Journal of Sound and Vibration (1999) **223**(1), 97–113
Article No. jsvi.1998.2096, available online at http://www.idealibrary.com on IDEAL®

A COMPACT MATRIX FORMULATION USING THE IMPEDANCE AND MOBILITY APPROACH FOR THE ANALYSIS OF STRUCTURAL–ACOUSTIC SYSTEMS

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(*Received* 20 *March* 1997, *and in final form* 16 *December* 1998)

This paper describes a compact matrix formulation for the steady-state analysis of structural–acoustic systems. A new approach to the problem is adopted that uses impedance and mobility methods commonly found in the analysis of purely structural or purely acoustic systems. The advantages of the approach are that an investigation into the coupling between the structural and acoustic systems is made easier, and it facilitates improved physical insight into the behaviour of structural–acoustic systems. In addition, because the equations describing the complete system are in matrix form, they can be solved easily using a computer. Due to the mismatch of dimensions between structural mobility and acoustic impedance, new terms are introduced for the coupled system analysis; the coupled acoustic impedance and the coupled structural mobility. *F–u* (force–velocity) and *p–Q* (pressure–source strength) diagrams are also introduced for impedance and mobility representations of a complete coupled system. Experimental work is presented, in which a simple rectangular acoustic enclosure with five rigid and one flexible side was used, to validate the analytical model and to investigate structural–acoustic coupling.

1. INTRODUCTION

The interaction between an acoustic space and its flexible boundaries is an important problem in the field of acoustics. Analysis of this interaction has been of interest to many researchers during the last half a century, as reviewed by Pan *et al.* [1, 2] and Hong and Kim [3]. A comprehensive theoretical model for coupled responses in a structural–acoustic coupled system has been presented by Dowell *et al.* [4]. They provided solutions for coupled responses in terms of the modal characteristics of the uncoupled structural and acoustic systems. This paper considers the analysis of a similar coupled system, but uses the impedance-mobility approach, which results in a compact matrix formulation. The impedance-mobility approach is well known to electrical engineers and physicists and is particularly applicable to the analysis of coupled systems, which are

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composed of several individual linear systems. Each system at the connection can be characterised by impedance or mobility, and the dynamics of a complete coupled system can be described at some or all of the points of interest. They are particularly useful concepts to judge the degree of coupling when two or more systems are connected, and are often used for the analysis of electrical systems [5]. In the 1950s, the method was adapted by mechanical engineers, who applied it to mechanical vibration problems [6]. A general theory of the approach and application examples to mechanical systems can be found in reference [7]. Furthermore, the approach has been successfully applied to various sound and vibration problems, such as the coupling between actuators and substructures, sound radiation from a plate to the acoustic free field, and wave propagation through media with different physical properties, as can be found in many textbooks, for example references [8–10].

In this paper, the classical theory by Dowell *et al.* [4] is re-examined from the impedance-mobility point of view, and a general method of structural–acoustic coupling analysis is presented. The basic theory of the impedance–mobility approach is considered in Section 2 with a simple conceptual structural–acoustic coupled system. In fact this represents the coupling between a single structural mode and a single acoustic mode. The approach is extended in Section 3 to

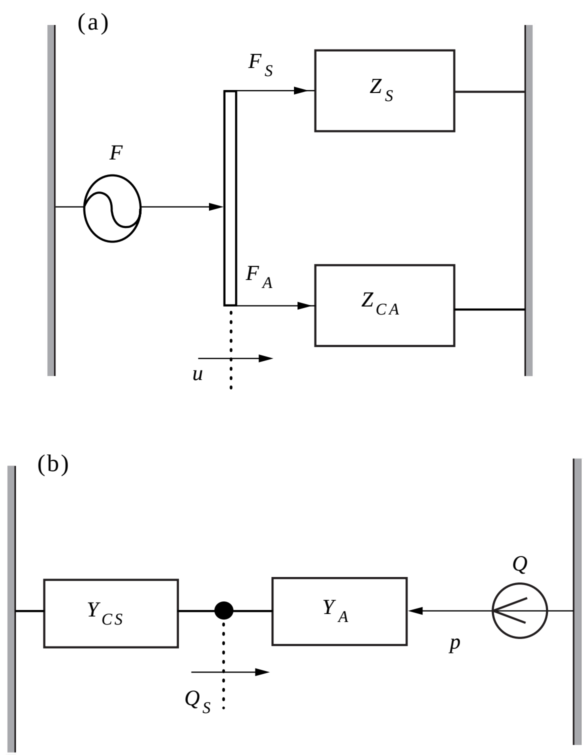

Figure 1. Impedance and mobility representation of a conceptual structural–acoustic system excited by (a) structural excitation, force F, and (b) acoustic excitation, source strength Q. (a) F–u representation for structural excitation; (b) p–Q representation for acoustic excitation.

analyse general structural–acoustic coupled systems in modal co-ordinates. The methodology can also be applied to structural–acoustic systems described by their physical co-ordinates, and this has been described in detail by Kim [11]. A criterion to establish whether or not a structural–acoustic system is strongly or weakly coupled is proposed in Section 3, and this criterion is presented in terms of acoustic impedance and structural mobility. In Section 4, some experimental results are presented that validate the analytical model developed, and illustrate the effects of structural–acoustic coupling. Finally, some conclusions are drawn in Section 5. There is also an Appendix to this paper that gives the relevant equations for the model problem used in the simulations and the experimental work.

2. BASIC THEORY OF THE IMPEDANCE-MOBILITY APPROACH

In this section a simple model of a conceptual structural–acoustic system is described, which forms the basis of the comprehensive model of a general structural–acoustic system discussed in Section 3. The conceptual model could, in fact, be used to describe the behaviour of a single structural mode coupled with a single acoustic mode.

In a single input structural system, the frequency domain quantities of mobility Y_S and impedance Z_S are defined as [7]:

$$Y_S = \frac{u}{F}, \quad Z_S = \frac{F}{u}, \tag{1a, b}$$

where the subscript S denotes the structural system and F and u are applied force and resulting velocity, respectively. In a single input acoustic system, the impedance and mobility are defined as [9]:

$$Z_A = \frac{p}{Q}, \quad Y_A = \frac{Q}{p}, \tag{2a, b}$$

where the subscript A denotes the acoustic system and Q and p are the source strength and acoustic pressure, respectively. It is important to note that the dimensions of impedance and mobility in structural and acoustic systems are different; the dimension of structural impedance being [Ns/m] and the dimension of acoustic impedance being [Ns/m^5]. This dimension difference makes the theoretical description for structural–acoustic coupled systems different from that for general mechanical systems considered in textbooks, for example reference [7].

