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WAVE PROPAGATION AT LIQUID/MICROPOLAR ELASTIC SOLID INTERFACE

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1. INTRODUCTION

Linear theory of micropolar elasticity developed by Eringen [l] is basically an extension of classical theory of elasticity. In classical theory of elasticity, the motion is characterized by translational degrees of freedom only, while in micropolar elasticity it is characterized by translational and rotational degrees of freedom. Physically, micropolar elastic solids can be thought of as being composed of dumbbell type molecules and these molecules in a volume element can undergo rotation about their center of mass along with the linear displacement and consequently, loads across a surface element are transmitted not only by a force stress vector but also by a coupled stress vector.

In the problems of waves and vibrations in a micropolar elastic medium, the effect of microstructure is found to be significant particularly in the case of high frequency waves. Many problems of reflection and refraction of micropolar elastic waves at an interface have been studied by several researchers. Notable among them are Parfitt and Eringen [2], Tomar *et al*. [3–7], Singh and Kumar [8, 9] among others.

This paper includes the study of two problems: (I) Reflection and refraction of longitudinal waves propagating through the liquid medium and impinging upon an interface between homogeneous, inviscid liquid half-space and a uniform micropolar elastic solid half-space. (II) Waves at the interface of liquid/micropolar elastic half-spaces. The problem of reflection and refraction of longitudinal waves incident from the liquid at an interface between liquid and uniform elastic solid half-spaces has been reduced as a special case of our problem by simply making the elastic constants corresponding to micropolarity equal to zero.

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2. FIELD EQUATIONS AND CONSTITUTIVE RELATIONS

Following Eringen [10], the micropolar elastodynamic equations in the absence of body forces and body couples are given by,

$$(c_1^2 + c_3^2)\nabla(\nabla\cdot\mathbf{u}) - (c_2^2 + c_3^2)\nabla\times(\nabla\times\mathbf{u}) + c_3^2\nabla\times\boldsymbol{\phi} = \ddot{\mathbf{u}},$$
$$(c_4^2 + c_5^2)\nabla(\nabla\cdot\boldsymbol{\phi}) - c_4^2\nabla\times(\nabla\times\boldsymbol{\phi}) + \omega_0^2\nabla\times\mathbf{u} - 2\omega_0^2\boldsymbol{\phi} = \ddot{\boldsymbol{\phi}}, \tag{1}$$

where

$$c_1^2 = (\lambda + 2\mu)/\rho, \qquad c_2^2 = \mu/\rho, \qquad c_3^2 = K/\rho,$$
$$c_4^2 = \gamma/\rho j, \qquad c_5^2 = (\alpha + \beta)/\rho j, \qquad \omega_0^2 = c_3^2/j.$$

$\mathbf{u}(\mathbf{x}, t)$ and $\boldsymbol{\phi}(\mathbf{x}, t)$ are displacement and rotation vectors respectively. λ, μ, K, α, β, γ are elastic moduli, ρ is density, and j is micro-inertia of the medium. The superposed dots on the right hand side of equations (1) denote the double time derivative.

The constitutive relations in micropolar medium are (cf. reference [10])

$$t_{kl} = \lambda u_{r,r}\delta_{kl} + \mu(u_{k,l} + u_{l,k}) + K(u_{l,k} - \varepsilon_{klr}\phi_r),$$
$$m_{kl} = \alpha\phi_{r,r}\delta_{kl} + \beta\phi_{k,l} + \gamma\phi_{l,k}, \tag{2}$$

where symbols have their usual meanings.

