Journal of Sound and Vibration (1999) **223**(1), 49–71
Article No. jsvi.1998.2106, available online at http://www.idealibrary.com on IDEAL®

ON THE STEADY STATE RESPONSE OF OSCILLATORS WITH STATIC AND INERTIA NON-LINEARITIES

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(*Received* 16 *July* 1998, *and in final form* 2 *December* 1998)

The concern of this work is the steady state periodic response having the same period as the excitation of strongly non-linear oscillators $\ddot{u} + \delta\dot{u} + mu + \varepsilon_1 u^2\ddot{u} + \varepsilon_1 u\dot{u}^2 + \varepsilon_2 u^3 = P\cos\Omega t$, where $m = 1$, 0 or -1, ε_1 and ε_1 are positive parameters which may be arbitrarily large. Single-mode and two-mode harmonic balance (HB) approximations, and second order perturbation-multiple time scales (MMS) with reconstitution version I and version II approximations to the steady state amplitude frequency response curves are compared, for the case $m = 1$ with each other, and with those obtained by numerically integrating the equation of motion. The transformation of time $T = \Omega t$ and detuning in the square of forcing frequency are used in the MMS with reconstitution version I and version II. The objective here is to assess the accuracy of these approximate solutions in predicting the system response over some range of system parameters by examining their ability or failure in establishing the correct qualitative behavior of the actual (numerical) solution. The cases $m = 0$ and $m = -1$, are studied for selected range of system parameter, using the single and two modes harmonic balance method and compared to those obtained numerically. It was shown that MMS version II, in addition to being appreciably simpler than MMS version I, leads to more accurate qualitative and quantitative results even when the non-linearity is not necessarily small.

1. INTRODUCTION

This work is concerned with the primary resonance response of a lightly damped strongly non-linear harmonically driven oscillators with inertia and static (displacement dependent) odd non-linearities for which the equation of motion has the following form:

0022–460X/99/210049 + 23 $30.00/0

$$\ddot{u} + \delta\dot{u} + mu + \varepsilon_1 u^2\ddot{u} + \varepsilon_1 u\dot{u}^2 + \varepsilon_2 u^3 = P\cos\Omega t. \quad (1)$$

Here dots denote the time derivatives, δ, ε_1, ε_2 and P are assumed to be positive, the displacement $u(t)$ is of order unity, and m is an integer which may take the values $m = 1$, 0, -1, in order to allow results to be obtained for which the associated linear unforced ($P = 0$) oscillator is, respectively, statically stable, neutrally stable, or statically unstable. In equation (1), the first of the non-linear terms is a softening inertia type, the second is a hardening inertia type, while the third is a hardening static spring type.

Results of free undamped vibration analysis of equation (1) ($P = \delta = 0$), obtained by Hamdan and Shabaneh [1] indicated that for the case $m = 1$, regardless the value of ε_1 and ε_2, the period–amplitude ($\tau - A$) relation exhibits a softening behavior when, roughly, $\varepsilon_1/\varepsilon_2 > 1{\cdot}6$, and a hardening type behavior when $\varepsilon_1/\varepsilon_2 < 1{\cdot}6$; when $\varepsilon_1 \sim 1{\cdot}6\,\varepsilon_2$ the period τ becomes nearly constant, independent of motion amplitude A, i.e., exhibits a nearly linear behavior, for all values of A.

For the cases in which the associated linear oscillator in neutrally stable ($m = 0$) or statically unstable ($m = -1$) the results obtained in reference [1] indicate that, the period τ exhibits a hardening behavior with amplitude **A** variations regardless of the relative strength of the inertia non-linearities with respect to the static hardening non-linearity; i.e., regardless of the value of ε_1 with respect to ε_2. It was also shown in [1] that the period of free oscillations of these types of oscillators described in equation (1) ($P = \delta = 0$) becomes nearly constant independent of motion amplitude A, i.e., exhibits a linear behavior, at relatively large values of A for all three cases $m = 1$, 0, -1. Furthermore, the results presented in reference [1] indicate that, for the case $m = 1$, a qualitative failure of the single term harmonic balance method (SHB) occurs when ε_1 and ε_2 are in the range $1{\cdot}5 < \varepsilon_1/\varepsilon_2 < 1{\cdot}8$.

The aim of this work is to extend the analysis presented in reference [1] by determining quantitative and qualitative information about the steady state response behavior of the harmonically forced oscillators of the type described in equation (1). The emphasis is on situations for which first order approximate solutions may lead to qualitative as well as appreciable quantitative errors in the steady state amplitude–frequency response.

A number of approximate analytic techniques are available for the analysis of non-linear oscillators, such as those under consideration in equation (1). The more commonly used of these methods are, the MMS, which in its standard form, is applicable only to the case $m = 1$, and the HB method which is applicable for all three cases $m = 1$, 0, -1.

In order to illustrate the problem of interest of this work, a brief summary of the procedural steps involved in the MMS and HB methods and limitations of these methods, discussed in details in references [2–6] are presented in the sequel. By applying the MMS method, which is applicable only to weakly non-linear systems, the equation of motion can be ordered by introducing arbitrarily a non-dimensional small gauge parameter ε in the appropriate terms of the equation [2]. For example, to analyze a resonance response, the non-linear terms, the forcing and, the usually assumed linear, damping terms are multiplied by the small gauge parameter ε so that these terms appear at the same time (same order) in the perturbation scheme (equations) [2]. Thus the range of system parameters and response amplitude over which the predicted perturbation solution is satisfactory is fixed in advance by the ordering scheme; however this range is usually left unspecified [6].

Furthermore, the amplitudes of different harmonics of the predicted approximate periodic response are assumed to satisfy the established ordering scheme which determines in advance the relative importance of each of these harmonics and assumes the rapid attenuation of higher ones for the weakly non-linear system [6]. Furthermore, the essence of the MMS perturbation method is to seek asymptotically valid, usually low order, approximations to the steady state periodic response by using a number of time scales and power series expansions for the dependent variables and parameters of the assumed weakly non-linear system in terms of a small positive gauge parameter ε. These series expansions are neither unique nor convergent, and several procedural steps have been devised by various authors in order to obtain consistently ordered (asymptotically valid) first and higher order MMS results. This has led to, so called, different "versions" of the MMS method which differ in, for example, whether or not a transformation of time, $T = \Omega t$, is used, the way the detuning parameters are introduced (i.e., the way the excitation frequency Ω is expanded in power series of the perturbation parameter ε), and in the way the partial time derivatives for the amplitude and phase of the response main harmonic component are used to obtain the second and higher order steady state response [3–5, 7, 8]. Consider, for example, the problem, discussed in detail in reference [3], of obtaining a uniformly valid second order MMS approximation to the steady state primary resonance response of the weakly non-linear oscillator

$$\ddot{v} + \varepsilon\delta\dot{v} + v + \varepsilon g(v) = \varepsilon\cos(\Omega t), \tag{2}$$

where ε is a small positive parameter ($0 < \varepsilon \leqslant 1$), v is of order unity, $\delta \geqslant 0$ and $g(v)$ is a static non-linear function. According to the MMS method [2], one defines a number of time scales $T_n = \varepsilon^n t$, $n = 0, 1, 2, \ldots$, where $T_0 = t$ is the fast time scale on which the main oscillatory behavior of the response occurs and T_n, $n \geqslant 1$, are slow time scales on which the amplitude and phase modulations, caused by the non-linearity, damping and response, take place. Then upon expressing the time t derivatives in terms of the new time scales T_n, and substituting assumed series expansions for the dependent variables, i.e.,

$$v(t, \varepsilon) = v_0(T_0, T_1, T_2, \ldots) + \varepsilon v_1(T_0, T_1, T_2, \ldots) + \varepsilon^2 v_2(T_0, T_1, T_2, \ldots) + \mathcal{O}(\varepsilon^3),$$

and in some cases, assumed series expansion for some or all system parameters into equation (2), and equating the coefficient of each power of ε to zero, one obtains a hierarchical set of linear partial differential equations:

$$D_0^2 v_0 + v_0 = 0, \quad D_0^2 v_1 + v_1 = f_1(v_0, T_0), \quad D_0^2 v_2 + v_2 = f_2(v_0, v_1, T_0, T_1);$$

$$D_0^2 v_n + v_n = \dot{f}_2(v_0, v_1, v_2, \ldots, v_{n-1}, T_0, T_1, T_2, \ldots, T_{n-1}), \tag{3(a–c, …, n)}$$

where $D_n = \partial/\partial T_n$, and the f_n, defined by the system non-linearity $g(v)$, are functions of the system parameters, i.e., damping and excitation amplitude and frequency. These linear differential equations are then solved in sequence to the desired order of approximation, where the solutions $v_0, v_1, \ldots, v_{n-1}$ appear as forcing terms in the differential equation for v_n. For the first of these equations, (equation (3a)), the solution is given by

