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LETTERS TO THE EDITOR

A PARTICULAR RECENSION ON TORSIONAL VIBRATIONS OF BEAMS OF THIN-WALLED OPEN SECTIONS

M. J. MAURIZI AND P. M. BELLÉS
Department of Engineering, Universidad Nacional del Sur, 8000 *Bahía Blanca, Argentina*

AND

H. D. MARTÍN
UAVT, Universidad Tecnológica Nacional, 2600 *Venado Tuerto, Argentina*

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The analysis of the free torsional vibrations of thin-walled beams with open cross-section, in which the shear center and the centroid are coincident, is of interest in modern technological applications and many structural configurations such as those performed by the aerospace industry. This is a special case of the general problem whose exact solution has been developed by Gere and Lin [1].

Recently Li *et al*. [2] proposed a versatile approach for calculating the uncoupled torsional natural frequencies of open section members that is applicable in the cited case.

The mentioned authors obtained the solutions for natural frequencies and the associated mode shape vectors of a simple beam for four combinations of end support conditions. Characteristic equations giving relationships between natural frequencies, the dimensions and the properties of a beam are determined by applying the conditions associated with three common types of end support. Furthermore, an other task of this study was to determine the influence of warping on torsional natural frequencies. This is due to the fact that the prevention of warping increases the torsional stiffness of a member.

The present letter is motivated by the fact that this type of set-up for an analysis of the torsional vibrations of beams of a thin-walled open section was derived earlier by Gere [3]. His study, published in 1954, was included in the dissertation submitted to the Stanford University (California, U.S.A.) in partial fulfillment of the requirements for the degree of Doctor of Philosophy† and has many points of similarity to that used subsequently by Li *et al*. [2].

In the investigation presented and for the case of a thin-walled bar of open cross-section, the equation of static equilibrium for torsion is

†Adviser: Professor Donovan H. Young.

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$$GC\frac{\partial\phi}{\partial x} - EC_w\frac{\partial^3\phi}{\partial x^3} = M_t, \qquad (1)$$

where E is the modulus of elasticity, G is the shear modulus, C is the St Venant torsional constant, C_w is the warping constant, M_t is the total torque acting at any cross-section, ϕ is the angle of twist at any cross-section, and x denotes the length co-ordinate of the bar.

This equation was first derived for the case of an I-beam by Timoshenko [4] and was then extended to all thin-walled open cross-sections by Wagner [5], and is valid for bars in which the length is large compared to the cross-sectional dimensions.

The principle of D'Alembert states that the equation of motion for a dynamic system may be obtained by substituting inertia forces into the equation of static equilibrium. At one time, the rate of change of the total torque equals the intensity of the inertia torque

$$\frac{\partial M_t}{\partial x} = mI_p\frac{\partial^2\phi}{\partial t^2}, \qquad (2)$$

in which I_p is the polar moment of inertia about the centroid, m is the mass density of the material of the bar, and t is the time.

Differentiating equation (1) and combining with equation (2) leads to the differential equation for free torsional vibrations

$$GC\frac{\partial^2\phi}{\partial x^2} - EC_w\frac{\partial^4\phi}{\partial x^4} = mI_p\frac{\partial^2\phi}{\partial t^2} \qquad (3)$$

Obviously, this equation coincides with equation (5) of reference [2]. Furthermore, the values of C_w for I and Z-sections were obtained by applying the formulas of reference [6]. If the shear center does not coincide with the centroid, the torsional vibrations will be accompanied by bending vibrations and the equations are more complicated. The solution of the general case was discussed in Gere's dissertation and it has also been demonstrated in reference [1]. In reference [3] the effect of warping on the frequency of vibration and the shapes of the normal modes of vibration are determined for bars of single span with various end conditions. In the case of a simply supported beam, a formula for the torsional frequencies and an expression for mode shape are derived. For other conditions of support (both ends clamped, both ends free, one end clamped and the other simply supported, cantilever beam with warping restrained and cantilever beam with unrestrained warping) the frequency equations are derived and their solutions presented in graphical fashion. In this respect, Maurizi *et al.* [7] corroborated these results and numerical values of the frequency coefficients βL were calculated for the first three torsional modes of vibration of a beam clamped at the ends (see Table 1) as a function of k, where

$$k = L\sqrt{(GC)/(EC_w)}, \qquad (4)$$

TABLE 1

Values of βL for the first three modes of vibration corresponding to a clamped–clamped beam

k	Mode 1	Mode 2	Mode 3
0	4·730041	7·853205	10·995607
0·001	4·730041	7·853205	10·995607
0·01	4·730038	7·853204	10·995607
0·1	4·729803	7·853124	10·995566
1	4·706582	7·845188	10·991484
5	4·299672	7·677924	10·900252
10	3·828357	7·352776	10·686142
100	3·205651	6·410968	9·615626
500	3·154209	6·308416	9·462617

$$\beta = \sqrt{\frac{-GC + \sqrt{(GC)^2 + 4EC_w m I_p p_n^2}}{2EC_w}}. \tag{5}$$

p_n is the natural frequency and the other coefficients are as defined previously. The parameter k, such as βL, is a dimensionless quantity which is constant for any given beam. It is interesting to observe that as k increases, which means C_w is relatively small, the solution is the same as for a simply supported beam and the mode shape approaches a sine curve.

On the other hand, Table 2 shows the values of βL for the fundamental mode of vibration of a cantilever beam considering warping and neglecting warping. In the first case, a cantilever beam was considered in which the supported end was fixed and offered complete restraint against warping. In the second case, the supported end offers restraint against rotation but warping is free to occur. Therefore, it is a fixed support with regard to bending. In both cases, if k is very large, the frequency coefficients approach $\pi/2$,

TABLE 2

Values of βL fundamental mode for a cantilever beam with warping restrained and with unrestrained warping

k	With unrestrained warping	With warping restrained
0	0	1·8751041
0·001	0·0416119	1·8751041
0·01	0·1314175	1·8751083
0·1	0·4101960	1·8755320
1	1·1350832	1·9080204
5	1·5450658	1·8667033
10	1·5623155	1·7351255
100	1·5707925	1·5866555
500	1·5707963	1·5769441

Finally, another basic issue was addressed by Gere [3]. If a bar has a cross-section such that $C_w = 0$ (for example, a cruciform section), the differential equation (3) of free torsional vibrations becomes

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{m I_p}{GC} \frac{\partial^2 \phi}{\partial t^2}. \tag{6}$$

For such shapes the formulas for torsional frequencies and modes of vibration are sufficiently simple. This application of the theory is also valid for bars of circular cross-section and the formulas may be used approximately for other solid sections [8].

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