Introducing the scalar and vector potentials q, ξ and $\mathbf{U}$, $\boldsymbol{\Phi}$ through Helmholtz's theorem as follows:

$$\mathbf{u} = \nabla q + \nabla\times\mathbf{U}, \qquad \nabla\cdot\mathbf{U} = 0, \qquad \boldsymbol{\phi} = \nabla\xi + \nabla\times\boldsymbol{\Phi}, \qquad \nabla\cdot\boldsymbol{\Phi} = 0, \tag{3}$$

and following the calculations given by Singh [11], one arrives at

$$(\nabla^2 + k_1^2)q = 0, \qquad (\nabla^2 + k_2^2)\xi = 0, \tag{4}$$

$$(\nabla^2 + k_3^2)(\mathbf{U}, \boldsymbol{\Phi}) = 0, \qquad (\nabla^2 + k_4^2)(\mathbf{U}, \boldsymbol{\Phi}) = 0, \tag{5}$$

where

$$k_1 = \frac{\omega}{V_i}, \qquad (i = 1, 2, 3, 4), \qquad V_1^2 = c_1^2 + c_3^2,$$
$$V_2^2 = (c_4^2 + c_5^2)(1 - 2\omega_0^2/\omega^2)^{-1}, \tag{6}$$

$$V_{3,4}^2 = \frac{1}{2a}[b \pm (b^2 - 4ac)^{1/2}],$$
$$a = 1 - 2\omega_0^2/\omega^2, \qquad b = c_2^2 + c_3^2 + c_4^2 - (2c_2^2 + c_3^2)\omega_0^2/\omega^2, \tag{7}$$
$$c = c_4^2(c_2^2 + c_3^2).$$

Note from equations (4) and (5) that scalar potentials q and ξ and vector potentials $\mathbf{U}$ and $\boldsymbol{\Phi}$ satisfy Helmholtz equations. Parfitt and Eringen [2] have shown that V_1 is the velocity of propagation of the longitudinal wave, V_2 that of a longitudinal microrotational wave and V_3, V_4 are the velocities of two sets of coupled transverse and microrotational waves. The set of coupled waves travelling with velocity V_3 propagates only if $\omega^2 > 2\omega_0^2$; otherwise they degenerate into distance decaying sinusoidal vibrations.

The equation of motion in terms of displacement potential for homogeneous, inviscid liquid is given by (cf reference [12]):

$$\nabla^2 \phi' = \frac{1}{\alpha'^2}\frac{\partial^2 \phi'}{\partial t^2},$$

where $\alpha' = (\lambda'/\rho')^{1/2}$ is the velocity of dilatational wave in liquid. λ' and ρ' are bulk modulus and density of the liquid, respectively. The displacement and pressure are given by

$$u_i' = \frac{\partial \phi'}{\partial x_i'}, \qquad p^* = \rho' \frac{\partial^2 \phi'}{\partial t^2}. \tag{8}$$

3. REFLECTION AND REFRACTION OF LONGITUDINAL WAVE (I)

Consider a two-dimensional problem in the y–z plane with the z-axis pointing downward into the micropolar solid half-space and a plane interface $z = 0$ between a homogeneous, inviscid liquid half-space and a uniform micropolar elastic solid half-space. Take

$$\mathbf{u} = (0, u_2, u_3), \qquad \mathbf{\Phi} = (\phi_1, 0, 0). \tag{9}$$

Consider a longitudinal wave propagating through the liquid medium $M_1[z < 0]$ and striking at the interface $z = 0$ making an angle θ_0 with the interface. The following reflected and refracted waves are assumed at the interface: (a) a reflected longitudinal wave in the medium M_1 travelling with speed α' and making an angle θ_1 with the interface; (b) a refracted longitudinal wave in the medium M_2 travelling with speed V_1 and making an angle θ_2 with the interface; (c) two sets of refracted coupled wave in medium M_2 travelling with speeds V_3 and V_4 making an angle θ_3 and θ_4 with the interface, respectively. The complete geometry of the problem is shown in Figure 1.

Assume the following form of potentials in the half-spaces:

In liquid half-space M_1:

$$\begin{aligned}\phi' = {} & A_0 \exp[\imath k_0(\cos\theta_0 y + \sin\theta_0 z) - \imath\omega_0' t] \\ & + A_1 \exp[\imath k_0(\cos\theta_1 y - \sin\theta_1 z) - \imath\omega_0' t],\end{aligned} \tag{10}$$

where A_0, A_1 are amplitudes of the incident and reflected longitudinal waves, respectively. ω_0' $(= k_0\alpha')$ is the circular frequency of the wave in liquid.

