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A NEW RANDOM BOUNDARY ELEMENT FORMULATION APPLIED TO HIGH FREQUENCY PHENOMENA

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The basis of a new boundary element method adapted to wide frequency range applications is proposed in this paper. The formulation employs the frequency domain dynamic fundamental solutions of the problems. Random geometrical parameters are introduced in the integral equations. These equations are then modified to exhibit the products of the different kinematic variables. The expectations of the new equations with respect to the latter random variables are considered, and the new variables of the formulation are the different stochastic moments force–displacement variables. This formulation is applied to different structures, such as beams, rods and assembled one-dimensional systems.

1. INTRODUCTION

It is generally impossible to compute analytical solutions of complex structural problems such as assembled beams, rods, membranes or plates. Numerical solutions like the finite element method FEM [1], and classical direct and indirect boundary element methods BEM [2], can deal with this kind of problem as long as the frequency of the studied phenomenon does not reach the "high frequency domain", for which the huge number of degrees of freedom is inappropriate considering the computing time that is required. In the early sixties, new solutions were developed in order to characterize the high frequency dynamic behaviour of mechanical structures. The most famous theory is the Statistical Energy Analysis (SEA) developed by Lyon [3]. The aim of the SEA is to evaluate the spread of the energy through complex systems divided into coupled subsystems. The relationships between the different subsystems are obtained using the concept of "coupling loss factors", while the energy dissipation of each subsystem is connected to the vibrational energy lost through "internal loss factors". From the early beginnings of SEA different extensions of the method have arisen. Langley [4] proposed an improvement of the classical SEA by getting rid of the diffuse field assumption. Indeed, this hypothesis is very

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restricting, especially for particular structures such as curved shells for which the wave group speeds depend greatly on the direction. This new energy formulation is called Wave Intensity Analysis (WIA). Another SEA improvement is the Simplified Energy Method (SEM), which is a power flow formulation developed on the original works by Belov *et al.* [5] who first derived a differential equation of the heat conduction type to characterize the spread of the energy throughout an absorbing structure. Nefske and Sung [6] applied the SEM to evaluate the power flow of a flexural beam. Wohlever and Bernhard [7] and Bouthier [8] gave further results concerning one-dimensional systems as well as membranes and Kirchhoff–Love plates. Lase *et al.* [9] also developed the General Energy Method (GEM) for rods and beams which gives an exact energy description of the structures using the Lagrangian density, the total energy density and the active and reactive energy flows. The SEM is given as a simplification of the GEM by removing the modal characteristics. Ichchou *et al.* [10] proposed some results concerning the use of the SEM applied to curved rods and beams. Lately, Moron *et al.* [11] showed appreciable accuracy improvements when implementing the SEM for assembled plates, compared to the results of the SEA and the WIA, especially for strongly damped structures.

Even if the different energy methods previously mentioned brought much progress in structural high frequency predictions in the last twenty years, these formulations involve many deficiencies. According to Fahy [12], one of the most penalizing facts is the lack of confidence in the predictions given by these methods. The comparative results of Moron *et al.* [11] for coupled plates were particularly revealing of this default.

In this paper, an alternative approach is proposed to deal with high frequency phenomena. The basic idea of this work relies on the fact that the high frequency structural behaviour is dictated by an intrinsic law of uncertainty. This fact has been developed by Keane and Manohar [13, 14] who showed that the introduction of a random variable on the mass density of the structures was creating a “statistical overlap factor” whose influence increases with the frequency.

The formulation presented in the following sections is based on a boundary element formulation. The classical integral equations are multiplied by well chosen variables in order to obtain a formulation over the product of the classical displacement variables. Gaussian random variables are then introduced on the geometrical parameters of the structures and the stochastic expectations of the new integral equations are taken into account. Some assumptions are introduced to limit the number of high order moment unknowns. The unknowns of the new formulations are the second order stochastic moments of the boundary force displacement variables. The formulation highlights a transition range from the low frequency deterministic behaviour to a smooth high frequency evolution. The efficiency of the theory is demonstrated on simple beams and rods, loaded by point harmonic forces. Several cases of junctions are also treated, such as rod/dashpop, rod/rod, beam/beam. The results are compared with those available from deterministic methods.