$$v_0(T_0, T_1, \ldots) = A(T_1, T_2, \ldots)e^{iT_0} + cc, \tag{4}$$

where cc is the complex conjugate of the preceding term, and $\mathbf{A} = A(T_1, T_2, \ldots)$ is the complex amplitude which is function of the slow time scales $T_1, T_2, \ldots$. One then solves equations (3b–n) sequentially for $v_1, v_2, \ldots, v_n$, where usually only the particular solutions to each v_i are considered. The fact that each of the forcing terms f_i in equations (3b–n) is a function of v_0, as well as derivatives of v_0 with respect to the different time scales T_n, where v_0 is given by equation (4), results in the appearance of resonance term of the form $R_i\, e^{iT_0} + cc$, in the driving function f_i in the equation for v_i. This resonance (called secular) term, leads to unbounded term in the solution for v_i; therefore for each of the v_i, and thus v, to be periodic one must eliminate these secular (unbounded) behavior producing terms in each of equations (3b–n) by setting each $R_i = 0$. The functions R_i are in general non-linear functions of the system parameters and complex amplitude $\mathbf{A}$, but are linear in $D_iA, D_{i-1}^2A, \ldots, D_1^2A, D_{i-1}\mathrm{A}, \ldots, D_1A$. One then solves R_i for the partial time derivative $D_i\mathrm{A}$ of the complex amplitude $\mathbf{A}$ with respect to the time scale T_i, and substitutes for D_{i-1}^2A and $D_{i-1}A$ from the solution of $R_{i-1} = 0$, to obtain an equation for $D_i\mathrm{A}$ in the form

$$D_iA = N_i(A, \bar{A}), \quad i = 1, \ldots, n, \tag{5}$$

where $N_i(A, \bar{A})$ are non-linear functions of system parameters and complex amplitude $\mathbf{A}$. The partial time derivative in equation (5) are then combined, to obtain the total time derivative $\mathrm{d}\mathbf{A}/\mathrm{d}t$ by noting that $\mathbf{A} = A(T_1, T_2, \ldots)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2, \ldots. \tag{6}$$

This process, known as reconstitution procedure [9–11], yields a power series for the evolution of the complex amplitude $\mathbf{A}$ in the form [3]

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \varepsilon N_1(A, \bar{A}) + \varepsilon^2 N_2(A, \bar{A}) + \cdots + \varepsilon^i N_i(A, \bar{A}) + \mathcal{O}(\varepsilon^{i+1}), \tag{7}$$

where $\bar{A}$ is the complex conjugate of the amplitude A, The ith order steady solution in the original non-linear problem (equation (2)) is then obtained from equation (7) by setting $\mathrm{d}\mathbf{A}/\mathrm{d}t = 0$. Noting that $\mathbf{A}$ is complex, one substitutes into the resulting $\mathrm{d}\mathbf{A}/\mathrm{d}T = 0$ equation the polar form $\mathbf{A} = (a/2)e^{i\phi}$, where the amplitude a and the phase ϕ are real functions of the slow scales $T_1, T_2, \ldots$. Then upon separating the real and imaginary parts of the resulting equation, one obtains two equations $\mathrm{d}a/\mathrm{d}t$ and $\mathrm{d}\phi/\mathrm{d}t$, known as modulation equations, defining the slow evolution of the amplitude a and the phase ϕ of the main Fourier component of the ith order steady state response of the original non-linear problem. The steady state response thus corresponds to the fixed points of these modulation equations, which are obtained by setting $\mathrm{d}a/\mathrm{d}t = 0$ and $\mathrm{d}\phi/\mathrm{d}t = 0$. This leads to two real algebraic equations relating the ith order steady state solution amplitude a and ϕ to system parameters. By eliminating the phase from these two algebraic equations, one then obtains a single algebraic equation, known as the frequency–amplitude equation, relating the fundamental harmonic amplitude and frequency of the ith order steady state periodic response to the system parameters. The local stability of the obtained steady state solutions with respect to infinitesimal disturbances δa and $\delta\phi$ can be analyzed by determining the eigenvalues of the linearized modulation equations.

It is to be noted that, when carrying calculations to second or higher order, there are two different ways, leading, in general, to two different results, to make $\mathrm{d}A/\mathrm{d}t$ in equation (7) equals to zero; i.e., there two different ways to obtain second or higher order steady state solutions from equation (7). According to the first approach, known as MMS with reconstitution version I [3], the condition $\mathrm{d}A/\mathrm{d}t = 0$ in equation (7) (also called solvability condition), is obtained by letting

$$N_1(A, \bar{A}) + \varepsilon N_2(A, \bar{A}) + \cdots = 0. \tag{8}$$

Another way to meet the solvability condition $\mathrm{d}A/\mathrm{d}t = 0$ in equation (7), according to the so called MMS with reconstitution version II [3], is to set each N_i equal to zero, e.g., $N_i = 0$, $i = 1, 2, \ldots$. Obviously the two procedures lead to the same steady state results at the first order. They also lead to the same steady state solution at the ith order if it happens that each of the functions $N_1, \ldots, N_i$ is identically zero [3]. Rahman and Burton [3] determined a second order perturbation solution for the primary resonance response of the harmonically driven, weakly non-linear ($0 < \varepsilon \leqslant 1$), Duffing oscillator

$$\ddot{u} + \varepsilon\delta\dot{u} + u + \varepsilon u^3 = \varepsilon P_0 \cos \Omega t. \tag{9}$$

They used transformation of time $T = \Omega t$ and found that formal application of the MMS with reconstitution version I used, i.e., in [9–11], can in passing from first to second and higher order generate additional spurious steady state solutions (i.e., solutions which do not exist in the actual numerically obtained response), and inconsistency in the ordering as the amplitude becomes relatively large. For example, for the Duffing oscillator in equation (9), it was shown in reference [3] that the MMS with reconstitution version I procedure leads to, at the second order, an amplitude–frequency response relation in the form of a polynomial of degree seven in the square of the steady state amplitude A of the dominant harmonic. Thus, depending on the considered ranges in frequency Ω, up to seven solutions may coexist where some of these solutions (spurious solutions), although predicted to be stable, could not be obtained numerically [3]. As a result of these extraneous (spurious) solutions, the second order steady state frequency curves obtained this way were shown to exhibit qualitative behavior that is substantially different (i.e., do not resemble) the actual (numerically obtained) ones, even though the first order solution is reasonably acurate [3]. This implies that the second order terms in these solutions do not represent, as they should, a small correction to the first order ones Rahman and Burton [3] argued that the "breakdown" of these results at second order (also at higher orders), is caused by a violation of the ordering requirement as a result of the explicit dependence on ε of the steady state solutions $v_i(T)$ generated at each level of approximations. For example, they argued that the MMS with reconstitution version I procedure in references [9–11] may yield solutions where each $v_i(T)$ in the dependant $v(T)$ variable series expansion $v(T) = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots$, is a function of the parameter ε, and thus violates the stipulation of the MMS perturbation method that each of these $v_i(T)$ is not explicitly dependent on ε. They showed that, for the Duffing oscillator in equation (9), as a result of this explicit dependence of the v_i's on ε, the lowest order in ε of the neglected $\mathcal{O}(\varepsilon^3)$ terms in equation (8) is the same as that of the retained $N_1(A, \bar{A})$ and $\varepsilon N_2(A, \bar{A})$ terms in this equation. This according to reference [3], violates the ordering requirement of the MMS perturbation method, since the essence of this method is to obtain a uniformly valid approximate solution where the neglected terms should be of higher order in ε than the retained ones. It was also shown in reference [3] that the amplitude of the second order spurious solutions obtained for the Duffing oscillator in equation (9) using the so called "MMS with reconstitution version I", becomes relatively large as ε is made small, and thus these spurious solutions are far removed from the small amplitude solutions (i.e., from the backbone of the frequency response curve); also these solutions, for small ε, appear over small bands of the excitation frequency. Therefore, by discarding these large amplitude spurious solutions one may obtain, for small ε, a second order steady state response curve which resembles the actual one. In other words, the MMS with reconstitution version I procedure, may yield qualitatively, as well as quantitatively, correct second (and higher) order aproximations for weakly non-linear (i.e., ε small), systems provided that one disregards the additional (spurious) large amplitude solutions. For the cases where the oscillator in equation (9) is strongly non-linear (i.e., ε is relatively large), the second order spurious solutions were observed to spread over a wider bands of the excitation frequency and move closer to the actual (numerical) solution (i.e., move closer to the backbone curve of the actual solution), [3–5]. Consequently, as ε is increased from small values, the second order spurious solutions begin to interfere with, i.e., become difficult to separate from, the actual solutions and thus begin to change (distort) the qualitative (as well as quantitative) feature of the steady state frequency response curve.

