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A SYMMETRIC FORMULATION FOR EXPERIMENTAL STATISTICAL ENERGY ANALYSIS

S. FINNVEDEN
Department of Vehicle Engineering, KTH, SE-100 44 Stockholm, Sweden

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1. INTRODUCTION

Statistical Energy Analysis (SEA) predicts random vibrations and vibration energy flow in built-up structures. Experimental SEA is a set of techniques for identifying SEA parameters [1]. The results aid transmission path analysis, identification of sources and confirmation and updating of predictive SEA models. The approach, probably first proposed by Woodhouse [2], has recently acquired greatly increased interest as manifest in numerous conference contributions, journal papers [3, 4], doctoral thesises [5, 6] and commercial software. It is believed the method may have many industrial applications, especially for vehicles.

Experimental SEA is most often based on the Power Injection Method (PIM) [7]. Thus, for a number of locations in a substructure, a force is applied, the input power measured and the kinetic energy estimated in each substructure. As in predictive SEA, it is assumed that dissipation powers and coupling powers are proportional to vibration energy. Upon this, the measured total input power to one substructure is related to dissipation and coupling powers in the entire structure. The scheme is repeated, applying force to the other substructures in turn; eventually the SEA parameters can be determined from a linear system of equations.

An advance by De Langhe suggests, for each excitation, dividing both response measurements and input power by the square of the applied force [5]. The resulting expression for the conservation of energy is equal to the one applying, if all forces have unit amplitude. One advantage is that the stability of the relation between input power and kinetic energy is assured (crucial when the measurements are made over several days or weeks). Another is that some noise reduction results when transfer functions, instead of spectral powers, are measured.

The PIM requires that vibration energies are estimated from measured response variables. For non-homogenous structures this is not a trivial problem. To handle it Lalor introduced the "equivalent mass" [1, 8]. This is a substructure property which when multiplied by the mean square velocity gives the vibration energy. Lalor showed that for weak coupling, the equivalent mass can be estimated in frequency bands from comparisons of PIM measurements and transient decay measurements. Hermans revealed that if experimental SEA is

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used for response prediction, the actual value of the equivalent mass affects only the scaling of the equations, not the predicted values of response [9]. However, it must be emphasised, the equivalent mass used in the PIM must be a property (as a function of frequency) of the substructure. If it has one value when the substructure is directly driven and another when it is indirectly driven, e.g., if the coupling discriminates some of the modes, the approach fails, imposing a severe limitation on experimental SEA.

Unfortunately experimental SEA equations are sometimes badly conditioned and occasionally results are not even physically correct. It appears as if weak coupling, many resonances and modal overlaps that are of the order of unity or greater favour the PIM [6]. The same criteria are believed to apply for predictive SEA [10]. Often poor results are attributed to the combination of inevitable measurement errors and badly scaled matrices. However, arguably, results can be disappointing when the SEA equations are invalid—because of strong coupling, non-diffuse fields, too large or low damping, low mode count, non-resonant vibrations, etc. Yet there is no quantitative knowledge of when SEA applies but it is known that there are limitations. Thus, since the PIM is based on the SEA assumption of coupling powers being proportional to vibration energy, one can not expect that experimental SEA is a suitable method unless predictive SEA applies.

Langley derives exact equations relating input power to variables proportional to kinetic energy [11]. Though not identical to the SEA equations, they are approximately equal for resonant vibrations and weak coupling. (It is not yet known whether indirect couplings are ultimately negligible for weak coupling.) To arrive at the weak coupling result, Langley says that coupling is weak when the Green function of a substructure is, within the required accuracy, unaltered when the substructure is connected to the rest of the structure.

Langley's equations are very general based only on the assumptions of linear vibrations, rain on the roof excitation and uniform density in each substructure. In addition but only to demonstrate their similarity with SEA, weak conservative couplings and resonant motion are assumed. In what follows Langley's formulation is developed for non-uniform systems assuming that an effective density can be defined as a substructure property, independent of excitation.

