping layer, and the constrained layer, respectively.

The real storage modulus G_R and the material loss factor β of the visco-elastic core are: $G_R(f) = 1{\cdot}7 \times 10^6 + 4{\cdot}768269 \times 10^5 f^{0{\cdot}654634}$, $\beta(f) = 1/(0{\cdot}8549449 - 2{\cdot}381036 \times 10^{-3} f + 1{\cdot}3066037 \times 10^{-5} f^2 - 3{\cdot}066037 \times 10^{-8} f^3 + 2{\cdot}419863 \times 10^{-11} f^4)$, where f denotes frequency and $f = \omega/2\pi$.