As indicated above, in the MMS with reconstitution version I procedure [9–11], the second and higher order steady state solutions are obtained from equation (8) by setting the entire right side of this equation to zero. On the other hand, according to the so called MMS with reconstitution version II, suggested by Rahman and Burton [3, 8], one may obtain second and higher order solutions from equation (8) by requiring that each of the functions N_i in the right side of this equation is set to zero, in conjuction with using either of the following two procedural steps: (1) retain the homogeneous solutions for each of the v_i or (2) suppress the homogeneous solutions for the v_i but expand both the excitation frequency Ω and damping ratio ζ in power series in ε, (i.e., use Ω–ζ expansions):

$$\Omega^2 = 1 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2, \quad \zeta = \zeta_1 + \varepsilon\zeta + \cdots.$$

Rahman and Burton [3, 8] explained that the use of either of these two procedural steps is necessary to ensure that in each of the functions $N_1(A, \bar{A})$, $N_2(A, \bar{A}), \ldots,$ exactly two new unknowns appear which are determined when each of $N_i(A, \bar{A})$ is set to zero to obtain the steady state solution. They argued that this so called MMS with reconstitution procedure version II leads to results where the v_n are not explicitly dependent on ε. Their analysis of the primary resonance response of the Duffing oscillator in equation (9) for the weakly non-linear case ($\varepsilon = 0{\cdot}2$, $\delta = 0{\cdot}1$, $P = 1$) indicates that the second order solutions obtained using the above MMS with reconstitution version II, with Ω–ζ expansion show the correct qualitative and reasonably good quantitative, behavior; however, this second order solution was found to overcorrect "slightly" at large response amplitudes. They indicated that they preferred using the MMS with reconstitution version II with the Ω–ζ expansion instead of the MMS with reconstitution version II in which one retains the homogenous solutions (i.e., with the first of the above two procedural steps), as the later was found in reference [12] to suffer a breakdown at saddle node bifurcation points in the steady state response of the Duffing oscillator in equation (9).

Nayfeh and Sanchez [10] used the MMS with reconstitution version I to obtain a second order approximation to the primary resonance response in their study of the onset of asymmetry-breaking and period doubling bifurcation, of the forced softening Duffing oscillator:

$$\ddot{u} + 2\delta\dot{u} + u - \alpha u^3 = P\cos\Omega t, \quad \delta, \alpha > 0 \tag{10}$$

They presented the amplitude–frequency response curves calculated using the first and second order approximate solutions for the case $P = 0{\cdot}3$, $\delta = 0{\cdot}2$, $\alpha = 1$. These curves showed that the second order approximations, for this case where P, δ and α are not relatively small, are in good (quantitative as well as qualitative) agreement with those obtained by numerically integrating equation (10); that is, these results indicate that the second order solutions obtained using the MMS with reconstitution in [9–11], represent merely a "slight" additive correction to the first order solutions up to relatively large response amplitude, i.e., for a response amplitude up to $\sim 0{\cdot}85$.

First and second order perturbation solutions for the primary resonance response of the softening oscillator in equation (10) were also obtained by Rahman and Burton [8] using a "modified MMS" procedure presented in reference [7] with reconstitution version II in conjuction with the Ω, instead of (Ω–ζ), expansion. In accordance with this MMS procedure, they used transformation of time $T = \Omega t$, and defined a new expansion parameter $\alpha = \alpha(\varepsilon)$ using the free oscillation frequency–amplitude relation (backbone of the steady state response curve) i.e., $\alpha = \varepsilon a^2/(4 - 3a^2)$, where a is the steady state amplitude of the response fundamental harmonic. They presented results for the relatively strongly non-linear case $P = 0{\cdot}23$, $\delta = 0{\cdot}2$, $\varepsilon = 1$, which showed good agreement between second order solutions and numerical results only for amplitudes less than 0·8. For larger amplitudes, the first as well as the second order solutions were found to breakdown and become sensitive to the way in which the steady state amplitude is defined.

The effects of using transformation of time (i.e., $T = \Omega t$) and those of using two different ways of introducing the frequency detuning, i.e., in the square of excitation frequency or in the excitation frequency itself, with or without transformation of time, as well as the effects of using the frequency expansion in the inertia term only and in both the inertia and damping terms, on the second order MMS reconstitution version I approximations to the primary resonance response of the weakly non-linear hardening Duffing oscillator in equation (9) were recently investigated by Hassan [4, 5]. It was shown that the combined effects of using transformation of time ($T = \Omega t$) and introducing a detuning parameter in the square of excitation frequency, i.e., $\Omega^2 = 1 + \varepsilon\sigma$, can induce spurious solutions and a non-uniform expansion at large response amplitudes. In fact his results indicate that regardless of which of the above procedural steps, or combination of, is used in the MMS with reconstitution version I, the second order results show extraneous solutions at large response amplitudes. However, the size of the gap(s) between the predicted backbone curve and the predicted extraneous banches, and the size of frequency band(s) over which these extraneous solutions appear were different for the above mentioned different procedural steps or combinations of steps. It was argued that the use of transformed time ($T = \Omega t$) with MMS with reconstitution version I leads to a non-uniform expansion and to spurious solutions, and that these spurious solutions and other effects of a non-uniform expansion can be eliminated by transforming the partial derivatives with respect to different time scales in the modulation equation(s) back to the real time t. It is to be noted at this point that the approximate analytic solution should at least establish the correct qualitative behavior of the actual non-linear steady state response in order to get non-erroneous and reasonably accurate stability analysis results, i.e., in order to get the correct nature of the bifurcation points.

The other commonly used method for obtaining approximate solution to non-linear oscillators is the harmonic balance (HB). Unlike the MMS method, the HB method does not place restriction on the strength of the non-linearity and is applicable to those cases where the linear oscillator is statically unstable ($m = -1$) or neutrally stable ($m = 0$). Therefore, the arbitrary reordering of the various terms in the non-linear equation when using the MMS is not necessary when using the HB method. It is to be noted that, in many cases the MMS and the HB method yield basically the first order approximate solutions. These first order approximations, however, do not always, even for a weakly non-linear system, give a satisfactory, or even yield inaccurate, description of the qualitative and/or quantitative behavior and stability of the actual system repsonse [6, 13]. For such cases, it is necessary to go to second or higher order approximations to obtain realistic analytical description of the actual response and stability analysis. Hamdan and Burton [13], for example, have shown that the qualitative nature of the HB solution and the predicted stability boundaries of the softening

Duffing oscillator in equation (4), change, for certain parameters ranges, if one uses a two-modes HB solution rather than a single-mode solution.

It is to be noted that in the HB method a periodic solution of the dependent variable is assumed in the form of a Fourier series, mostly truncated to only a few leading harmonics which are assumed to be dominant and of equal level of importance over the full range of system parameters [6]. Upon substituting the assumed series solution in the equation of motion, and equating the coefficients of different harmonics to zero, one obtains a set of coupled non-linear algebraic equations in the coefficients of the assumed harmonics and frequency of motion. These coupled non-linear equations, are then solved simultaneously to obtain an approximation to the steady state periodic response. The number of these coupled equations which need to be solved simultaneously is equal to the number of harmonics in the assumed series solution.

Therefore, the use of a sufficiently large number of harmonics to improve accuracy results in a messy non-linear algebraic problem. Furthermore, intuitively one expects the assumed HB solution to converge to the actual solution as the number of harmonics in the assumed solution is increased. This is however generally true provided that the harmonics in the assumed truncated series solution are the dominant ones and the neglected harmonics are small compared to the retained ones [6]. Furthermore, since the retained harmonics, as indicated above, enjoy equal level of importance and are allowed to interact with each other in a non-linear way, the problem of selecting the "right" combination of these leading harmonics which will lead to the correct quantitative behavior of the predicted response becomes a difficult task, especially when the non-linearities are not small [6]. Hassan and Burton [6], as well as others, have shown that for the forced oscillation of the hardening Duffing oscillator given by equation (9) with $\varepsilon = 1$, the HB method in which only a few leading harmonics are used can fail to predict some actually existing periodic solutions and/or it may predict some spurious solutions which do not exist in the actual system response. It was shown [6] that this qualitative failure of the HB method, which may not be predicted by stability analysis, is neither restricted to the single approximation nor to systems with asymmetric potential wells. It was suggested [6] that to avoid erroneous results the approximate HB results should be checked against numerical or experimental or a higher order HB approximation, i.e., by increasing the number of harmonics in the assumed series solution.

In light of the above review, the objective of this work is to analyze the periodic steady state response of the type of oscillators described in equation (1). Interest here is in situations where the first order approximate solutions may lead to qualitative as well as appreciable quantitative errors is steady state amplitude–frequency response curves as, i.e., when the response is strongly non-linear. Single-mode and two-mode HB approximations, and second order MMS with reconstitution version I and version II approximations to the steady-state amplitude–frequency response curves are compared, for the case $m = 1$ with each other, and with those obtained by numerically integrating the equation of motion. The transformation of time $T = \Omega t$ and detuning in the square of forcing frequency are used in the MMS with reconstitution version I and version II. The objective here is to assess the accuracy of these approximate solutions in predicting the system response over some range of system parameters by examining their ability or failure in establishing the correct qualitative behavior of the actual (numerical) solution.

The qualitative behavior of the primary resonance response of the oscillator in equation (1) for the cases $m = 0$ and $m = -1$ are studied, for selected range of system parameter, using the single and two modes harmonic balance method and compared to those obtained numerically. And finally, regions of first order stability are also presented.

To the authors' knowledge, despite its physical importance, studies dealing with the forced vibration of oscillators having inertia and hardening static non-linearities of the type described in equation (1) are not commonly available.

2. ANALYSIS

Here, approximate analytical solutions for the periodic steady state response, having the same period as the excitation, of the non-linear oscillator described by equation (1) are presented. These solutions are obtained using single mode (HB), two modes (HB) and second order MMS with reconstitution version I and version II.