It is proposed that Langley's formulation be used as a basis for experimental SEA. The advantages are: it gives symmetric equations—reducing the measurement effort by almost a factor of two; it is based on transfer function measurements; it gives non-dimensional equations. Most important, the equations are less restrictive than the PIM.

2. ENERGY–INPUT POWER RELATIONSHIP

Langley derives equations for the conservation of vibration energy in structures excited by random, rain-on-the-roof, forces [11]. Linear theory and harmonic motion of the form $e^{j\omega t}$ are assumed, without further limitation, the response of the coupled system can be expressed in the form

$$\nu_i(\mathbf{x}, \omega) = \sum_j \int_{V_j} G_{ij}(\mathbf{x}, \mathbf{y}, \omega) F_j(\mathbf{y}, \omega)\, \mathrm{d}\mathbf{y}, \tag{1}$$

where v_i is the vibration velocity in subsystem i, F_j is the applied force in subsystem j, V_j is the "volume" of subsystem j and the Green function $G_{ij}(\mathbf{x}, \mathbf{y}, \omega)$ represents the response at location $\mathbf{x}$ on subsystem i to a harmonic point load situated at location $\mathbf{y}$ on subsystem j. It follows from reciprocity that

$$G_{ij}(\mathbf{x}, \mathbf{y}, \omega) = G_{ji}(\mathbf{y}, \mathbf{x}, \omega). \tag{2}$$

The frequency averaged kinetic energy in element i is given by

$$T_i = \frac{1}{2\Omega} \int_\Omega \sum_j \sum_k \int_{V_i} \int_{V_j} \int_{V_k} \rho_i(\mathbf{x}) G^*_{ij}(\mathbf{x}, \mathbf{y}, \omega) G_{ik}(\mathbf{x}, \mathbf{z}, \omega)$$

$$\times F^*_j(\mathbf{y}, \omega) F_k(\mathbf{z}, \omega)\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{z}\, \mathrm{d}\omega \tag{3}$$

where ρ is the "density" of subsystem i, Ω is the frequency band and F^* is the complex conjugate of F.

Assuming rain on the roof excitation, the applied forces are statistically independent and delta correlated in space. Thus, the cross-spectrum, S_{jk}, of F_j and F_k has the form

$$S_{jk}(\mathbf{y}, \mathbf{z}, \omega) = \delta_{jk} \delta(\mathbf{y} - \mathbf{z}) \alpha_j. \tag{4}$$

This also incorporates the assumption that the cross-spectrum is independent of ω over the frequency band Ω. Upon these assumptions, the statistical expectations of the frequency averaged kinetic energies in the subsystems can be calculated [11]. The result is conveniently expressed in matrix form,

$$\mathbf{T} = \mathbf{M}\alpha, \tag{5}$$

where the vector $\mathbf{T}$ contains the frequency averaged kinetic energies of the subsystem and the entries of $\mathbf{M}$ are given by

$$M_{ij} = \frac{1}{2\Omega} \int_\Omega \int_{V_i} \int_{V_j} \rho_i |G_{ij}(\mathbf{x}, \mathbf{y}, \omega)|^2\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\omega. \tag{6}$$

By similar calculations, the frequency averages of the input powers are given by

$$\mathbf{P} = \mathbf{q}\alpha, \tag{7}$$

where the diagonal matrix $\mathbf{q}$ has the entries

$$q_i = \frac{1}{\Omega} \mathrm{Re}\left(\int_\Omega \int_{V_i} G_{ii}(\mathbf{x}, \mathbf{x}, \omega)\, \mathrm{d}\mathbf{x}\, \mathrm{d}\omega \right). \tag{8}$$