2. THE CLASSICAL BOUNDARY INTEGRAL EQUATIONS FOR SIMPLE ONE-DIMENSIONAL STRUCTURES

The boundary integral representations of the simple rod and beam are formulated in this section. Different ways are proposed in the literature to obtain these integral equations. One of the possible formulations is derived from the dynamic reciprocal theorem [15]. It is simply stated: if two distinct elastic equilibrium states exist in a bounded region, then the work done by the forces and moments of the first system on the displacement and slope of the second is equal to the work done by the forces and moments of the second system on the displacement and slope of the first one. The first system is the actual state of displacements, slopes, body and boundary forces and moments, and the second one corresponds to a unit force system in an infinite solid [16].

2.1. CASE OF THE ROD

At first, the frequency domain governing equation of the rod is recalled. The mechanical system is considered to be homogeneous, isotropic, linear, elastic, and the hypothesis of small displacement is taken into account. The longitudinal loading as well as the displacements is assumed to be harmonic. The equation of motion for the rod in the frequency domain, can be written [17]:

$$\partial^2 w/\partial x^2 + k^2 w = -q(x)/ES, \tag{1}$$

where $w(x)$ is the longitudinal deflection, $E = E_0(1 + \mathrm{i}\eta)$ is the complex modulus of elasticity, η is the loss damping factor, ρS is the mass density per unit length, S is the cross-sectional area, and $q(x)$ is the longitudinal loading. k represents the wave number, and may be written:

$$k^2 = \omega^2\rho/E_0(1+\mathrm{i}\eta) \approx k_0^2(1-\mathrm{i}\eta), \tag{2}$$

where ω denotes the circular frequency of vibration. The fundamental solution of the unit force system for the infinite rod is the Green Kernel G, whose expression is:

$$G(x, \xi) = (1/2\mathrm{i}k)\,\mathrm{e}^{-\mathrm{i}k|\xi - x|}. \tag{3}$$

The variable ξ is the loading location while x denotes the spatial position. The integral formulation may finally be written for ξ matching successively with the two boundary locations x_1 and x_2.

Equation for the rod:

$$\begin{aligned} w(\xi) = {} & \int_{x_1}^{x_2} G(\xi - x)\frac{q(x)}{ES}\,\mathrm{d}x + \frac{\partial w(x_2)}{\partial x}G(\xi - x_2) - \frac{\partial w(x_1)}{\partial x}G(\xi - x_1) \\ & - w(x_2)\frac{\partial G(\xi - x_2)}{\partial x} + w(x_1)\frac{\partial G(\xi - x_1)}{\partial x}. \end{aligned} \tag{4}$$

Finally, one must give boundary conditions in order to be able to solve the boundary integral equations. If a clamped/clamped rod is considered, the boundary conditions are, $w(x_1) = w(x_2) = 0$.

2.2. Case of the beam

Considering an homogeneous, isotropic, linear, elastic beam transversally loaded by a harmonic force, and assuming small displacements, one can write its equation of motion in the frequency domain:

$$\partial^4 w/\partial x^4 - k^4 w = q(x)/EI, \tag{5}$$

where I denotes the moment of inertia. The parameter k represents the wave number and may be written:

$$k^4 = \omega^2 \rho S/E_0 I(1 + \mathrm{i}\eta) \approx k_0^4(1 - \mathrm{i}\eta). \tag{6}$$

For the beam, G has the following expression:

$$G(x, \xi) = -(1/4k^3)\{\mathrm{i}\mathrm{e}^{-\mathrm{i}k|\xi - x|} + \mathrm{e}^{-k|\xi - x|}\}. \tag{7}$$

In relation to the beam, two equations are required in order to solve the entire set of boundary unknowns.

Equations for the beam:

$$w(\xi) = \int_{x_1}^{x_2} \frac{q(x)}{EI} G(\xi, x)\,\mathrm{d}x - \left[\frac{\partial^3 w(x)}{\partial x^3} G(\xi, x)\right]_{x_1}^{x_2} + \left[\frac{\partial^2 w(x)}{\partial x^2}\frac{\partial G(\xi, x)}{\partial x}\right]_{x_1}^{x_2}$$
$$- \left[\frac{\partial w(x)}{\partial x}\frac{\partial^2 G(\xi, x)}{\partial x^2}\right]_{x_1}^{x_2} + \left[w(x)\frac{\partial^3 G(\xi, x)}{\partial x^3}\right]_{x_1}^{x_2},$$

$$\frac{\partial w(\xi)}{\partial \xi} = \int_{x_1}^{x_2} \frac{q(x)}{EI}\frac{\partial G(\xi, x)}{\partial \xi}\,\mathrm{d}x - \left[\frac{\partial^3 w(x)}{\partial x^3}\frac{\partial G(\xi, x)}{\partial \xi}\right]_{x_1}^{x_2} + \left[\frac{\partial^2 w(x)}{\partial x^2}\frac{\partial^2 G(\xi, x)}{\partial x \partial \xi}\right]_{x_1}^{x_2}$$
$$- \left[\frac{\partial w(x)}{\partial x}\frac{\partial^3 G(\xi, x)}{\partial x^2 \partial \xi}\right]_{x_1}^{x_2} + \left[w(x)\frac{\partial^4 G(\xi, x)}{\partial x^3 \partial \xi}\right]_{x_1}^{x_2}. \tag{8}$$