A new time $T = \Omega t$ is first introduced so that equation (1) becomes;

$$\Omega^2 \ddot{u} + \Omega\delta\dot{u} + mnu + \varepsilon_1\Omega^2 u^2\ddot{u} + \varepsilon_1\Omega^2 u\dot{u}^2 + \varepsilon_2 u^3 = P\cos(T + \phi) \qquad (11)$$

where dots are T derivative and the unknown phase ϕ has been added to the excitation so that one may obtain a fundamental harmonic response containing a single trigonometric term.

2.1. SINGLE MODE HARMONIC BALANCE (SHB)

According to the SHB method, an approximate solution of equation (11), takes the form;

$$u(T) = A\cos T, \qquad (12)$$

where A is the steady state response amplitude. Substituting equation (12) into equation (11), neglecting third harmonics which arise, and equating coefficients of the first harmonics, one obtains the following equations:

$$(\tfrac{3}{4}\varepsilon_2 - (\varepsilon_1/2)\Omega^2)A^3 + (m - \Omega^2)A = P\cos\phi, \quad -\Omega\delta A = {}^{-}P\sin\phi. \qquad (13, 14)$$

The steady state frequency response is obtained by squaring and adding equations (13) and (14) and solving for Ω^2 as a function of A; this yields

$$\Omega^2 = R_1 \pm \sqrt{R_1^2 - R_2}, \qquad (15)$$

where

$$R_1 = -(\delta^2 - \tfrac{3}{2}\varepsilon_2 A^2 - \tfrac{3}{4}\varepsilon_1\varepsilon_2 A^4 - 2m - \varepsilon_1 m A^2)/(2 + 2\varepsilon_1 A^2 + \varepsilon_1^2 A^4/2), \qquad (16)$$

$$R_2 = (\tfrac{9}{16}\varepsilon_2^2 A^4 + \tfrac{3}{2}mE_2A^2 + m^2 - P^2/A^2)/(1 + \varepsilon_1 A^2 + \varepsilon_1^2 A^4/4). \tag{17}$$

Equation (15), yields two real solutions for Ω provided that the radical term is real and less than R_1; a single real solution is obtained when the radical term is zero or greater the R_1, and no real solution exists when $R_1^2 - R_2 < 0$.

The steady state frequency response curves obtained using equation (15) are, for convenience, presented and discussed in section 3.

2.2. TWO MODES HARMONIC BALANCE (2HB)

In order to improve the accuracy of SHB approximation one includes more higher harmonics in the assumed solution in equation (12). In this work, only one more mode is added to this equation, whereby the two-modes approximation, having the same period as the excitation, to the steady state solution of the system in equation (11) with odd non-linearities takes the form;

$$u(T) = A_1 \cos T + A_3 \cos 3T + B_3 \sin 3T. \tag{19}$$

Substituting equation (19) into equation (12) and using the same procedure followed previously and neglecting the higher order harmonics, one obtains the following coupled non-linear algebraic equations for A_1, A_3, B_3 and the phase ϕ;

$$\begin{aligned}&\tfrac{3}{4}\varepsilon_2 A_1^3 + \tfrac{3}{4}\varepsilon_2 A_1^2 A_3 + \tfrac{3}{2}\varepsilon_2 A_1 A_3^2 + \tfrac{3}{2}\varepsilon_2 A_1 B_3^2 + Am - A\Omega^2 - (\varepsilon_1/2)\Omega^2 A^3\\ &\quad - \tfrac{3}{2}\varepsilon_1 \Omega^2 A_1^2 A_3 - 5\varepsilon_1 \Omega^2 A_1^2 A_3 - 5\varepsilon_1 \Omega^2 A_1^2 B_3 = P\cos\phi,\end{aligned} \tag{20}$$

$$\tfrac{3}{4}\varepsilon_2 A_1^2 B_3 - \Omega\delta A_1 - \tfrac{3}{2}\varepsilon_1 \Omega^2 A_1^2 B_3 = {}^{-}P\sin\phi, \tag{21}$$

$$\begin{aligned}&\tfrac{3}{2}\varepsilon_2 A_1^2 B_3 + \tfrac{3}{4}\varepsilon_2 A_3^2 B_3 + \tfrac{3}{4}\varepsilon_2 B_3^3 + B_3 m - 3A_3\delta\Omega - 9\Omega^2 B_3\\ &\quad - 5\varepsilon_1 \Omega^2 A_1^2 B_3 - \tfrac{9}{2}\varepsilon_1 \Omega^2 A_3^2 B_3 - \tfrac{9}{2}\varepsilon_1 \Omega^2 B_3^3 = 0,\end{aligned} \tag{22}$$

$$\begin{aligned}&(\varepsilon_2/4)A_1^3 + \tfrac{3}{2}\varepsilon_2 A_1^3 A_3 + \tfrac{3}{4}\varepsilon_2 A_3^2 + \tfrac{3}{4}\varepsilon_2 A_3 B_3^2 + A_3 m + 3B_3\delta\Omega - 9A_3\Omega^2\\ &\quad - (\varepsilon_1/2)\Omega^2 A_1^3 - 5\varepsilon_1 \Omega^2 A_1^2 A_3 - \tfrac{9}{2}\varepsilon_1 \Omega^2 A_3^2 - \tfrac{9}{2}\varepsilon_1 \Omega^2 A_3 B_3^2 = 0,\end{aligned} \tag{23}$$

These equations may be expressed in a more convenient form as follows. First, squaring and adding equations (20) and (21) and solving for Ω^2 leads to

$$a\Omega^4 + b\Omega^2 + c = 0 \tag{24}$$

where:

$$\begin{aligned}a = {}&1 + A_1^2\varepsilon_1 + 3\varepsilon_1 A_1 A_3 + 10\varepsilon_1 A_3^2 + 10\varepsilon_1 B_3^2 + \tfrac{1}{4}\varepsilon_1 A_1^4 + \tfrac{3}{2}\varepsilon_1^2 A_1^3 A_3\\ &+ \tfrac{29}{4}\varepsilon_1^2 A_1^2 A_3^2 + 15\varepsilon_1^2 A_1 A_3^2 + 25\varepsilon_1^2 A_3^4\\ &+ \tfrac{29}{4}\varepsilon_1^2 A_1^2 B_3^2 + 15\varepsilon_1^2 A_1 A_3 B_3^2 + 50\varepsilon_1^2 A_3^2 + 25\varepsilon_1^2 B_3^4,\end{aligned} \tag{25}$$

$$b = \delta^2 - \tfrac{3}{2}\varepsilon_2 A_1^2 - \tfrac{3}{2}\varepsilon_2 A_1 A_3 - 3\varepsilon_2 A_3^2 - 3\varepsilon_2 B_3^2 - \tfrac{3}{4}\varepsilon_1\varepsilon_2 A_1^4 - 3\varepsilon_1\varepsilon_2 A_1^3 A_3$$
$$- \tfrac{45}{4}\varepsilon_1\varepsilon_2 A_1^2 A_3^2 - 12\varepsilon_1\varepsilon_2 A_1 A_3^2 - 15\varepsilon_1\varepsilon_2 A_3^4 - \tfrac{45}{4}\varepsilon_1\varepsilon_2 A_1^2 B_3^2 - 12\varepsilon_1\varepsilon_2 A_1 A_3 B_3^2$$
$$- 30\varepsilon_1\varepsilon_2 A_3^2 B_3^2 - 15\varepsilon_1\varepsilon_2 B_3^4 - 2m - \varepsilon_1 m A_1^2 - 3\varepsilon_1 m A_1 A_3 - 10\varepsilon_1 m A_3^2 - 10\varepsilon_1 m B_3^2, \tag{26}$$

$$c = \tfrac{9}{16}\varepsilon_2^2 A_1^4 + \tfrac{9}{8}\varepsilon_2^2 A_1^3 A_3 + \tfrac{45}{16}\varepsilon_2^2 A_1^2 A_3^2 + \tfrac{9}{4}\varepsilon_2^2 A_1 A_3^3 + \tfrac{9}{3}\varepsilon_2^2 A_3^4 + \tfrac{45}{16}\varepsilon_2^2 A_1^2 B_3^2 + \tfrac{9}{4}\varepsilon_2^2 A_1 A_3 B_3^2$$
$$+ \tfrac{9}{2}\varepsilon_2^2 A_3^2 B_3^2 + \tfrac{9}{4}\varepsilon_2^2 B_3^4 + \tfrac{3}{2}\varepsilon_2 m A_1^2 + \tfrac{3}{2}\varepsilon_2 m A_1 A_3 + 3\varepsilon_2 m A_3^2 + 3\varepsilon_2 m B_3^2$$
$$+ m^2 + 3\varepsilon_1 \delta\Omega^3 A_1 B_3 - \tfrac{3}{2}\varepsilon_2 \delta\Omega A_1 B_3 - P^2/A_1^2. \tag{27}$$

Next, equations (22) and (23) are solved implicitly for A_3 and B_3, respectively:

$$B_3 = \frac{[-\tfrac{3}{4}\varepsilon_2 B_3(A_3^2 + B_3^2) + \tfrac{9}{2}\varepsilon_1\Omega^2 B_3(A_3^2 + B_3^2) + 3A_3\delta\Omega]}{[\tfrac{3}{2}\varepsilon_2 A_1^2 + (m - 9\Omega^2 - 5\varepsilon_1\Omega^2 A_1^2)]}, \tag{28}$$

$$A_3 = \frac{[A_1^3((\varepsilon_1/2)\Omega^2 - \varepsilon_2/4) - \tfrac{3}{4}\varepsilon_2 A_3(A_3^2 + B_3^2) - 3\delta\Omega B_3 + \tfrac{9}{2}\varepsilon_1\Omega^2 A_3 + B_3^2)]}{[\tfrac{3}{2}\varepsilon_2 A_1^2 + m - 9\Omega^2 - 5\varepsilon_1\Omega^2 A_1^2]}. \tag{29}$$

Equation (24) can be written using the form

$$\Omega^2 = R_3 \pm \sqrt{R_3^2 - R_4}, \tag{30}$$

where R_3 and R_4 can be calculated from equation (27), so that $R_3 = (-b/2a)$ and $R_3 = (c/a)$. Equation (30) has two real solutions provided that $R_3^2 > R_4$ and $\sqrt{R_3^2 - R_4} < R_3$. A single real solution exists provided that $R_3^2 > R_4$ and $\sqrt{R_3^2 - R_4} > R_3$, and no real solution exists when $R_3^2 < R_4$. Equations (28–30) were solved iteratively with an accuracy of 10^{-6} to define the steady state solution. The steady state frequency response curves obtained using these equations are, for convenience, presented and discussed in section 3.