From equations (5) and (7), the relation between frequency averaged kinetic energies and input powers is

$$\mathbf{T} = \mathbf{M}\mathbf{q}^{-1}\mathbf{P}. \tag{9}$$

To further simplify this, Langley considers subsystems that have constant density, however, simplification may be found for more general systems. Thus, an average, or effective, density is defined for each element and for each excitation. These effective densities are collected in the matrix $\mathbf{R}$ with entries

$$R_{ij} = \frac{1}{2\Omega}\int_{\Omega}\int_{V_i}\int_{V_j} \rho_i |G_{ij}(\mathbf{x}, \mathbf{y}, \omega)|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} \, \mathrm{d}\omega / \overline{M}_{ij}, \tag{10}$$

where

$$\overline{M}_{ij} = \frac{1}{2\Omega}\int_{\Omega}\int_{V_i}\int_{V_j} |G_{ij}(\mathbf{x}, \mathbf{y}, \omega)|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} \, \mathrm{d}\omega. \tag{11}$$

Now it is assumed that the effective density is independent of which substructure is excited. This is valid when either the density is uniform within each subsystem or the form of the vibrations, the relative amplitudes of the modes or of the vibration waves, is independent of excitation. The first assumption is Langley's, the latter is in accordance with the SEA assumptions of equipartition of modal energies or of diffuse wave fields within each subsystem. Consequently, in mathematical terms, it is assumed that

$$R_{ij} = R_{ii}, \qquad \forall j. \tag{12}$$

Upon this basis equation (9) may be written

$$\mathbf{T} = \mathbf{r}\overline{\mathbf{M}}\mathbf{q}^{-1}\mathbf{P}, \tag{13}$$

where the entries of the symmetric and positive matrix $\overline{\mathbf{M}}$ are given in equation (11) and the entries of the positive and diagonal matrix $\mathbf{r}$ are R_{ii}, equation (10).

Finally, to arrive at a symmetric equation similar to the one used in SEA, variables $\mathbf{E}$ are defined

$$\mathbf{E} = \pi\mathbf{r}^{-1}\mathbf{q}^{-1}\mathbf{T}, \tag{14}$$

and upon inserting this definition into equation (13) this can be inverted to yield

$$\mathbf{C}\mathbf{E} = \mathbf{P}, \tag{15}$$

where

$$\mathbf{C} = (\pi\mathbf{q}^{-1}\overline{\mathbf{M}}\mathbf{q}^{-1})^{-1} = (1/\pi)\mathbf{q}\overline{\mathbf{M}}^{-1}\mathbf{q}. \tag{16}$$

$\mathbf{C}$ is a non-dimensional symmetric matrix. Moreover, for conservative coupling it can be written as the governing matrix in SEA—in terms of dissipation and coupling loss factors [11].

Following Lyon and DeJong [12], it is proposed here that the elements of the vector $\mathbf{E}$ be termed "vibration power potentials", or, when the context is given, "potentials".

Langley's definition of weak coupling is that the Green function G_{ii} for the coupled substructure is, within the accuracy required, equal to the one for the uncoupled substructure [11]. For weak coupling, defined in this manner, and for a frequency band containing resonances, the potentials **E** are

$$\mathbf{E} \approx 2\mathbf{n}^{-1}\mathbf{T}, \tag{17}$$

where the diagonal matrix **n** has the entries n_i—the modal densities (modes per rad/s) in the substructures. Consequently, for weak coupling and resonant vibrations, **E** contains the total vibration energies divided by the modal densities, i.e., the modal energies, sometimes referred to as "modal power potential" [12]. These are the variables most often used in SEA.

The derivation differs from Langley's in two respects. First, in not being restricted to systems with uniform density but applying to all systems for which effective densities can be defined. Second, in not being for frequency bands but for frequency band averages. Consequently, the matrix **C** is non-dimensional and the matrices **q** and $\overline{\mathbf{M}}$ are frequency averages. For high frequencies and large substructures, these can reach asymptotic values that are quite independent of bandwidth of the analysis. Possibly, upon an ergodic assumption, their estimated values in frequency bands may be equal to their ensemble average values, this however, is beyond the scope of the present investigation.