Consider the conceptual structural–acoustic coupled system consisting of impedances Z_S and Z_{CA} excited by a single known structural force F as shown in Figure 1(a). The impedance Z_S is defined as the *uncoupled structural impedance* and is the ratio of the effective force applied to the structure F_S to the velocity u. The impedance Z_{CA} represents the acoustic reaction force F_A to the structural input velocity u and may be defined as the *coupled acoustic impedance*. Thus,

$$Z_S = \frac{F_S}{u}, \quad Z_{CA} = \frac{F_A}{u}. \tag{3a, b}$$

Using the force equilibrium condition, $F = F_S + F_A$, one gets an expression for the velocity of the structure u in terms of the structural mobility Y_S and the coupled acoustic impedance Z_{CA}.

$$u = \frac{Y_S}{1 + Y_S Z_{CA}} F, \tag{4}$$

where $Y_S = 1/Z_S$. When a single acoustic source of strength Q excites the conceptual structural–acoustic coupled system, it can be represented by the series combination of mobilities Y_A and Y_{CS} as shown in Figure 1(b). Hereafter it is called the *p–Q representation* since the physical parameters are pressure and source strength, while the diagram in Figure 1(a) is called the *F–u representation*, i.e., the force–velocity representation. The mobility Y_A is defined as the *uncoupled acoustic mobility* and is the ratio of the effective source strength Q_A acting on the acoustic system to the acoustic pressure p. The mobility Y_{CS} represents the induced structural source strength Q_S to the acoustic pressure p and is defined as the *coupled structural mobility*. thus,

$$Y_A = \frac{Q_A}{p}, \quad Y_{CS} = -\frac{Q_S}{p}. \tag{5a, b}$$

Note the minus sign of Y_{CS} because the direction of Q_S is defined opposite to the

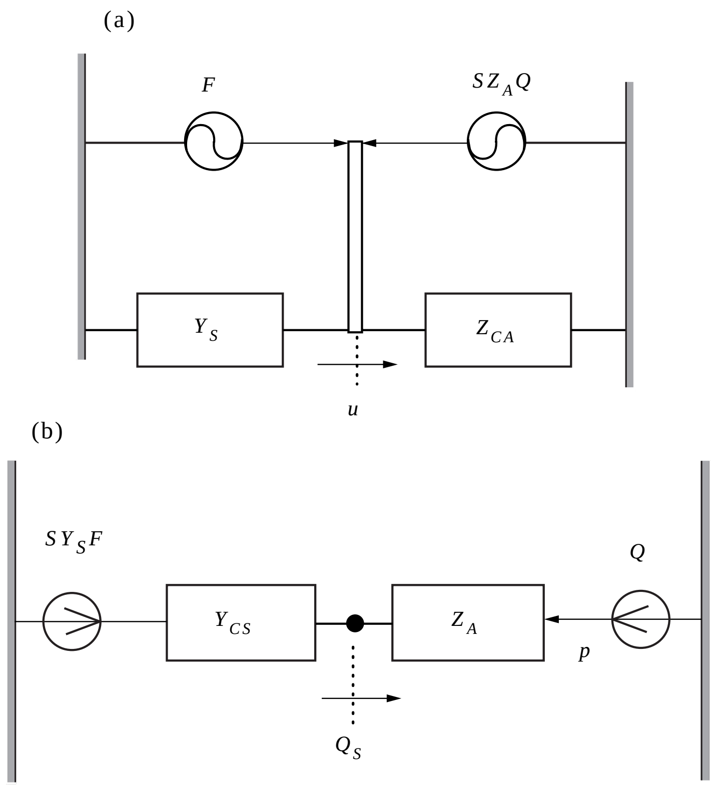

Figure 2. *F–u* and *p–Q* representation for the conceptual structural–acoustic system with both structural and acoustic excitation. (a) *F–u* representation; (b) *p–Q* representation.

acoustic pressure. Since both source strengths are acting toward the acoustic system, the effective source strength acting on this system is $Q_A = Q + Q_S$. Thus,

$$p = \frac{Z_A}{1 + Z_A Y_{CS}} Q, \tag{6}$$

where $Z_A = 1/Y_A$. When there is both force and acoustic excitation, a coupling factor, which connects the *F–u* and the *p–Q* representations is required. For the conceptual system studied in this section an area S may be simply used to match the dimensions. Thus, the relationship between the coupled and uncoupled acoustic impedances and the coupled and uncoupled structural mobilities are given by:

$$Z_{CA} = S^2 Z_A, \quad Y_{CS} = S^2 Y_S. \tag{7a, b}$$

Conversions between the *F–u* representation and the *p–Q* representation can be achieved by using Thevenin and Norton's theorems [7]. The *F–u* and the *p–Q* representations for the conceptual structural–acoustic system subject to both structural and acoustic excitation are given in Figure 2(a) and (b), respectively. The equations relating the structural velocity and the acoustic pressure to the applied force and acoustic strength are given by:

$$u = \frac{1}{1 + Y_S Z_{CA}} Y_S(F - S Z_A Q), \quad p = \frac{1}{1 + Z_A Y_{CS}} Z_A(Q + S Y_S F). \tag{8a, b}$$

These are the key equations for the analysis of general coupled systems, and can be extended to vector and matrix forms to deal with multi-degree-of-freedom systems with several excitation points. In Section 3 these equations are expanded to model a general structural–acoustic system.

If the system is excited by a structural source and the structure responds predominantly as though it was *in vacuo* then the coupled acoustic impedance has a negligible effect on the structure. In this case the system is said to be weakly coupled. Moreover, if the system is excited acoustically and the cavity responds predominantly as though the structure were infinitely rigid it is also said to be weakly coupled. These conditions can be examined mathematically using equations (8a) and (8b). If one sets $Q = 0$ in equation (8a) then $u = Y_S F$ provided that $|Y_S Z_{CA}| \ll 1$ so that one can set $Y_S Z_{CA} = 0$. If one sets $F = 0$ in equation (8b) then $p = Z_A Q$ provided that $|Z_A Y_{CS}| \ll 1$ so that one can set $Z_A Y_{CS} = 0$. Noting the relationship between the coupled and the uncoupled acoustic impedance and structural mobility in equations (7a) and (7b), one can see that $Y_S Z_{CA} = Z_A Y_{CS}$, and thus the condition for weak coupling is independent of the type of excitation. If there is *both* structural and acoustic excitation in a weakly coupled conceptual system then the equations for the structural velocity and acoustic pressure are given by

$$u = Y_S(F - S Z_A Q), \quad p = Z_A(Q + S Y_S F). \tag{8c, d}$$

The *F–u* and the *p–Q* representations for a weakly coupled conceptual structural–acoustic system are shown in Figure 3(a) and (b), respectively.