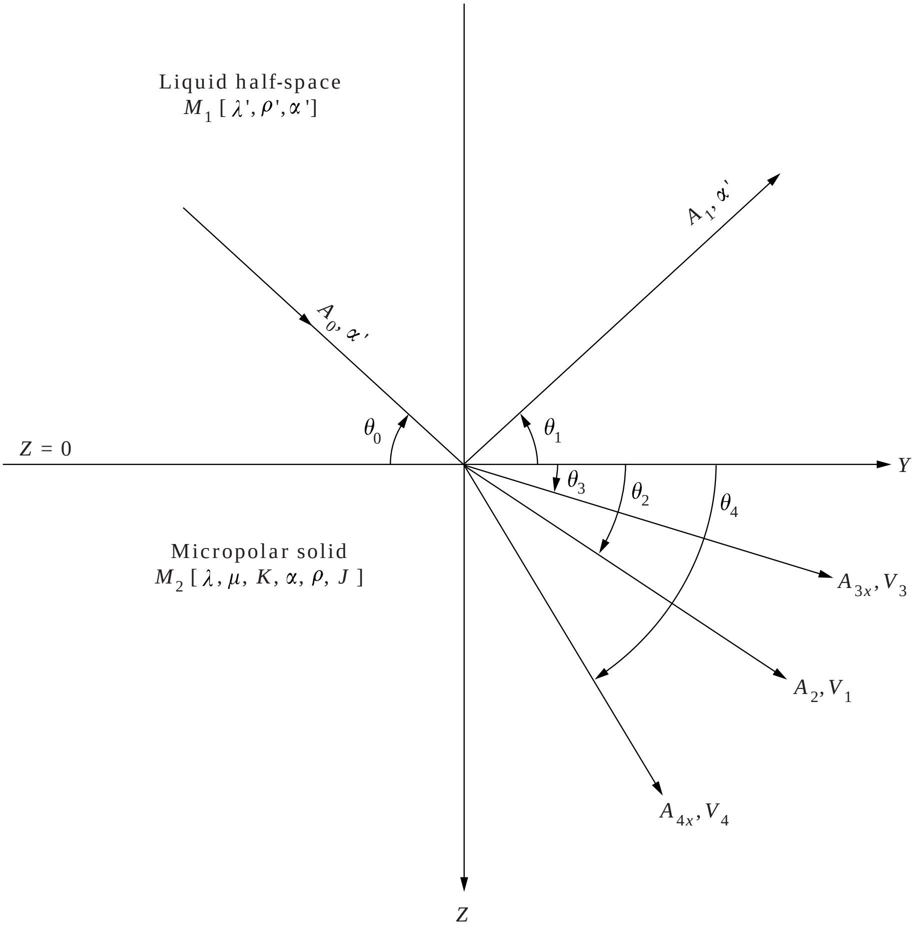

Figure 1. Geometry of the problem.

In micropolar elastic solid half-space M_2:

$$q = A_2 \exp[\imath k_1(\cos\theta_2 y + \sin\theta_2 z) - \imath\omega_1 t],$$

$$\mathbf{U}_p = \hat{\mathbf{I}}[A_{px}] \exp[\imath k_p(\cos\theta_p y + \sin\theta_p z) - \imath\omega_p t], \tag{11}$$

$$\mathbf{\Phi}_p = [B_{py}\hat{\mathbf{J}} + B_{pz}\hat{\mathbf{K}}] \exp[\imath k_p(\cos\theta_p y + \sin\theta_p z) - \imath\omega_p t],$$

where $p = 3, 4$. A_2, A_{3x} and A_{4x} are the amplitudes of the refracted longitudinal wave, refracted coupled wave at an angle θ_3, and refracted coupled wave at an angle θ_4, respectively. $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$, $\hat{\mathbf{K}}$ are unit cartesian base vectors. ω_p $(= k_p V_p)$ is the frequency of the respective wave.