3. EXPERIMENTAL SEA

It is proposed that equation (15) be taken as the basis for experimental investigation of structural (and acoustic) motion in built-up structures. The matrix **C** is defined in equation (16) by the matrices **q** and $\overline{\mathbf{M}}$ which are determined by the structure's Green function. In an experimental procedure this function is estimated by measured Frequency Response Functions (FRFs). Thus, the input mobilities and the square magnitude of transfer mobilities are measured:

$$Y_i(x_n, \omega) = \frac{\nu_i(x_n, \omega)}{F_i(x_n, \omega)}; \qquad H_{ij}^2(x_n, y_m, \omega) = \left|\frac{\nu_i(x_n, \omega)}{F_j(y_m, \omega)}\right|^2, \tag{18}$$

where $\nu_i(x_n, \omega)$ is the velocity in substructure i at observation point x_n and $F_j(y_m, \omega)$ is the applied force in substructure j at position y_m. The entries of the matrix **q**, equation (8), are estimated by

$$q_i = \frac{V_i}{N}\,\mathrm{Re}\left(\sum_{n=1}^{N}\frac{1}{\Omega}\int_{\Omega} Y_i(x_n, \omega)\,\mathrm{d}\omega\right), \tag{19}$$

and, similarly, those of the symmetric matrix $\overline{\mathbf{M}}$, equation (11), are estimated by

$$\overline{M}_{ij} = \frac{V_i V_j}{N\,M}\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{1}{2\Omega}\int_{\Omega} H_{ij}^2(x_n, y_m, \omega)\,\mathrm{d}\omega, \tag{20}$$

where V_i is the "volume" of element i. From the previous section it is clear that

$\mathbf{q}$ and $\overline{\mathbf{M}}$ are independent entities, so there is no requirement that the same points be used for the approximate evaluation of the integrals, while perhaps this is most favourable. The number of sampling points required to obtain convergence of the integrals depends on the structure and frequency. Possibly, when SEA applies and there are many resonant modes within the frequency band all having roughly the same amplitude, the averages converge rapidly and only a few (say five to ten?) points in each substructure are needed. In other situations much more effort may be required.

When the matrices $\mathbf{q}$ and $\overline{\mathbf{M}}$ have been estimated with sufficient accuracy, the matrix $\mathbf{C}$ can be calculated by equation (16). Equation (15) can then be used for calculations of the potentials $\mathbf{E}$, if the operating forces can be approximated by rain on the roof. If it can be assumed that couplings are non-dissipative, and indirect couplings are negligible, the results can be used to find the coupling powers in a structure.

3.1. EQUIVALENT MASS AND EFFECTIVE DENSITY

To find the kinetic energies of the substructures, the effective densities defined in equation (10) are needed. When SEA applies, and the effective densities are independent of excitation, these can be found by Lalor's method for the equivalent mass [1, 8].

In reference [1] the equivalent mass, M_{eq}, is defined by

$$(E_{tot})_i = T_i + U_i = M_{eq\cdot i}\langle v_i^2 \rangle, \tag{21}$$

where T_i is the kinetic energy in element i, U_i is the strain energy, $\langle\ \rangle$ denotes spatial averaging and where all quantities are implicit time averages. In reference [1] it is, implicitly, assumed that equation (21) is equally valid when the element is directly and indirectly excited. This assumption is similar to the one in equation (12). Now, assuming that equation (12) is valid, it follows from equation (10) that $T_i = R_{ii} V_i \langle v_i^2 \rangle / 2$. Upon this, if it is also assumed that the strain energy is on average equal to the kinetic energy, it follows that

$$M_{eq\cdot i} = 2 V_i R_{ii}, \tag{22}$$

so that estimates of M_{eq}, based on reverberation time and input power measurements [1], can be used to determine the effective density.