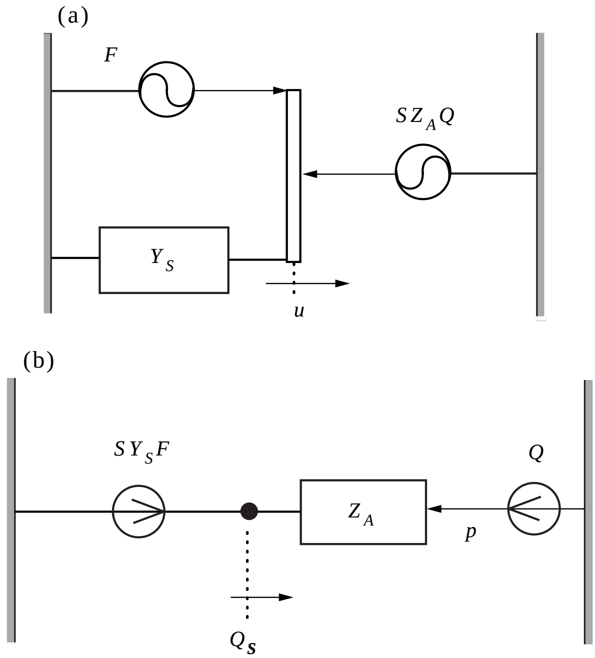

Figure 3. F–u and p–Q representations for a *weakly coupled* conceptual structural–acoustic system with both structural and acoustic excitation. (a) F–u representation; (b) p–Q representation.

3. STRUCTURAL–ACOUSTIC COUPLING THEORY IN MODAL COORDINATES

In this section the impedance and mobility approach described in Section 2 is used to analyse the dynamic behaviour of an arbitrary shaped enclosure surrounded by a flexible structure and an acoustically rigid wall such as that shown in Figure 4. The acoustic source strength density function $s(\mathbf{x}, \omega)$ and the force distribution function $f(\mathbf{y}, \omega)$ excite the cavity and the flexible structure, respectively. Co-ordinate $\mathbf{x}$ is used for the acoustic field in the cavity, and co-ordinate $\mathbf{y}$ is used for vibration on the structure.

It is assumed that coupled responses can be described by finite sets of uncoupled acoustic and structural modes. The uncoupled modes are the *rigid-walled* acoustic modes of the cavity and the *in vacuo* structural modes of the structure. Full coupling is considered between the flexible structure and the acoustic cavity system. However, weak coupling is assumed between the flexible structure and the acoustic field outside the cavity. This is because it is assumed that the vibration of the structure is not influenced by the radiated acoustic field outside the cavity.

The acoustic pressure and the structural vibration are described by the summation of N and M modes, respectively. Hence, both the acoustic pressure p at $\mathbf{x}$ inside the enclosure and the structural vibration velocity u at $\mathbf{y}$ are given by [4]:

$$p(\mathbf{x}, \omega) = \sum_{n=1}^{N} \psi_n(\mathbf{x}) a_n(\omega) = \mathbf{\Psi}^{\mathrm{T}} \mathbf{a}, \tag{9a}$$

$$u(\mathbf{y}, \omega) = \sum_{m=1}^{M} \phi_m(\mathbf{y}) b_m(\omega) = \mathbf{\Phi}^{\mathrm{T}} \mathbf{b}, \tag{9b}$$

where, the N length column vectors $\mathbf{\Psi}$ and $\mathbf{a}$ consist of the array of uncoupled acoustic mode shape functions $\psi_n(\mathbf{x})$ and the complex amplitude of the acoustic pressure modes $a_n(\omega)$, respectively. Likewise the M length column vectors $\mathbf{\Phi}$ and $\mathbf{b}$ consist of the array of uncoupled vibration mode shape functions $\phi_m(\mathbf{y})$ and the complex amplitude of the vibration velocity modes $b_m(\omega)$, respectively. The superscript T denotes the transpose. The mode shape functions $\psi_n(\mathbf{x})$ and $\phi_m(\mathbf{y})$ satisfy the orthogonal property in each uncoupled system, and are normalised as follows:

$$V = \int_V \psi_n^2(\mathbf{x})\, \mathrm{d}V, \quad S_f = \int_{S_f} \phi_m^2(\mathbf{y})\, \mathrm{d}S, \tag{10a, b}$$

where V and S_f are the volume of the cavity and the surface area of the flexible structure, respectively. The complex amplitude of the nth acoustic mode under structural *and* acoustic excitation is given by [4, 12]:

$$a_n(\omega) = \frac{\rho_o c_o^2}{V} A_n(\omega) \left(\int_V \psi_n(\mathbf{x}) s(\mathbf{x}, \omega)\, \mathrm{d}V + \int_{S_f} \psi_n(\mathbf{y}) u(\mathbf{y}, \omega)\, \mathrm{d}S \right) \tag{11}$$

where ρ_0 and c_o denote the density and the speed of sound in air, respectively.

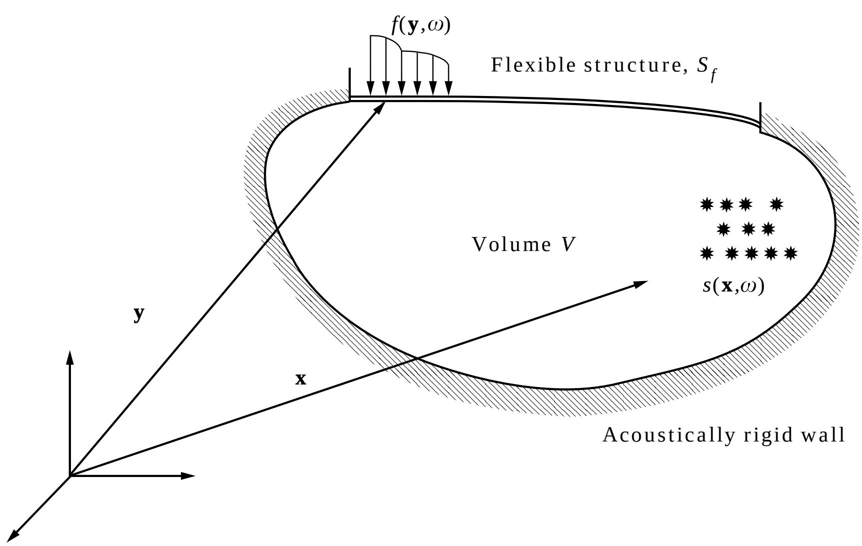

Figure 4. A structural–acoustic coupled system with structural excitation $f(\mathbf{y}, \omega)$ and acoustic excitation $s(\mathbf{x}, \omega)$.