2.3. PERTURBATION MULTIPLE TIME SCALES APPROXIMATIONS (MMS)

In this subsection, second order approximation to the steady state response, having the same period as the excitation, of equation (1) for the case $m = 1$ are obtained using MMS with reconstitution versions I and II. In order to apply the MMS perturbation technique, it is necessary to reorder various terms in equation (11), for example, since the concern is with primary resonance, the non-linear terms, the damping and forcing are multiplied by a small gauge parameter, ε, so that they appear at the same order in the perturbation scheme. Accordingly equation (11) becomes;

$$\Omega^2\ddot{u} + \varepsilon k\dot{u} + mu + \varepsilon\varepsilon_1\Omega^2 u^2\ddot{u} + \varepsilon\varepsilon_1\Omega^2 u\dot{u}^2 + \varepsilon\varepsilon_2 u^3 = \varepsilon P\cos T, \tag{31}$$

where dots are T derivatives and $k = \Omega\delta$. A frequency detuning is next introduced by noting that, for the primary resonance, the driving frequency differs from the linear natural frequency by terms of $\mathcal{O}(\varepsilon)$, so;

$$\Omega^2 = 1 + \varepsilon\sigma. \tag{32}$$

2.3.1. *MMS version* I

Upon substituting equation (32) into equation (31) and according to the procedural steps discussed in section (1), equating like powers of ε to zero, one obtains;

$$D_0^2 u_0 + u_0 = 0, \tag{33}$$

$$D_0^2 u_1 + u_1 = -2D_1 D_0 u_0 - kD_0 u_0 - \sigma D_0^2 u_0 - \varepsilon_1 u_0^2 D_0^2 u_0 - \varepsilon_1 u_0 (D_0 u_0)^2 - \varepsilon_2 u_0^3 + (p/2)\mathrm{e}^{iT_0}, \tag{34}$$

$$\begin{aligned} D_0^2 u_2 + u_2 = {} & -D_1^2 u_0 - 2D_2 D_0 u_0 - 2D_0 D_1 u_1 - 2\sigma D_1 D_0 u_0 - -kD_1 u_0 \\ & - 2D_1 D_0 u_1 - kD_0 u_1 - \sigma D_0^2 u_1 - 3\varepsilon_2 u_0^2 u_1 - \sigma\varepsilon_1 u_0^2 D_0^2 u_0 \\ & - \sigma\varepsilon_1 u_0 (D_0 u_0)^2 - 2\varepsilon_1 u_0^2 D_1 D_0 u_0 - \varepsilon_1 u_0^2 D_0^2 u_1 - 2\varepsilon_1 u_0 u_1 D_0^2 u_0 \\ & - 2\varepsilon_1 u_0 D_0 u_0 D_1 u_0 - \varepsilon_1 u_1 (D_0 u_0)^2. \end{aligned} \tag{35}$$

The solution of equation (33) is:

$$u_0(T_0, T_1, T_2) = A(T_1, T_2)\mathrm{e}^{iT_0} + \bar{A}(T_1, T_2)\mathrm{e}^{-iT_0}, \tag{36}$$

where the T_0, T_1 and T_2 are time scales and are used to obtain second order solutions, and $\bar{A}$ is the complex conjugate of the complex amplitude $\mathbf{A} = (a/2)\mathrm{e}^{i\phi}$ which is a function of the slow time scale. Substituting equation (36) in equation (34), yields the following inhomogenous equation for $u_1(T_0, T_1, T_2)$;

$$D_0^2 u_1 + u_1 = \mathrm{e}^{iT_0}[-2\mathrm{i}D_1 A + \sigma A - \mathrm{i}kA + 2\varepsilon_1 A^2 \bar{A} - 3\varepsilon_2 A^2 \bar{A} + p/2] + \mathrm{e}^{3iT_0}[A^3(2\varepsilon_1 - \varepsilon_2)] + cc, \tag{37}$$

where *cc* stands for the complex conjugate of the preceding terms. Eliminating the terms in equation (37) that produce secular terms in u_1 yields

$$2\mathrm{i}D_1 A = (\sigma - \mathrm{i}k)A + (2\varepsilon_1 - 3\varepsilon_2)A^2\bar{A} + p/2. \tag{38}$$

Equation (38) defines the rate of change of A on the slow time scale T_i. The steady state first order solution can be obtained from equation (38) by equating $D_1 A$ to zero, so the frequency response is giving by:

$$\sigma = (a^2/4)(3\varepsilon_2 - 2\varepsilon_1) \pm \sqrt{(p/a)^2 - k^2}. \tag{39}$$

According to MMS version I, the second order solution can be obtained by solving equation (37) for u_1, retaining the particular solution only, so that

$$u_1 = (A^3/8)(\varepsilon_2 - 2\varepsilon_1)\mathrm{e}^{3iT_0} + cc. \tag{40}$$

Substituting equations (40) and (36) into equation (35), one obtains the following equation for $u_2(T_0, T_1, T_2)$;

$$\begin{aligned} D_0^2 u_2 + u_2 = {} & \mathrm{e}^{iT_0}[-D_1^2 A - 2\mathrm{i}D_2 A - 2\mathrm{i}\sigma D_1 A - kD_1 A - 4\mathrm{i}\varepsilon_1 A\bar{A}D_1 A \\ & + \tfrac{9}{8}\varepsilon_1(\varepsilon_2 - 2\varepsilon_1)A^3\bar{A}^2 - \tfrac{3}{8}\varepsilon_2(\varepsilon_2 - 2\varepsilon_1)A^3\bar{A}^2 + 2\sigma\varepsilon_1 A^2\bar{A}] \\ & + \mathrm{e}^{3iT_0}[-\mathrm{i}(\tfrac{18}{8}(\varepsilon_2 - 2\varepsilon_1) + 4\mathrm{i}\varepsilon_1)A^2 D_1 A + \tfrac{9}{8}\sigma(\varepsilon_2 - 2\varepsilon_1)A^3 - \tfrac{3}{8}\mathrm{i}k(\varepsilon_2 - 2\varepsilon_1) \\ & + 2\sigma\varepsilon_1 A^3 + \tfrac{18}{8}\varepsilon_1(\varepsilon_2 - 2\varepsilon_1)A^4\bar{A} - \tfrac{3}{4}\varepsilon_2(\varepsilon_2 - 2\varepsilon_1)A^4\bar{A}] \\ & + \mathrm{e}^{5iT_0}[\tfrac{15}{8}\varepsilon_1(\varepsilon_2 - 2\varepsilon_1)A^5 - \tfrac{3}{8}\varepsilon_2(\varepsilon_2 - 2\varepsilon_1)A^5] + cc. \end{aligned} \tag{41}$$

The annulment of the secular terms in u_2 requires that,

$$\begin{aligned} 2\mathrm{i}D_2 A = {} & -D_1^2 A - 2\mathrm{i}\sigma D_1 A - kD_1 A - 4\mathrm{i}\varepsilon_1 A\bar{A}D_1 A + \tfrac{9}{8}\varepsilon_1(\varepsilon_2 - 2\varepsilon_1)A^3\bar{A}^2 \\ & + 2\sigma\varepsilon_1 A^2\bar{A} - \tfrac{3}{8}\varepsilon_2(\varepsilon_2 - 2\varepsilon_1)A^3\bar{A}^2. \end{aligned} \tag{42}$$