Parfitt and Eringen [2] have shown that the coefficients $\mathbf{A}_p$ and $\mathbf{B}_p$ are connected to each other through the relation

$$\mathbf{B}_p = -\left[\frac{\imath\omega_0^2 A_{px}}{k_p(V_p^2 - 2\omega_0^2 k_p^{-2} - c_4^2)}\right](\sin\theta_p\hat{\mathbf{J}} - \cos\theta_p\hat{\mathbf{K}}). \tag{12}$$

4. BOUNDARY CONDITIONS

The boundary conditions at the liquid–solid interface are the continuity of displacement and stresses. Note that the shear stress and couple stress must vanish at the interface, since inviscid liquid cannot support these. Mathematically, these boundary conditions can be expressed as: at $z=0$,

$$t_{zz} = p^*, \qquad 0 = t_{zy} = m_{mx}, \qquad u_3 = u_3'. \tag{13}$$

The requisite components of displacement and stresses for both the mediums M_1 and M_2 in terms of potentials can be obtained from equations (2), (3), (8) and (9) as follows:

$$u_3 = \frac{\partial q}{\partial z} - \frac{\partial U_x}{\partial y}, \tag{14a}$$

$$u_3' = \frac{\partial \phi'}{\partial z}, \qquad p^* = \rho' \frac{\partial^2 \phi'}{\partial t^2}, \tag{14b}$$

$$\begin{aligned} t_{zz} &= \lambda \nabla^2 q + (2\mu + K)\left(\frac{\partial^2 q}{\partial z^2} - \frac{\partial^2 U_x}{\partial y\, \partial z}\right), \\ t_{zy} &= (2\mu + K)\frac{\partial^2 q}{\partial y\, \partial z} + (\mu + K)\frac{\partial^2 U_x}{\partial z^2} - \mu \frac{\partial^2 U_x}{\partial y^2} + K\left(\frac{\partial \Phi_z}{\partial y} - \frac{\partial \Phi_y}{\partial z}\right), \\ m_{zx} &= \gamma\left(\frac{\partial^2 \Phi_z}{\partial y\, \partial z} - \frac{\partial^2 \Phi_y}{\partial z^2}\right). \end{aligned} \tag{15}$$

Using equations (10) and (11) in equations (14) and (15) and the Snell's law given by

$$\frac{\cos\theta_0}{\alpha'} = \frac{\cos\theta_1}{\alpha'} = \frac{\cos\theta_2}{V_1} = \frac{\cos\theta_3}{V_3} = \frac{\cos\theta_4}{V_4}, \tag{16}$$

in the above boundary conditions (13) and the assumptions that at the interface $z=0$

$$\omega_0' = \omega_1 = \omega_3 = \omega_4 = \omega, \tag{17}$$

these boundary conditions reduce to a system of four equations in five unknown as follows:

$$\begin{aligned} &-\lambda' k_0^2 (A_0 + A_1) + [\lambda + (2\mu + K)\sin^2\theta_2] k_1^2 A_2 \\ &\quad - (2\mu + K) k_3^2 \cos\theta_3 \sin\theta_3 A_{3x} - (2\mu + K) k_4^2 \cos\theta_4 \sin\theta_4 A_{4x} = 0, \\ &(2\mu + K) k_1^2 \cos\theta_2 \sin\theta_2 A_2 + k_3^2\left[-\mu\cos 2\theta_3 + K\sin^2\theta_3 + \frac{K}{R_1}\right] A_{3x} \\ &\quad + k_4^2\left[-\mu\cos 2\theta_4 + K\sin^2\theta_4 + \frac{K}{R_2}\right] A_{4x} = 0, \end{aligned} \tag{18}$$

$$k_3^3 R_2 \sin\theta_3 A_{3x} + k_4^3 R_1 \sin\theta_4 A_{4x} = 0, \tag{19a}$$

$$k_0 \sin\theta_0 (A_0 - A_1) - k_1 \sin\theta_2 A_2 + k_3 \cos\theta_3 A_{3x} + k_4 \cos\theta_4 A_{4x} = 0. \tag{19b}$$