The function $s(\mathbf{x}, \omega)$ denotes the acoustic source strength density function in the cavity volume V, and $u(\mathbf{y}, \omega)$ denotes the normal velocity of the surrounding flexible structure of surface area S_f. The two integral expressions inside the bracket represent the nth acoustic modal source strength contributed from $s(\mathbf{x}, \omega)$ and $u(\mathbf{y}, \omega)$, respectively. The acoustic mode resonance term $A_n(\omega)$ is given by:

$$A_1(\omega) = \frac{1}{1/T_a + \mathrm{j}\omega}, \quad \text{when} \quad n = 1 \tag{12a}$$

and

$$A_n(\omega) = \frac{\mathrm{j}\omega}{\omega_n^2 - \omega^2 + \mathrm{j}2\zeta_n\omega_n\omega}, \quad \text{when} \quad n \neq 1, \tag{12b}$$

where T_a is the time constant of the first mode [13], and ω_n and ζ_n are the natural frequency and damping ratio of the nth acoustic mode, respectively. Substituting equation (9b) into equation (11) and introducing the generalised acoustic source strength $q_n = \int_V \psi_n(\mathbf{x}) s(\mathbf{x}, \omega) \mathrm{d}V$, gives:

$$a_n(\omega) = \frac{\rho_o c_0^2}{V} A_n(\omega) \left(q_n + \sum_{m=1}^{M} C_{n,m} \cdot b_m(\omega) \right), \tag{13}$$

where $C_{n,m}$ represents the geometric coupling relationship between the uncoupled structural and acoustic mode shape functions on the surface of the vibrating structure S_f and is given by:

$$C_{n,m} = \int_{S_f} \psi_n(\mathbf{y}) \phi_m(\mathbf{y}) \, \mathrm{d}S. \tag{14}$$

Thus, the modal acoustic pressure vector $\mathbf{a}$ can be expressed as:

$$\mathbf{a} = \mathbf{Z}_a(\mathbf{q} + \mathbf{q}_s), \tag{15}$$

where $\mathbf{q}$ is the N length modal source strength vector and $\mathbf{q}_s = \mathbf{Cb}$ is the modal source strength vector due to vibration of the structure, which acts as a set of acoustic sources on the flexible structure. The M length $\mathbf{b}$ is the complex vibration modal amplitude vector and the $(N \times M)$ matrix $\mathbf{C}$ is the structural–acoustic mode shape coupling matrix. $\mathbf{Z}_a = \mathbf{A}\rho_o c_o^2/V$ is an $(N \times N)$ diagonal matrix defined as the *uncoupled acoustic modal impedance matrix*, which determines the relationship between the acoustic source excitation and the resultant acoustic pressure in modal co-ordinates of the acoustic system. The uncoupled modal impedance matrix is diagonal because of the orthogonal property of uncoupled modes. The matrix $\mathbf{A}$ is a $(N \times N)$ diagonal matrix in which each (n, n) diagonal term consists of A_n.

If one assumes that the flexible structure in Figure 4 is an isotropic thin plate, the complex vibration velocity amplitude of the mth mode can be expressed as [4]:

$$b_m(\omega) = \frac{1}{\rho_s h S_f} B_m(\omega) \left(\int_{S_f} \phi_m(\mathbf{y}) f(\mathbf{y}, \omega) \, \mathrm{d}S - \int_{S_f} \phi_m(\mathbf{y}, \omega) \, p(\mathbf{y}, \omega) \, \mathrm{d}S \right), \tag{16}$$

where, ρ_s is the density of the plate material, h is the thickness of the plate, S_f is the area of the plate, and $f(\mathbf{y}, \omega)$ and $p(\mathbf{y}, \omega)$ denote the force distribution function and the cavity acoustic pressure distribution on the surface of the plate, respectively. The two integral equations in the bracket represent the generalised mth vibration modal force due to $f(y, \omega)$ and $p(\mathbf{y}, \omega)$, respectively. Since the directions of the external force and acoustic pressure are defined to be opposite, there is a minute sign in front of the second integral term in the bracket. The structural mode resonance term $B_m(\omega)$ can be expressed as:

$$B_m(\omega) = \frac{j\omega}{\omega_m^2 - \omega^2 + \mathrm{j}2\zeta_m\omega_m\omega}, \tag{17}$$

where ω_m and ζ_m are the natural frequency and the damping ratio of the mth mode, respectively. Substituting equation (9a) into equation (16) and introducing the generalised modal force $g_m = \int_{S_f} \phi_m(\mathbf{y}) f(\mathbf{y}, \omega) \mathrm{d}S$, gives:

$$b_m(\omega) = \frac{1}{\rho_s h S_f} B_m(\omega) \left(g_m - \sum_{n=1}^{N} C_{n,m}^{\mathrm{T}} \cdot a_n(\omega) \right) \tag{18}$$

where $C_{n,m}^{\mathrm{T}} = C_{m,n}$. Thus, the modal vibration amplitude vector $\mathbf{b}$ can be expressed as:

$$\mathbf{b} = \mathbf{Y}_s(\mathbf{g} - \mathbf{g}_a), \tag{19}$$

where $\mathbf{g}$ is the generalised modal force vector due to the external force distribution $f(\mathbf{y}, \omega)$, and $\mathbf{g}_a = \mathbf{C}^{\mathrm{T}}\mathbf{a}$ is the modal force vector acting on the acoustic system, which is the reaction force due to the acoustic pressure fluctuation. $\mathbf{Y}_s = \mathbf{B}/(\rho_s h S_f)$ is the $(M \times M)$ diagonal matrix defined as the *uncoupled structural modal mobility matrix* which determines the relationship between structural excitation and the resultant structural velocity response in modal co-ordinates of the uncoupled structural system. As with the uncoupled acoustic impedance matrix $\mathbf{Z}_a$, note that $\mathbf{Y}_s$ is a diagonal matrix. The matrix $\mathbf{B}$ is a $(M \times M)$ size diagonal matrix in which each (m, m) diagonal term consists of B_m, $\mathbf{C}^{\mathrm{T}}$ is the transpose matrix of $\mathbf{C}$, and the M length vector $\mathbf{g}$ is the generalised modal force vector due to the external force distribution $f(\mathbf{y}, \omega)$. Combining equations (15) and (19), the acoustic and structural modal amplitude vectors $\mathbf{a}$ and $\mathbf{b}$ can be expressed in terms of the modal excitation vectors $\mathbf{q}$ and $\mathbf{g}$:

$$\mathbf{a} = (\mathbf{I} + \mathbf{Z}_a\mathbf{C}\mathbf{Y}_s\mathbf{C}^{\mathrm{T}})^{-1}\mathbf{Z}_a(\mathbf{q} + \mathbf{C}\mathbf{Y}_s\mathbf{g}), \tag{20a}$$

$$\mathbf{b} = (\mathbf{I} + \mathbf{Y}_s\mathbf{C}^{\mathrm{T}}\mathbf{Z}_a\mathbf{C})^{-1}\mathbf{Y}_s(\mathbf{g} - \mathbf{C}^{\mathrm{T}}\mathbf{Z}_a\mathbf{q}). \tag{20b}$$