One now combines equation (38) and equation (42) to form a single equation for the slow evolution of A in real time T,

$$\mathrm{d}A/\mathrm{d}T = \varepsilon D_1 A + \varepsilon^2 D_2 A + \mathcal{O}(\varepsilon^3) \tag{43}$$

where equation (38) is used to determine the term $D_1^2 A$ which appears in equation (42). This procedure known as reconstitution [9], leads to the explicit form of equation (43);

$$\begin{aligned} 2\mathrm{i}\ \mathrm{d}A/\mathrm{d}T = {} & \varepsilon[A(\sigma - \mathrm{i}k) + (2\varepsilon_1 - 3\varepsilon_2)A^2\bar{A} + p/2] + \varepsilon^2[-D_1^2 A - 2\mathrm{i}\sigma D_1 A \\ & - kD_1 A - 4\mathrm{i}\varepsilon_1 A\bar{A}D_1 A + \tfrac{9}{8}\varepsilon_1(\varepsilon_2 - 2\varepsilon_1)A^3\bar{A}^2 + 2\sigma\varepsilon_1 A^2\bar{A} \\ & - \tfrac{3}{8}\varepsilon_2(\varepsilon_2 - 2\varepsilon_1)A^3\bar{A}^2] + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{44}$$

Following the procedure of MMS version I, one obtains the steady state solutions by setting $\mathrm{d}A/\mathrm{d}T$ to zero in equation (44) and then solving for **A**. This leads to the second order frequency amplitude (a, σ) relation, which using Cramer's rule, becomes

$$\Delta^2 = (R_1 a_{22} - R_2 a_{12})^2 + (R_2 a_{11} - R_1 a_{21})^2, \tag{45}$$

where

$$\Delta = a_{11}a_{22} - a_{12}a_{21}, \quad a_{11} = \varepsilon kp/4, \quad a_{12} = p[1 - \varepsilon(\tfrac{3}{4}\sigma + (\tfrac{6}{16}\varepsilon_1 + \tfrac{3}{16}\varepsilon_2)a^2)],$$

$$a_{21} = -p[1 - \varepsilon(\tfrac{3}{4}\sigma + (\tfrac{2}{16}\varepsilon_1 + \tfrac{9}{16}\varepsilon_2)a^2)], \quad a_{22} = a_{11},$$

$$R_1 = -a\sigma + \sigma(\tfrac{3}{4}\varepsilon_2 - \varepsilon_1/2)a^3 + \varepsilon[a(\tfrac{3}{4}\sigma^2 - k^2/4) + \sigma a^3(\varepsilon_1/4 - \tfrac{3}{8}\varepsilon_2) + a^5(\tfrac{9}{32}\varepsilon_1^2 - \tfrac{9}{32}\varepsilon_1\varepsilon_2 - \tfrac{15}{128}\varepsilon_2^2)],$$

$$R_2 - ak - \varepsilon[k\sigma a + (\tfrac{2}{16}\varepsilon_1 + \tfrac{9}{16}\varepsilon_2)a^2].$$

Equation (45) can be written in the form:

$$\sum_{n=0}^{7} C_n(\varepsilon, p, k, \sigma)a^{2n} = 0, \tag{46}$$

which is polynomial of degree seven in a^2. It is clear that equation (45) is algebraically complicated and one can obtain more than one solution for the second order approximation of the steady state frequency response. Examples of the results obtained using this equation are presented and discussed in section (3).

2.3.2. *MMS version* II

In this subsection, the second order solution to the equation (31) is obtained using MMS version II, with the frequency Ω and damping k expansions:

$$\Omega^2 = 1 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \cdots, \quad k = k_1 + \varepsilon k_2 + \varepsilon^2 k_3 + \cdots. \tag{47}$$

Accordingly, as explained in the introduction, the analysis proceeds as in the MMS version II, but with the Ω–k expansions (47) instead of (32). This leads to;

$$D_0^2 u_0 + u_0 = 0, \tag{48}$$

$$D_0^2 u_1 + u_1 = -2D_1 D_0 u_0 - k_1 D_0 u_0 - \sigma_1 D_0^2 u_0 - \varepsilon_1 u_0^2 D_0^2 u_0 - \varepsilon_1 u_0 (D_0 u_0)^2 - \varepsilon_2 u_0^3 + (p/2)\mathrm{e}^{iT_0}, \tag{49}$$

$$\begin{aligned} D_0^2 u_2 + u_2 = {} & -D_1^2 u_0 - 2D_2 D_0 u_0 - 2D_0 D_1 u_1 - 2\sigma_1 D_1 D_0 u_0 - k_1 D_1 u_0 \\ & - 2D_1 D_0 u_1 - k_1 D_0 u_1 - k_2 D_0 u_0 - \sigma_1 D_0^2 u_1 - \sigma_2 D_0^2 u_0 \\ & - 3\varepsilon_2 u_0^2 u_1 - \sigma_1 \varepsilon_1 u_0^2 D_0^2 u_0 - \sigma_1 \varepsilon_1 u_0 (D_0 u_0)^2 - 2\varepsilon_1 u_0^2 D_1 D_0 u_0 \\ & - \varepsilon_1 u_0^2 D_0^2 u_1 - 2\varepsilon_1 u_0 u_1 D_0^2 u_0 - 2\varepsilon_1 u_0 D_0 u_0 D_1 u_0 - \varepsilon_1 u_1 (D_0 u_0)^2. \end{aligned} \tag{50}$$

The solution to equation (48) is given by equation (36), also equation (40) still holds. Annulment of secular terms in u_1, from equation (49), yields

$$2\mathrm{i}D_1 A = (\sigma_1 - \mathrm{i}k_1)A + (2\varepsilon_1 - 3\varepsilon_2)A^2\bar{A} + p/2. \tag{51}$$

To obtain steady state solutions and according to MMS version II, one equates each $D_i A$ to zero. Thus, the steady state first order solution of equation (51), may be expressed as;

$$\sigma_1 = (a^2/4)(3\varepsilon_2 - 2\varepsilon_1) \pm \sqrt{(p/a)^2 - k_1^2}. \tag{52}$$

Substituting equation (40) particular solution for u_1 and equation (36) into equation (50) and the annulment of the secular terms in u_2 yields;

$$\begin{aligned} 2\mathrm{i}D_2A = {} & -D_1^2A - 2\mathrm{i}\sigma_1 D_1 A - k_1 D_1 A + \sigma_2 A - \mathrm{i}k_2 A - 4\mathrm{i}\varepsilon_1 A\bar{A}D_1 A \\ & + \tfrac{9}{8}\varepsilon_1(\varepsilon_2 - 2\varepsilon_1)A^3\bar{A}^2 + 2\sigma_1\varepsilon_1 A^2\bar{A} - \tfrac{3}{8}\varepsilon_2(\varepsilon_2 - 2\varepsilon_1)A^3\bar{A}^2. \end{aligned} \tag{53}$$

Upon setting each D_iA in equation (53) to zero, one obtains

$$\sigma_2 = \tfrac{3}{128}(\varepsilon_2^2 - 5\varepsilon_1\varepsilon_2 + 6\varepsilon_1^2)a^4 - \sigma_1\varepsilon_1 a^2/2, \quad k_2 = 0 \tag{54}$$

The frequency response in terms of system parameters can now be obtained by combining equations (52) and (54) and is expressed as follows;

$$\Omega^2 = 1 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2. \tag{55}$$

For convenience the results obtained using equation (55) are presented and discussed in section (3).

2.4. STABILITY OF THE STEADY STATE RESPONSE

The stability of the steady state response of the fundamental harmonic approximation (12), is examined by introducing a small perturbation $v(T)$, i.e., by substituting $u(T) = A\cos(T) + v(T)$, into the equation (11), followed by use of the steady state conditions (13) and (14). This leads to the following non-linear variational equation

$$\begin{aligned} & \ddot{v}\Omega^2(1 + \varepsilon_1 A^2/2 + \varepsilon_1 v^2 + 2\varepsilon_1 vA\cos T + \varepsilon_1(A^2/2)\cos 2T) \\ & \quad + \dot{v}(\delta\Omega - 2\varepsilon_1\Omega^2 Av\sin T - \varepsilon_1\Omega^2A^2\sin 2T) \\ & \quad + v(\tfrac{3}{2}\varepsilon_2A^2 + m - \varepsilon_1\Omega^2A^2/2 + \varepsilon_1\Omega^2\dot{v}^2 + \tfrac{3}{2}\varepsilon_2A^2\cos 2T - \tfrac{3}{2}\varepsilon_1A^2\cos 2T) \\ & \quad + \varepsilon_1\Omega^2A^2\dot{v}^2\cos T + v^2A\cos T(3\varepsilon_2 - \varepsilon_1\Omega^2) \\ & \quad + \varepsilon_2 v^3 = (A^2/4)(2\varepsilon_1\Omega^2 - \varepsilon_2)\cos 3T. \end{aligned} \tag{56}$$

The stability is governed by the linear version of equation (56). In addition, the excitation term on the right-side of the equation is deleted, because it has no influence on the stability of the response $v(T)$. The linear stability is governed by the standard form of the damped Mathieu equation.

$$\ddot{v} + \mu\dot{v} + v(\alpha - 2q\cos 2T) = 0 \tag{57}$$

where

$$\begin{aligned} & \alpha = (m + (A^2/2)(3\varepsilon_2 - \varepsilon_1\Omega^2))/\Omega^2(1 + \varepsilon_1A^2/2), \quad \mu = \delta\Omega/\Omega^2(1 + \varepsilon_1A^2/2), \\ & q = \tfrac{3}{4}A^2(\varepsilon_1\Omega^2 - \varepsilon_2)/\Omega^2(1 + \varepsilon_1A^2/2). \end{aligned} \tag{58}$$