Equations (18) and (19) can be written in matrix form as

$$AX = B,$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ \sin\theta_0 \end{bmatrix}, \tag{20}$$

$$X = [x_1 \ x_2 \ x_3 \ x_4]^T.$$

The non-vanishing elements of matrix $[A]$ in non-dimensional form can be written as

$$a_{11} = 1, \qquad a_{12} = -\left[\frac{\lambda}{\lambda'}\left(\frac{\alpha'}{V_1}\right)^2 + \frac{K}{\lambda'} D D_1^2\right],$$

$$a_{13} = D D_3 \frac{K}{\lambda'} \cos\theta_0, \qquad a_{14} = D D_4 \frac{K}{\lambda'} \cos\theta_0,$$

$$a_{23} = -\cos^2\theta_0 + \frac{1}{D}\left(\frac{\alpha'}{V_3}\right)^2 \left[\frac{\mu}{K} + 1 + \frac{1}{R_1}\right],$$

$$a_{24} = -\cos^2\theta_0 + \frac{1}{D}\left(\frac{\alpha'}{V_4}\right)^2 \left[\frac{\mu}{K} + 1 + \frac{1}{R_2}\right], \tag{21}$$

$$a_{33} = D_3, \qquad a_{34} = D_4 \frac{R_1}{R_2}\left(\frac{\alpha'}{V_4}\right)^2 \left(\frac{\alpha'}{V_3}\right)^{-2}, \qquad a_{22} = D_1 \cos\theta_0,$$

$$a_{41} = \sin\theta_0, \qquad a_{42} = D_1, \qquad a_{43} = a_{44} = -\cos\theta_0,$$

where

$$D_i^2 = \left(\frac{\alpha'}{V_i}\right)^2 - \cos^2\theta_0, \qquad i = 1, 3, 4$$

$$R_1 = \frac{\omega^2}{\omega_0^2} - 2 - \frac{c_4^2 k_3^2}{\omega_0^2}, \qquad R_2 = \frac{\omega^2}{\omega_0^2} - 2 - \frac{c_4^2 k_4^2}{\omega_0^2},$$

$$D = 2\frac{\mu}{K} + 1.$$

The elements of matrix $[X]$ are as follows:

$$x_1 = \frac{A_1}{A_0}, \qquad x_2 = \frac{A_2}{A_0}, \qquad x_3 = \frac{A_{3x}}{A_0}, \qquad x_4 = \frac{A_{4x}}{A_0}.$$

x_1 to x_4 represent the amplitude ratios of reflected longitudinal wave, refracted longitudinal wave, refracted coupled wave at an angle θ_3 and refracted coupled wave at an angle θ_4, respectively.

5. SPECIAL CASE

If the elastic constants corresponding to micropolarity of the half-space M_2 vanish, the problem should reduce to that of reflection and refraction of the longitudinal wave at the liquid–solid interface. This is verified as follows: When $K=\alpha=\beta=\gamma=0$, the velocities of various waves given in equations (6) and (7) with the help of relations given just below equation (1) in the half-space M_2 reduce to

$$V_1 = c_1, \qquad V_3 = c_2, \qquad V_2 = V_4 = 0,$$

and the Snell's law becomes

$$\frac{\cos\theta_0}{\alpha'} = \frac{\cos\theta_1}{\alpha'} = \frac{\cos\theta_2}{c_1} = \frac{\cos\theta_3}{c_2}.$$

With these considerations, note that the boundary condition (19a) yields $A_{4x}=0$, i.e., the coupled wave at an angle θ_4 does not exist, as was expected beforehand. Replacing α', c_2, k_0, k_1, k_3, A_1/A_0, A_2/A_0 and A_{3x}/A_0 by c, b_1, k, K_1, k_1, V, W and P, respectively, and also changing θ_0, θ_2 and θ_3 by $90-\theta$, $90-\theta_1$ and $90-\gamma_1$, respectively, in the remaining boundary conditions (18) and (19b), one can easily obtain the equations (4.19), (4.20), (4.22) of reference [13] (pp. 30–31) for the relevant problem. Consequently, on solving these reduced boundary conditions, one can obtain the same reflection and refraction coefficients as given in reference [13] (pp. 31).