Equations (20a) and (20b) can be rewritten using the impedance-mobility approach to enable a comparison with the simple conceptual system discussed in Section 2. If one assumes that there is only structural excitation ($\mathbf{q} = 0$) and substitute for $\mathbf{q}_s = \mathbf{C}\mathbf{b}$ in equation (15) one gets $\mathbf{a} = \mathbf{Z}_a\mathbf{C}\mathbf{b}$. Combining this with the expression for the acoustic reaction force vector, $\mathbf{g}_a = \mathbf{C}^{\mathrm{T}}\mathbf{a}$, gives:

$$\mathbf{g}_a = \mathbf{Z}_{ca}\mathbf{b}, \tag{21}$$

where $\mathbf{Z}_{ca} = \mathbf{C}^{\mathrm{T}}\mathbf{Z}_a\mathbf{C}$ is defined as the $(M \times M)$ size symmetric *coupled acoustic modal impedance matrix*, which determines the acoustic reaction force $\mathbf{g}_a$ in modal co-ordinates induced by structural vibration $\mathbf{b}$.

Likewise, if one substitutes for $\mathbf{g}_a = \mathbf{C}^{\mathrm{T}}\mathbf{a}$ in equation (19), and assumes that there is only acoustic excitation $(\mathbf{g} = \mathbf{0})$, one gets $\mathbf{b} = -\mathbf{Y}_s\mathbf{C}^{\mathrm{T}}\mathbf{a}$. Combining this with the expression for the reaction source strength vector, $\mathbf{q}_s = \mathbf{C}\mathbf{b}$, gives:

$$\mathbf{q}_s = -\mathbf{Y}_{cs}\mathbf{a}, \tag{22}$$

where $\mathbf{Y}_{cs} = \mathbf{C}\mathbf{Y}_s\mathbf{C}^{\mathrm{T}}$ is defined as the $(N \times N)$ size symmetric *coupled structural modal mobility matrix* which determines the induced source strength on the flexible structure $\mathbf{q}_s$ induced by acoustic excitation $\mathbf{a}$. $\mathbf{Z}_{ca}$ and $\mathbf{Y}_{cs}$ are the equivalent matrix forms of the expressions used in the conceptual model given in equations (3b) and (5b), respectively. Unlike $\mathbf{Z}_a$ and $\mathbf{Y}_s$, note that the coupled matrices $\mathbf{Z}_{ca}$ and $\mathbf{Y}_{cs}$ are symmetric but non-diagonal. The coupled responses of the structure–acoustic system given in equation (20a,b) can thus be rewritten using the definitions of $\mathbf{Z}_{ca}$ and $\mathbf{Y}_{cs}$ to give:

$$\mathbf{a} = (\mathbf{I} + \mathbf{Z}_a\mathbf{Y}_{cs})^{-1}\mathbf{Z}_a(\mathbf{q} + \mathbf{C}\mathbf{Y}_s\mathbf{g}), \tag{23a}$$

$$\mathbf{b} = (\mathbf{I} + \mathbf{Y}_a\mathbf{Z}_{ca})^{-1}\mathbf{Y}_s(\mathbf{g} - \mathbf{C}^{\mathrm{T}}\mathbf{Z}_a\mathbf{q}). \tag{23b}$$

The above equations are a compact description of the classical theory for

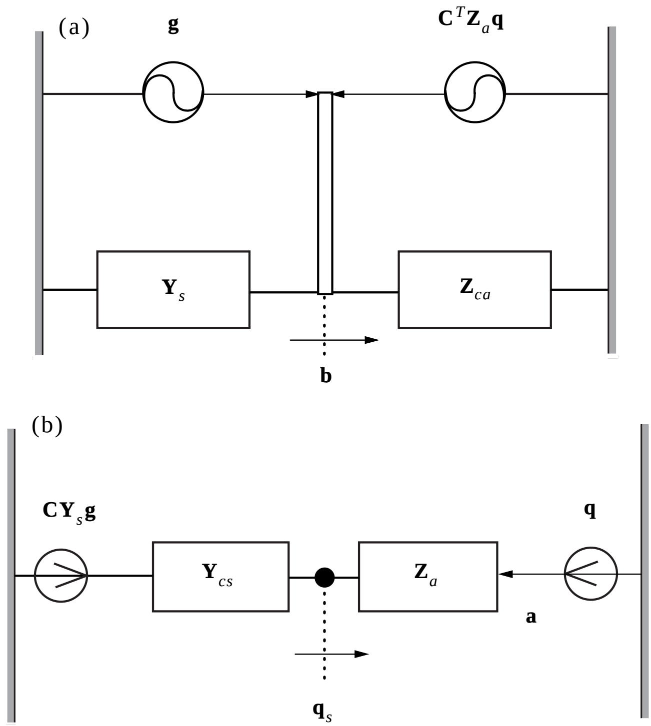

Figure 5. *F–u* and *p–Q* representations in modal co-ordinates of a structural–acoustic system with both structural and acoustic excitation. (a) *F–u* representation; (b) *p–Q* representation.

calculating the coupled responses in a structural–acoustic coupled system subject to both structural and acoustic excitation. Because they are of vector–matrix form these equations can be readily solved using a computer. When the coupling between a single acoustic mode and a single structural mode is considered, it can be easily seen that the coupled responses are the same as for the conceptual model described by equations (8a) and (8b). Equivalent *F–u* and *p–Q* representations of the coupled system are shown in Figure 5. The coupling factor which corresponds to the area **S** in Section 2 is the structural–acoustic mode shape coupling matrix $\mathbf{C}$ or $\mathbf{C}^{\mathrm{T}}$, whose elements have the dimensions of area [m^2]. To calculate the acoustic pressure or the structural velocity the modal amplitude vectors determined using equations (23a) and (23b) have to be substituted into equations (9a) and (9b), respectively.