The leading term approximation to the stability boundaries of (57) associated with the principal parametric resonance is given by [2].

$$\alpha(q) \cong 1 \pm (q^2 - \mu^2)^{1/2}. \qquad (59)$$

If A, Ω, δ, m, ε_1, ε_2 in equation (58) are such that the point $\alpha(q)$ lies between the curves (59), the steady state solution is unstable to small disturbances. Thus the conditions for *instability* may be stated in term of A, Ω, δ, m, ε_1, ε_2 as follows:

$$\left[m + \frac{A^2}{2}(3\varepsilon_2 - \varepsilon_1\Omega^2)\right] > \left[\Omega^2\left(1 + \varepsilon_1\frac{A^2}{2}\right) - \left(\left(\tfrac{3}{4}A^2(\varepsilon_1\Omega^2 - \varepsilon_2)\right)^2 - (\delta\Omega)^2\right)^{1/2}\right], \qquad (60)$$

$$\left[m + \frac{A^2}{2}(3\varepsilon_2 - \varepsilon_1\Omega^2)\right] < \left[\Omega^2\left(1 + \varepsilon_1\frac{A^2}{2}\right) + \left(\left(\tfrac{3}{4}A^2(\varepsilon_1\Omega^2 - \varepsilon_2)\right)^2 - (\delta\Omega)^2\right)^{1/2}\right]. \qquad (61)$$

These instability conditions are identical to those one obtains using the MMS results, i.e., by determining the eigenvalues of the linearized amplitude and phase modulation equations to the first MMS solution.

3. RESULTS AND DISCUSSION

The steady state frequency response of the non-linear oscillators governed by equation (1) was calculated approximately, for given values of system parameters ε_1, ε_2, δ, m and excitation amplitude p, by using the single and two modes harmonic balance method (SHB and 2HB), equations (15) and (24) respectively,

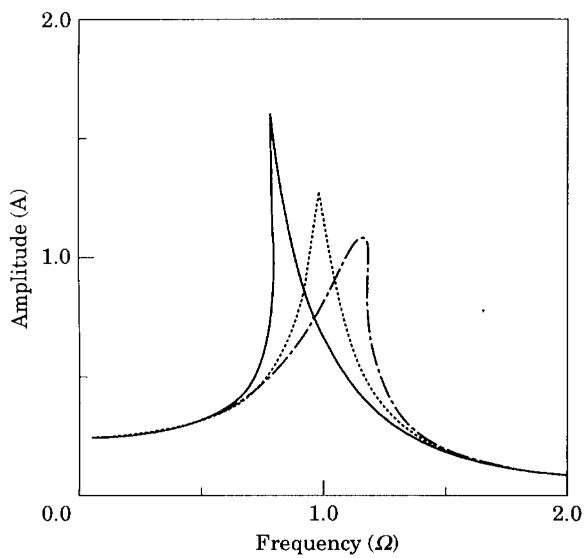

Figure 1. Steady state frequency response of the oscillator using SHB. $m = 1$, $P = 0{\cdot}25$, $\delta = 0{\cdot}2$: ——, $\varepsilon_1 = 3{\cdot}0$ and $\varepsilon_2 = 1{\cdot}0$; - - - - - -, $\varepsilon_1 = 1{\cdot}6$ and $\varepsilon_2 = 1{\cdot}0$; –·–·–, $\varepsilon_1 = 0{\cdot}1$ and $\varepsilon_2 = 1{\cdot}0$.

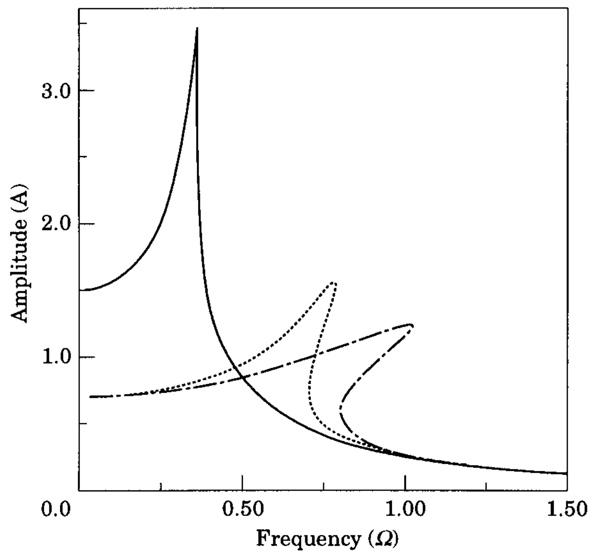

Figure 2. Steady state frequency response of the oscillator using SHB. $m = 0$, $P = 0{\cdot}25$, $\delta = 0{\cdot}2$: ——, $\varepsilon_1 = 1{\cdot}0$ and $\varepsilon_2 = 0{\cdot}1$; ------, $\varepsilon_1 = 1{\cdot}6$ and $\varepsilon_2 = 1{\cdot}0$; —·—·—, $\varepsilon_1 = 0{\cdot}1$ and $\varepsilon_2 = 1{\cdot}0$.

MMS with reconstitution version I (equation (45)) and MMS with reconstitution version II (equation (55)). The concern of this work is on the strongly non-linear cases; therefore, the steady state frequency response was calculated for cases in which ε_1 and/or ε_2 are not small. Examples of the results of these calculations for selected cases of system parameters ε_1, ε_2, δ, m and excitation amplitude p are displayed in Figures (1–9).

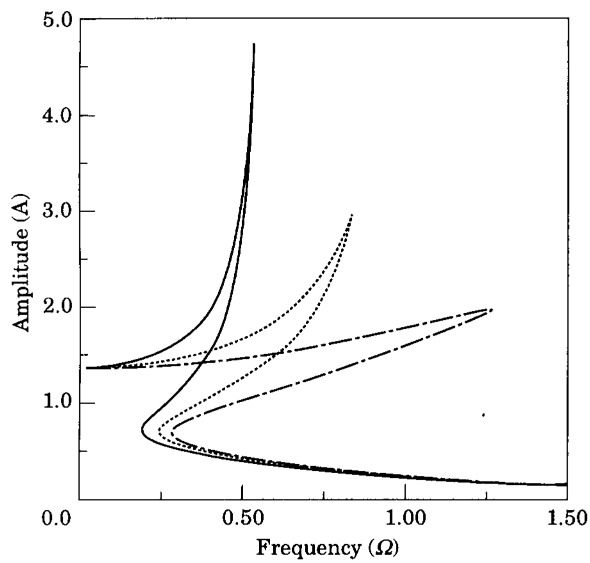

Figure 3. Steady state frequency response of the oscillator using SHB. $m = -1$, $P = 0{\cdot}5$, $\delta = 0{\cdot}2$: ——, $\varepsilon_1 = 5{\cdot}0$ and $\varepsilon_2 = 1{\cdot}0$; ------, $\varepsilon_1 = 1{\cdot}6$ and $\varepsilon_2 = 1{\cdot}0$; —·—·—, $\varepsilon_1 = 0{\cdot}1$ and $\varepsilon_2 = 1{\cdot}0$.

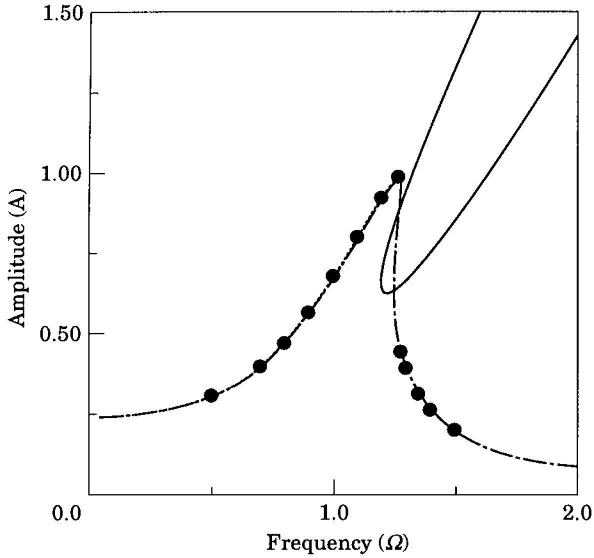

Figure 4. Steady state frequency response of the oscillator using SHB, 2HB and numerical integration: $m = 1$, $P = 0{\cdot}25$, $\delta = 0{\cdot}2$, $\varepsilon_1 = 0{\cdot}1$ and $\varepsilon_2 = 1{\cdot}0$; –·–·–, SHB; ------, 2HB; ●, numerical; ——, instability boundary.