6. WAVES AT THE LIQUID/SOLID INTERFACE (II)

Here, the solutions in the half-spaces M_1 and M_2 are considered such that the radiation condition is satisfied. The appropriate form of the solution in liquid medium M_1 $[z<0]$ is:

$$\phi' = B_0 \exp[\imath k(z\xi_0 + y - ct)]$$

where

$$\xi_0 = \left[\frac{c^2}{\alpha'^2} - 1\right]^{1/2}. \qquad (22)$$

B_0 is the arbitrary constant and k is the wave number. And in the micropolar medium M_2 $[z>0]$ the solutions are,

$$\begin{aligned} u_2 &= G\exp[\imath\{k(y-ct) - V_1'z\}], \\ u_3 &= S\exp[\imath\{k(y-ct) - V_1''z\}], \\ \phi_1 &= Q\exp[\imath\{k(y-ct) - V_1'''z\}], \end{aligned} \tag{23}$$

where

$$V_1' = k\left[\frac{\rho c^2}{\lambda + 2\mu + K} - 1\right]^{1/2}, \qquad V_1'' = k\left[\left(1 - \frac{K}{\mu}\right)\frac{\rho c^2}{\mu} - 1\right]^{1/2},$$
$$V_1''' = \left[k^2\left(\frac{\rho c^2 J}{\gamma} - 1\right) - \frac{2K}{\gamma}\right]^{1/2}, \tag{24}$$

and G, S and Q are arbitrary constants.

The propagation of surface waves along the interface is possible if the quantities under the square root in the expressions (22) and (24) are positive. Substituting the values from equations (22) and (23) in the boundary conditions given by equation (13) and making use of equations (14b) and (2), one obtains the following period equation for waves along the interface between liquid/micropolar elastic solid half-spaces, which is analogous to the Stoneley wave,

$$\sqrt{x^2 - 1}[a_1 a_2 \sqrt{a_3 x^2 - 1}\sqrt{a_4 x^2 - 1} - 1] + a_1 a_5 x^2 \sqrt{a_3 x^2 - 1} = 0, \tag{25}$$

where

$$x = \frac{c}{\alpha'}, \qquad a_1 = 1 + \frac{K}{\mu}, \qquad a_2 = 1 + \frac{2\mu}{\lambda} + \frac{K}{\lambda},$$
$$a_3 = \frac{\alpha'^2}{c_1^2 + c_3^2}, \qquad a_4 = \left(1 - \frac{K}{\mu}\right)\frac{\alpha'^2}{c_2^2}, \qquad a_5 = \frac{\lambda'}{\lambda}. \tag{26}$$

Clearly period equation (25) is independent of wave number. Hence, surface waves along the liquid–solid interface are non-dispersive.

7. NUMERICAL RESULTS AND DISCUSSION

In order to study these problems numerically, the following values of relevant elastic parameters have been taken: for micropolar elastic solid:

$$\begin{gathered} \lambda = 7{\cdot}59\ \text{GPa}, \qquad \mu = 1{\cdot}89\ \text{GPa}, \qquad c_1 = 2{\cdot}28\ \text{mm}/\mu\text{s}, \\ K = 0{\cdot}0149\ \text{GPa}, \qquad \gamma = 2{\cdot}93\ \text{kN}, \qquad c_2 = 0{\cdot}929\ \text{mm}/\mu\text{s}, \\ \rho = 2{\cdot}192\ \text{g/cm}^3, \qquad j = 0{\cdot}196\ \text{mm}^2, \qquad c_3 = 0{\cdot}0825\ \text{mm}/\mu\text{s}, \\ c_4 = 2{\cdot}48\ \text{mm}/\mu\text{s}, \qquad \omega^2/\omega_0^2 = 10. \end{gathered} \tag{27}$$

for liquid half-space:

$$\alpha' = 1{\cdot}48\ \text{mm}/\mu\text{s}, \qquad \rho' = 1{\cdot}01\ \text{g/cm}^3, \tag{28}$$

Equation (20) is solved for amplitude ratios. It is found that various amplitude ratios depend on the angle of emergence and frequency of the incident longitudinal wave, and the nature of dependence of these amplitude ratios is different for different values of angle of emergence and frequency.