Following the analysis in Section 2, the criteria for weak coupling is given as $[\mathbf{I} + \mathbf{Y}_s\mathbf{Z}_{ca}] \approx [\mathbf{I}]$ for structural response and $[\mathbf{I} + \mathbf{Z}_a\mathbf{Y}_{cs}] \approx [\mathbf{I}]$ for acoustic response. This means that for weakly coupled systems one can set $\mathbf{Y}_s\mathbf{Z}_{ca} = \mathbf{0}$ or $\mathbf{Z}_a\mathbf{Y}_{cs} = \mathbf{0}$ in equations (23a) and (23b), respectively. These two criteria generally have different matrix dimensions. However, it does not necessarily mean that both criteria must be satisfied for the responses considered to be weakly coupled when the system is excited by both structural and acoustic sources; one is just a description for the *F–u* representation and the other is for the *p–Q* representation.

The criteria for weak coupling can be rewritten by extracting physical quantities of the system, i.e., the acoustic bulk stiffness K_a and the mass of the structure M_s. Noting that $\mathbf{Y}_s = \mathbf{B}/(\rho_s h S_f)$, $\mathbf{Z}_{ca} = \mathbf{C}^{\mathrm{T}}\mathbf{Z}_a\mathbf{C}$, $\mathbf{Z}_a = \mathbf{A}\rho_o c_o^2/V$ and $\mathbf{Y}_{cs} = \mathbf{C}\mathbf{Y}_s\mathbf{C}^{\mathrm{T}}$, the coupling terms can be written as:

$$\mathbf{Y}_s\mathbf{Z}_{ca} = \frac{K_a}{M_s}\mathbf{B}\mathbf{C}'^{\mathrm{T}}\mathbf{A}\mathbf{C}', \quad \mathbf{Z}_a\mathbf{Y}_{cs} = \frac{K_a}{M_s}\mathbf{A}\mathbf{C}'\mathbf{B}\mathbf{C}'^{\mathrm{T}}, \tag{24a, b}$$

where $\mathbf{C}' = \mathbf{C}/S_f$. K_a is the bulk stiffness of the cavity when the flexible structure vibrates as a rigid body and is given by $K_a = (\rho_o c_o^2/V)S_f^2$. For a flexible plate, $M_s = \rho_s h S_f$.

It can be seen that the degree of coupling is dependent upon three factors: (i) the ratio of the acoustic bulk stiffness to structure mass K_a/M_s; (ii) the normalised geometric mode shape coupling term $\mathbf{C}'$ and $\mathbf{C}'^{\mathrm{T}}$; (iii) frequency dependent terms **A** and **B**. The first factor states that the coupling in a system becomes weaker as K_a/M_s gets smaller, which is consistent with other descriptions in the literature, for example references [1–4]. The second factor is a non-dimensional quantity, and is determined by the geometric coupling between the acoustic and structural modes. The third factor **A** and **B**, however, are frequency dependent and each diagonal term of the matrices are typical of the mobility of a single-degree-of-freedom mass–spring–damper system. The second and third factors are combined together in equation (24a,b), and describe the complicated frequency dependent behaviour. The resulting equations for weakly coupled systems are given by:

$$\mathbf{a} = \mathbf{Z}_a(\mathbf{q} + \mathbf{C}\mathbf{Y}_s\mathbf{g}), \quad \mathbf{b} = \mathbf{Y}_s(\mathbf{g} - \mathbf{C}^{\mathrm{T}}\mathbf{Z}_a\mathbf{q}). \tag{25a, b}$$

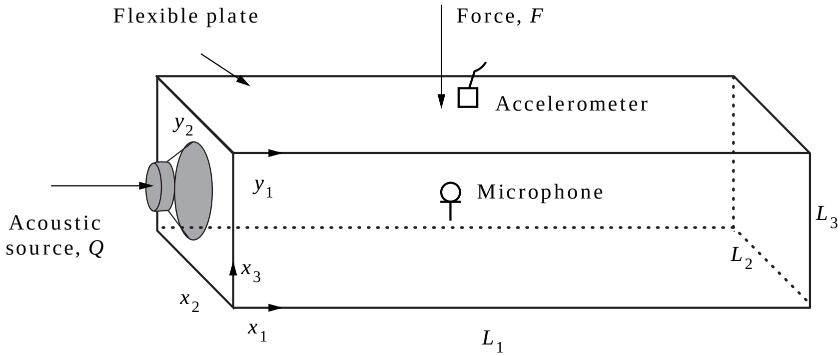

Figure 6. Experimental set-up to validate the analytical model for the structural–acoustic system .

In a similar way to Figure 3, the *F–u* and *p–Q* representations for a weakly coupled system can be drawn by removing the coupled impedance and mobility blocks.

4. EXPERIMENTAL WORK

To demonstrate the validity of the analytical model, an experiment was performed using a rectangular enclosure as shown in Figure 6, and the experimental results were compared with simulations generated using the model described in Section 3. The enclosure consisted of five acoustically rigid walls and a simply supported flexible plate on the remaining side. To make the acoustically rigid boundary condition, 25-mm thick plywood walls were used

TABLE 1
Material properties of the experimental rig

Material	Density (kg/m^3)	Phase speed (m/s)	Young's modulus (N/m^2)	Poisson's ratio (ν)	Damping ratio (ζ)
Air	1·21	340	–	–	0·01
Al	2770	–	71 × 10^9	0·33	0·01

TABLE 2
The natural frequencies and geometric mode shape coupling coefficients of each uncoupled system of the experimental rig

Order (Plate) / Cavity	Type	Frequency (Hz)	1	2	3	4	5	6
			(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
			141	157	184	222	270	330
1	(0,0,0)	0	1·0000	0	0·3333	0	0·2000	0
2	(1,0,0)	113	0	0·9428	0	0·3771	0	0·2424
3	(2,0,0)	227	−0·4714	0	0·8485	0	0·3367	0
4	(3,0,0)	340	0	−0·5657	0	0·8081	0	0·3143

which were surrounded by 75-mm deep sand layers packed by an extra container. To achieve the simply supported boundary condition for the plate, 1·25-mm steel strips were bolted around the perimeter of the plate. The design concept is that the thin strip is relatively rigid to in-plane motion but flexible to rotation and thus restricts out-of-plane motion of the plate but does not apply a moment at the plate edges. The dimensions of the cavity were $L_1 \times L_2 \times L_3$,