In Figure 1, the steady state frequency response of the statically stable ($m = 1$) non-linear oscillator governed by equation (1) was obtained using single mode harmonic balance method (SHB) for cases in which ε_1 is relatively large or small with respect to ε_2. The results of Figure (1) shows that the steady state frequency

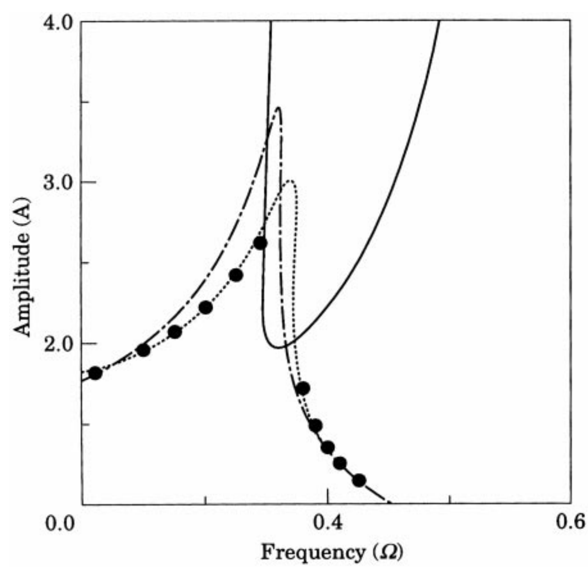

Figure 5. Steady state frequency response of the oscillator using SHB, 2HB and numerical integration: $m = 0$, $P = 0{\cdot}25$, $\delta = 0{\cdot}2$, $\varepsilon_1 = 0{\cdot}1$ and $\varepsilon_2 = 1{\cdot}0$; –·–·–, SHB; ------, 2HB; ●, numerical; ——, instability boundary.

response exhibits a softening behavior when, roughly, $\varepsilon_1/\varepsilon_2 > 1{\cdot}6$, hardening behavior when $\varepsilon_1/\varepsilon_2 < 1{\cdot}6$, and resembles linear behavior when $\varepsilon_1 \cong 1{\cdot}6\varepsilon_2$.

In Figures 2 and 3, the frequency responses for the neutrally stable ($m = 0$) and statically unstable ($m = -1$) cases were obtained using SHB. These results show that the steady state frequency response is of hardening type regardless of the value of ε_1 relative to ε_2.

In Figures 4, 5 and 6, for the three cases ($m = 1, 0, -1$) the instability boundaries are shown as well as the frequency response curves obtained by using single and two modes harmonic balance method (SHB and 2HB) and by integrating equation (1) numerically by the fourth order Runge–Kutta method with an integration step of 10^{-3}. The 2HB frequency response was obtained by a direct iteration technique with accuracy of 10^{-6} of the non-linear coupled equations (28–30). It is shown that the 2HB improves the accuracy of the SHB frequency response curves.

In Figures 7 and 8, the steady state frequency response curves for the case ($m = 1$) were obtained by using SHB, 2HB and MMS with reconstitution version II "first and second order", all of the three methods predict the correct qualitative "hardening" behavior when compared to the actual response "numerically integrated".

In Figure 9, the steady state frequency responses are shown for the Duffing oscillator ($m = 1$ and $\varepsilon_1 = 0$), the solutions are obtained using MMS with reconstitution version I, version II and numerical integration. As mentioned in section (1), the MMS version I leads to, at the second order, an amplitude–frequency response relation in the form of a polynomial of degree seven in the square of the steady state amplitude, as shown in equation (45). It is clear from Figure (9), that MMS with reconstitution version I introduces incorrect results

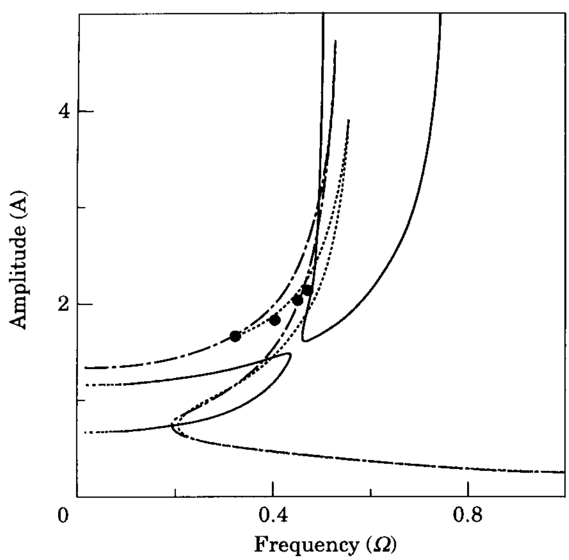

Figure 6. Steady state frequency response of the oscillator using SHB, 2HB and numerical integration: $m = -1$, $P = 0{\cdot}5$, $\delta = 0{\cdot}2$, $\varepsilon_1 = 5{\cdot}0$ and $\varepsilon_2 = 1{\cdot}0$; –·–·–, SHB; ------, 2HB; ●, numerical; ——, instability boundary.

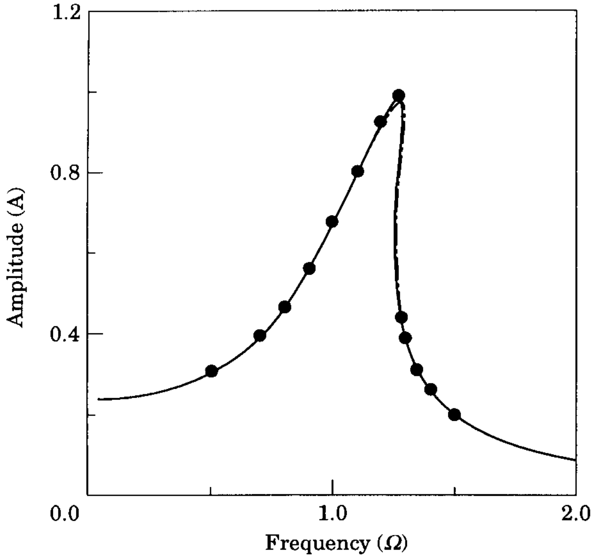

Figure 7. Steady state frequency response of the oscillator using SHB, 2HB, MMS version II and numerical integration: $m = 1$, $P = 0{\cdot}25$, $\delta = 0{\cdot}2$, $\varepsilon_1 = 0{\cdot}1$ and $\varepsilon_2 = 1{\cdot}0$; ——, SHB; ------, 2HB; –·–·–, MMS II first order; –··–, MMS II second order; ●, numerical integration.

"additional spurious solutions have appeared", and the frequency response are distorted compared to MMS version II.

4. CONCLUSIONS

The results presented in this work indicate that for the class of strongly non-linear oscillators governed by equation (1), first order approximation obtained by using harmonic balance (SHB) and perturbation (MMS) methods may lead

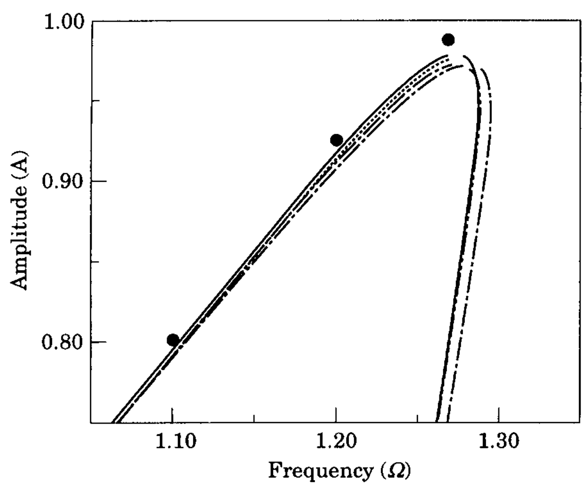

Figure 8. Expanded view of a portion of Figure 7.

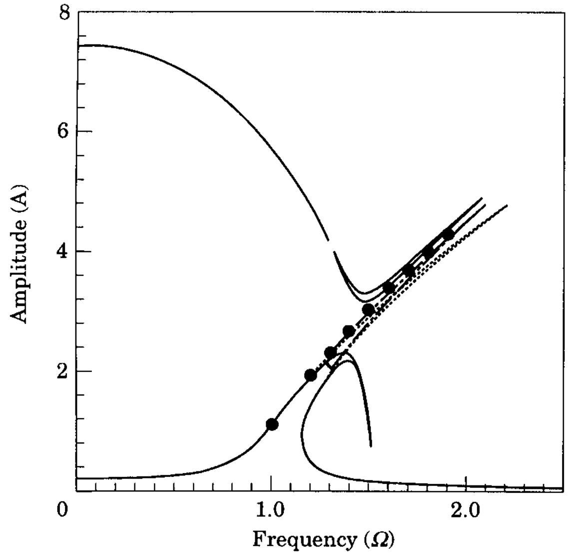

Figure 9. Steady state frequency response of the Duffing oscillator using MMS version I, MMS version II and Numerical integration. $m = 1$, $P = 1{\cdot}0$, $\delta = 0{\cdot}1$, $\varepsilon_1 = 0{\cdot}0$, $\varepsilon_2 = 1{\cdot}0$ and $\varepsilon = 0{\cdot}2$. ——, MMS I second order; –·–·–, MMS II first order; - - - - - -, MMS II second order; ●, numerical integration.

to appreciable, not only quantitative, but also qualitative errors in the predicted response.

From the results presented in this work it appears that Multiple Time Scales (MMS) with reconstitution version II in addition to being algebraically simpler, can lead to elimination of the spurious solutions of the steady state response of the oscillators governed by the equation (1), which may appear when one applies MMS version I.

Appreciable improvement of the accuracy of the predicted response were obtained using Two Modes Harmonic Balance (2MHB) and MMS version II even when the non-linearity is relatively strong.

Further analysis of the qualitative behavior of the resonance curves for the oscillators governed by equation (1) would require a more detailed stability analysis, which is currently under consideration by the authors.

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