Note from Figure 2 that for the reflected longitudinal wave, the modulus of amplitude ratio x_1 has a value of one at zero degree angle of incidence. This value increases monotonically with an increase in the value θ_0, and attains its maximum value equal to 1·3616 at $\theta_0 = 17°$. Beyond $\theta_0 = 17°$, it decreases monotonically achieving its minimum value equal to 0·0254 at $\theta_0 = 43°$. Thereafter, there exist local maxima and minima in the range $44° < \theta_0 < 49°$ and $50° < \theta_0 < 58°$. Finally, beyond $\theta_0 = 58°$, it increases gently with an increase in θ_0 and approaches the value 0·4450 as θ_0 approaches 90°.

For a refracted longitudinal wave, the modulus of amplitude ratio x_2 has a value of zero at zero degree angle of incidence. It increases monotonically with increase in θ_0 and attains its maximum value 1·1635 at $\theta_0 = 49°$. Beyond $\theta_0 = 49°$, it decreases monotonically and attains its minimum value 0·7849 at $\theta_0 = 61°$ and then increases slowly towards the value 0·8190 as θ_0 approaches 90°.

For the refracted coupled wave at angle θ_3, the modulus of amplitude ratio x_3 has a value of zero at zero degree angle of incidence. It increases monotonically,

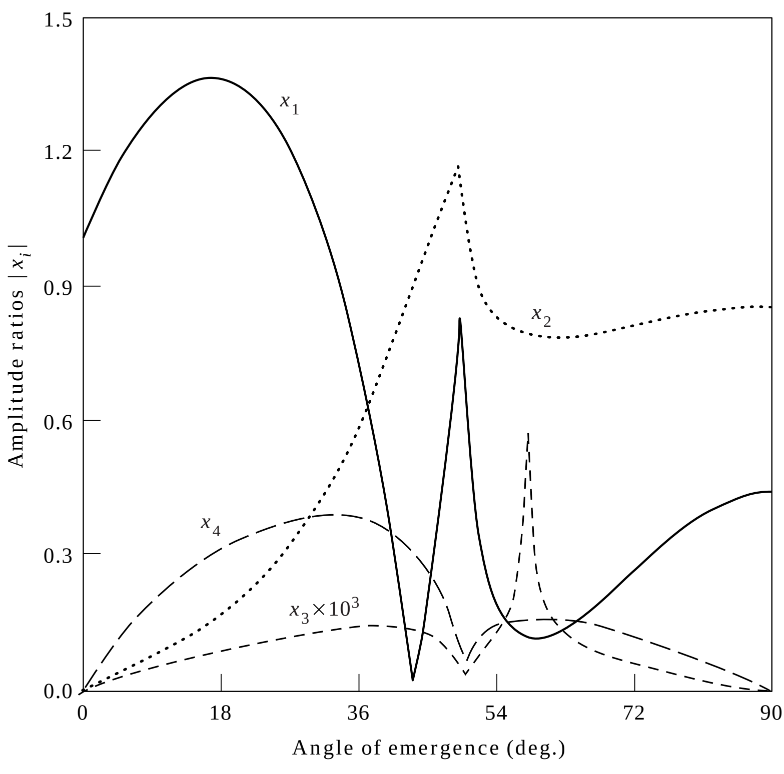

Figure 2. Variation of amplitude ratios with angle of emergence of longitudinal wave (when $\omega^2/\omega_0^2 = 10$).