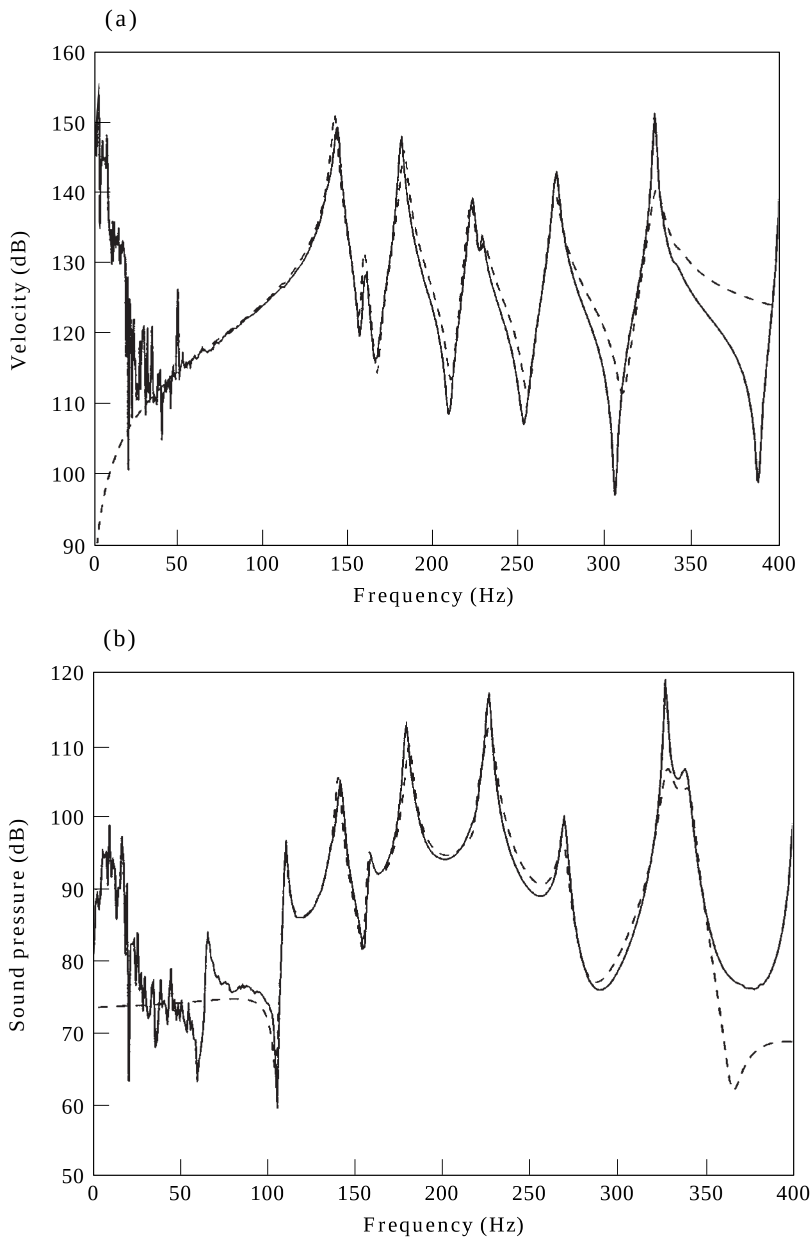

Figure 7. Experimental (—) and predicted (– – –) responses to a point force excitation of the structural–acoustic system shown in Figure 6. The predicted results were calculated using equations (9) and (23). (a) Structural velocity (dB ref 10^{-9} m/s); (b) acoustic pressure (dB ref 20 μ Pa).

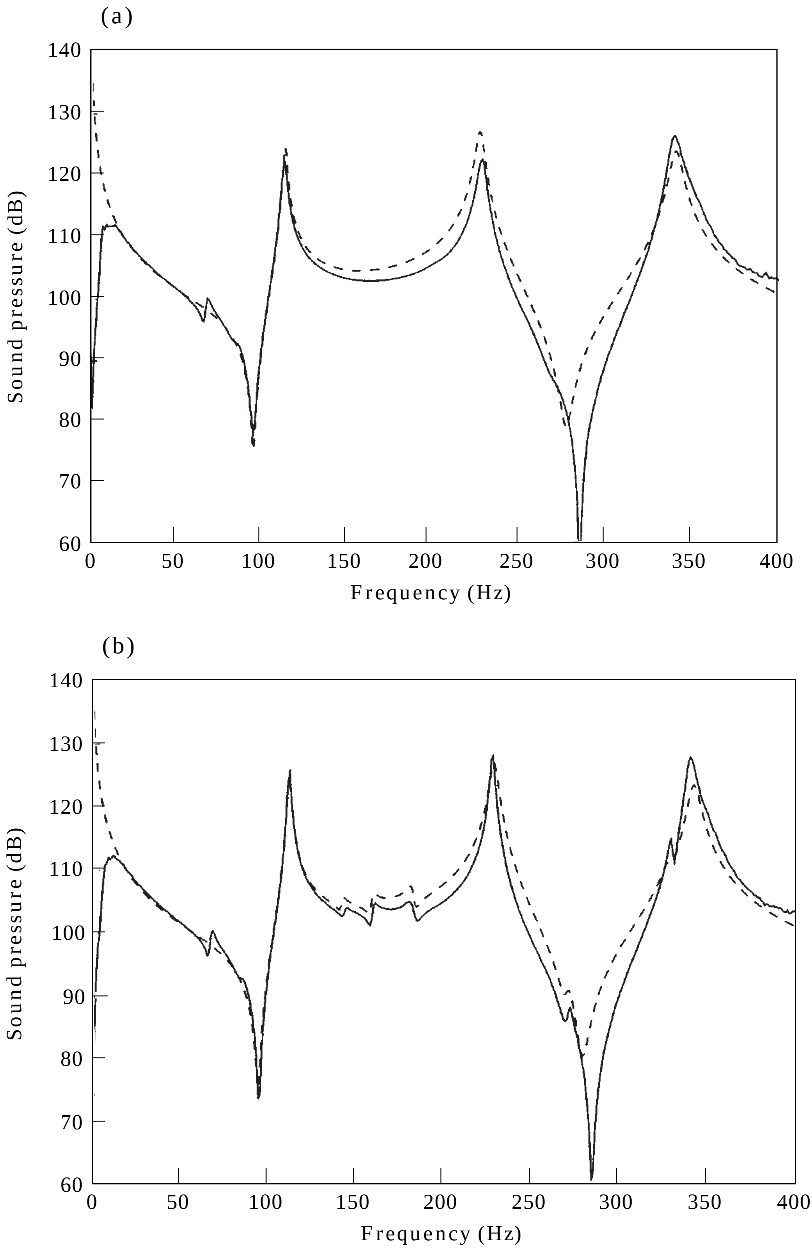

Figure 8. Experimental (—) and predicted (– – –) acoustic pressure due to acoustic excitation of the structural–acoustic system shown in Figure 6. The predicted results were calculated using equations (9) and (25) for (a) and equations (9) and (23) for (b). (a) Acoustic response with limp masses placed on the plate (dB ref 20 μ Pa). A rigid-walled condition was assumed in the simulations; (b) acoustic response with the masses removed from the plate (dB ref 20 μ Pa). Full coupling was assumed in the simulations.

where $L_1 = 1{\cdot}5$ m, $L_2 = 0{\cdot}3$ m, and $L_3 = 0{\cdot}4$ m, and the thickness of the aluminium plate was 5 mm. The system was excited by either a loudspeaker installed at the left-hand end of the enclosure, or a point force generated by a PCB piezo-actuator located at $(13L_1/30,\ L_2/2)$ on the plate, over the frequency range 0–400 Hz. The plate acceleration was measured using an accelerometer adjacent to the shaker and the acoustic pressure was measured using a microphone, positioned approximately at position ($4L_1/10$, $L_2/2$, $L_3/2$). An HP3566A frequency response analyser was used to collect the data. The material properties of air and aluminium (A1) used in the simulations are listed in Table 1.