The effective densities can also be estimated from the measured response using equation (10). Then, a part of the total mass in the substructure is assigned to each response point x_n [6, 13, 14]. For complicated structures this is non-trivial. However, if the kinetic energy is dominated by measurable out-of-plane motion and if great care is taken and a sufficient number of response points are used, the results should be reasonable. Thus, the entries of the matrix $\mathbf{R}$ are estimated from

$$R_{ij} = \frac{V_i V_j}{N M} \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{1}{2\Omega} \int_{\Omega} m_{in} H_{ij}^2(x_n, y_m, \omega)\, \mathrm{d}\omega / \overline{M}_{ij}, \tag{23}$$

where the estimate of $\overline{\mathbf{M}}_{ij}$ is defined in equation (20) and m_{in} is the mass assigned to response point x_n in substructure i.

There is no extra measurement effort required for evaluating $\mathbf{R}$—it is found by "guessing" the m_{in} and performing some computer calculations. It is believed that even if the results of equation (23) are not reliable for estimating effective densities they may still be useful for verifying, or otherwise, the assumption (12) since if it holds, e.g., if there is modal equipartition or a diffuse wave field, it should also hold for a sample of response points. Thus, it is proposed that $\mathbf{R}$ should always be calculated to check that effective densities can be defined for the substructures.

3.2. RESPONSE CALCULATIONS

If effective densities can be defined, then their actual values are not needed for response calculations. From the definitions of the vibration potentials $\mathbf{E}$, (14), and of the effective densities, (10) and (12)

$$E_i = \pi q_i^{-1} r_i^{-1} T_i = \frac{\pi q_i^{-1}}{2\Omega} \int_{\Omega} \int_{V_i} |\nu_i(\mathbf{x}, \omega)|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}\omega. \tag{24}$$

Consequently, when the "volume" of the substructure, the frequency and spatial averaged input mobility, q_i and the potential E_i are known, the mean square velocity of the substructure can be found. Similarly, from measurements of the mean square velocities, and the non dimensional matrix $\mathbf{C}$, the operating input powers can be found using equation (15). If indirect couplings are negligible, measurements of $\mathbf{C}$ and of the operating input powers can be used to determine the important paths of transmission in the investigated structure. Finally, if coupling is weak according to Langley's definition [11], equation (17) holds and the proposed formulation is equal to an SEA formulation, so that the measurements can be used for updating predictive SEA models.

In summary, if equation (12) is valid then many of the objectives for experimental SEA can be met with the proposed formulation, even if the values of the effective densities are not known. The assumption (12) is trivially true for uniform systems whereas for non-uniform systems it can be validated by equation (23).

4. CONCLUSIONS

The energy–input power relations previously derived by Langley [11] are developed for application to inhomogenous structures. This application is valid provided that effective densities can be defined as substructure properties, relating a substructure's mean square vibration velocity to its kinetic energy. It may be a function of frequency but must not depend on which substructure is excited. Measurement procedures for validating this assumption are presented.

It is proposed that Langley's energy–input power relations be used for experimental investigations of vibrations in built-up structures. Compared to procedures commonly applied in experimental SEA the advantages are: (1) it is a symmetric formulation; (2) the energy–input power relations are non-

dimensional; (3) it is based on FRFs; (4) it is derived only upon the assumptions of linear equations of motion, the existence of effective densities and stochastic excitation.

Statistical Energy Analysis is built on a number of assumptions. The validity of these and the criteria by which they can be assessed are still open to discussion [10]. In contrast to this, for homogenous structures, the energy–input power relations in equation (15) are based on first principles. Moreover, this work demonstrates that these relations are formulated in terms of directly measurable quantities. In situations where SEA should provide good answers equation (15) is equal to the SEA equations for energy conservation [11]. However, the applicability of equation (15) is much wider: consequently, these equations form a sound and versatile basis for investigating vibrations in a built-up structure.

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