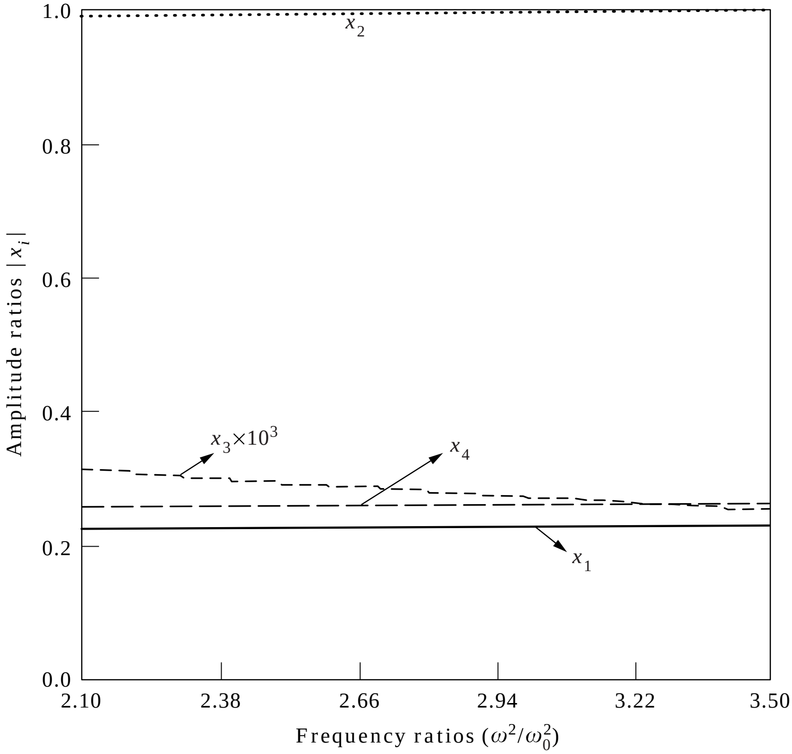

Figure 3. Variation of amplitude ratios with frequency ratio (ω^2/ω_0^2) (when $\theta_0 = 45°$).

attaining a value equal to 0·00142 at $\theta_0 = 39°$. Thereafter, it decreases to a value approaching zero at $\theta_0 = 49°$. Then, there is a sharp increase in the value and it reaches 0·00591 at nearly 57°. Further it decreases rapidly and finally approaches the value zero as θ_0 approaches 90°. The graph of x_3 has been shown to be 10^3 times the original value.

For the refracted coupled wave at an angle θ_4, the modulus of amplitude ratio x_4 has value zero at zero degree angle of incidence. It increases monotonically as it increases beyond $\theta_0 = 0°$, and attains its maximum value equal to 0·3837 at $\theta_0 = 32°$. Thereafter, its value decreases and then increases slowly and again decreases gently approaching zero as θ_0 approaches 90°.

Figure 3 shows the variation of various amplitude ratios with frequency ratio (ω^2/ω_0^2). It can be observed from this figure that the values of all amplitude ratios decreases very slowly to a small extent except x_4, which is constant for all values of frequency ratio. Coupled waves in micropolar medium are excited only if $\omega^2/\omega_0^2 > 2$.

Finally, the value of c/α' has been calculated from the period equation (25) and is found to be equal to 1·576848, This clarifies that the propagation of surface waves along the interface (which is analogous to the Stoneley wave) between the two half spaces is possible.

In conclusion, a mathematical study of reflection/refraction coefficients of incident longitudinal wave at a plane interface between M_1 and M_2 is made. It is found that at $\theta_0 = 0°$, i.e., grazing incidence, the modulus of $x_1 = 1$. In this case, refracted waves are not excited. On the other hand, at $\theta_0 = 90°$ i.e., normal incidence, coupled waves in the solid medium are not excited. Both reflection and refraction occur together for $0° < \theta_0 < 90°$, having ups and downs in their values. It is also important to note that the refraction of the coupled wave at an angle θ_3 is negligibly small compared to other waves.

In the majority of practical cases, the velocity of sound α' in the liquid is less than the velocity of longitudinal wave V_1. It may also be less than the velocities V_3 and V_4. One can consider the case, $V_3 < V_4 < \alpha' < V_1$. From equation (16)

$$\cos\theta_2 = \frac{V_1}{\alpha'}\cos\theta_0, \qquad \cos\theta_1 = \frac{V_i}{\alpha'}\cos\theta_0, \qquad (i = 3, 4).$$

Hence, it is clear that for $\cos\theta_0 > \alpha'/V_1$, the angle θ_2 will be complex. However, the angles θ_3, θ_4 will be real for all θ_0. Thus, the longitudinal wave in the micropolar solid medium will be an inhomogeneous wave, "gliding" along the boundary, while the other waves will be ordinary plane waves. Other cases can be discussed in a similar manner as given in reference [13] for the liquid–solid boundary.

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