A total of four acoustic and six structural modes were assumed to contribute to the coupled responses within the frequency range of interest and this number was used in the simulations. Table 2 shows the calculated natural frequencies of both uncoupled systems and their geometric mode-shape coupling coefficients normalised by their maximum value. The (m_1, m_2) and (n_1, n_2, n_3) indicate the indices of the mth plate mode and the nth cavity mode. Equations for the analysis of the rectangular enclosure are given in the Appendix. The modal damping ratios of the plate and the cavity were both assumed to be 0·01, and the time constant of the first acoustic mode was taken to be 0·2 s.

With the force actuator exciting the plate, the velocity at the excitation point and the acoustic pressure response at the microphone location are compared with their analytical results in Figure 7(a) and (b), respectively. The results presented are per unit input force. Above about 80 Hz there is good agreement between the experimental and theoretical results. In Figure 7(a), the small peak after the 4th structural mode at 221·6 Hz is due to the strong coupling with the second acoustic mode at 226·6 Hz. The poor experimental results at low frequencies were because an inertial shaker was used that did not deliver a large force at these frequencies, and hence there was a small signal to noise ratio.

The system was then excited using the loudspeaker. For one experiment heavy limp masses were placed on the top plate in an attempt to realise a weakly coupled system. Figure 8(a) shows the measured and predicted acoustic pressure at the microphone position per unit source strength. The predicted response was calculated using the equations for weak coupling, (9a) and (25a). It can be seen that there is reasonably good agreement between the model and the experimental results.

The final experiment involved removing the limp masses from the plate and repeating the measurement above. The results are shown in Figure 8(b) where the predicted pressure was calculated using the fully coupled equations (9a) and (23a). Examination of Figures 8(a) and (b) shows that both theoretical and experimental results clearly demonstrate structural coupling effects near the structural natural frequencies uncoupled. This result demonstrates the difference between weak and full coupling, and validates the theoretical model.

5. CONCLUSIONS

This paper has presented a compact matrix formulation of the analytical steady-state solution for a structural–acoustic coupled system. It is based on the impedance-mobility approach using the uncoupled mobility of the structure and the uncoupled acoustic impedance, both in modal co-ordinate systems. The formulations are expressed in terms of vectors and matrices that are convenient for physical interpretation as well as numerical computation. Due to the mismatch of the dimensions of impedance and mobility between structural and acoustic systems new mechanical terms were introduced for the coupled system analysis; coupled acoustic impedance and coupled structural mobility. The *F–u* (force–velocity) and *p–Q* (pressure–source strength) diagrammatic representations have also been introduced to represent the dynamic behaviour of the coupled system in terms of impedance and mobility. A criterion for weak coupling between a structure and an adjacent acoustic space has been investigated both theoretically and experimentally. The experimental work conducted has also validated the analytical model presented in this paper.

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APPENDIX: EQUATIONS FOR THE ANALYSIS OF A RECTANGULAR ENCLOSURE

For a rigid-walled rectangular enclosure, the acoustic (n_1, n_2, n_3) mode shape functions normalised by its volume is given by [14]:

$$\psi_n(\mathbf{x}) = \sqrt{e_1 e_2 e_3} \cos\left(\frac{n_1 \pi x_1}{L_1}\right) \cos\left(\frac{n_2 \pi x_2}{L_2}\right) \cos\left(\frac{n_3 \pi x_3}{L_3}\right), \tag{A1}$$

where n_1, n_2 and n_3 are integers and L_1, L_2 and L_3 are the dimensions of the rectangular enclosure in the $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ co-ordinate directions. The normalisation factors are given by $e_i = 1$ if $n_i = 0$ and $e_i = 2$ if $n_i \geqslant 1$ where the subscript i can be 1, 2, and 3. The corresponding acoustic natural frequency is given by:

$$\omega_n = \pi c_o \sqrt{\left(\frac{n_1}{L_1}\right)^2 + \left(\frac{n_2}{L_2}\right)^2 + \left(\frac{n_3}{L_3}\right)^2}, \tag{A2}$$

where c_o is the speed of sound in air. For a simply supported isotropic rectangular plate of dimensions ($L_1 \times L_2$), the plate (m_1, m_2) mode shape function normalised by its surface area can be written as

$$\phi_m(\mathbf{y}) = 2 \sin\left(\frac{m_1 \pi y_1}{L_1}\right) \sin\left(\frac{m_2 \pi y_2}{L_2}\right), \tag{A3}$$

where m_1 and m_2 are positive integers. The corresponding structural natural frequency is given by:

$$\omega_m = \left(\frac{D}{\rho_p h}\right)^{1/2} \left[\left(\frac{m_1 \pi}{L_1}\right)^2 + \left(\frac{m_2 \pi}{L_2}\right)^2\right], \tag{A4}$$

where the bending stiffness D is given by:

$$D = \frac{E h^3}{12(1 - \nu^2)} \tag{A5}$$

where, E is Young's modulus, ν is Poisson's ratio of the plate, and h is the thickness of the plate.

For the rectangular enclosure with the simply supported flexible plate wall, the coupling coefficient $C_{n,m}$ between the nth acoustic mode (n_1, n_2, n_3) and mth structural mode (m_1, m_2) is given by [15]:

$$C_{n,m} = \begin{cases} -1^{n_3} 2 S_f \sqrt{e_1 e_2 e_3} \dfrac{m_1 m_2 (-1^{n_1+m_1} - 1)(-1^{n_2+m_2} - 1)}{\pi^2 (n_1^2 - m_1^2)(n_2^2 - m_2^2)}, & n_1 \neq m_1, n_2 \neq m_2 \\ 0, \quad \text{otherwise} & \end{cases} \tag{A6}$$