For a clamped/clamped beam, the boundary conditions become: $w(x_1) = w(x_2) = 0$ and $\partial w(x_1)/\partial x = w(x_2)/\partial x = 0$.

3. THE RANDOM FORMULATION

3.1. A general overview of the method

Using the boundary equations developed in the previous section to predict the behaviour of structures in the high frequency field is numerically possible but physically unrealistic. The reason is that the deterministic response of any mechanical system is more and more sensitive to small perturbations of the geometrical and mechanical parameters of the structures, when the frequency increases. This phenomenon has been illustrated by Keane and Manohar [13, 14] who calculated the successive probability density functions of the eigen-frequencies of a beam, for which a random parameter is introduced in the definition of its mass density. Within this context, the aim of the authors is to introduce a relevant random description of the intrinsic structural parameters, in the boundary equations.

There are numerous possibilities with regards to the choice of structural parameters on which can be applied the random variables. Actually, the randomness of the structure studied in the high frequency field is resulting from small errors occurring on the global description of the system. In the following analytical formulations, the random parameters are introduced on the geometrical description of the structures. Each of the boundary equations is then multiplied by one kinematic variable, and the expectation of the equations is considered. A set of equations containing the different statistical moments of the variables of the formulation, is obtained. Some assumptions concerning the statistical independence of the different force–displacement variables must then be introduced to obtain a consistent number of equations. The unknowns of the formulation are the statistical moments of the force–displacement variables of first and second order. The first order moments do not give interesting information in the high frequency field, because their values vanish to zero. On the other hand, the second order moments may be connected to an energy description of the vibrational behaviour, whose expectation does not converge to zero since the energy is always positive.

3.2. THE RANDOM FORMULATION FOR ISOLATED STRUCTURES

In order to illustrate the method, the formulation is written for the simple example of a clamped/clamped rod. The equations are completely developed, and the approximations required to obtain a finite sequence of equations with a consistent number of unknowns is justified. The boundary equations for a clamped/clamped rod established in section 2 are recalled, in the case of a point loading F_0:

$$w(x_1) = \frac{F_0}{ES}G(x_1 - x_f) + \frac{\partial w(x_2)}{\partial x}G(x_1 - x_2) - \frac{\partial w(x_1)}{\partial x}G(x_1 - x_1),$$

$$w(x_2) = \frac{F_0}{ES}G(x_2 - x_f) + \frac{\partial w(x_2)}{\partial x}G(x_2 - x_2) - \frac{\partial w(x_1)}{\partial x}G(x_2 - x_1). \tag{9}$$

The geometrical parameters encountered in equations (9) are x_1, x_2 and x_f, corresponding to the position of the boundaries and the location of the loading. These parameters are considered to be randomly known and are written:

$$\tilde{x}_1 = x_1 + \varepsilon_1, \quad \tilde{x}_2 = x_2 + \varepsilon_2, \quad \tilde{x}_f = x_f + \varepsilon_f, \tag{10}$$

where ε_1, ε_2 and ε_f are assumed to be independent zero mean random variables. Even if the position of the boundaries are assumed to be random parameters, the boundary conditions are deterministic; that is to say for the case of the clamped/clamped rod: $w(\tilde{x}_1) = w(\tilde{x}_2) = 0$. Using the random notations, the integral formulation of the rod, equations (9) become:

$$0 = F_0 G(\tilde{x}_1 - \tilde{x}_f) + \frac{\partial w(\tilde{x}_2)}{\partial x} G(\tilde{x}_1 - \tilde{x}_2) - \frac{\partial w(\tilde{x}_1)}{\partial x} G(\tilde{x}_1 - \tilde{x}_1),$$

$$0 = F_0 G(\tilde{x}_2 - \tilde{x}_f) + \frac{\partial w(\tilde{x}_2)}{\partial x} G(\tilde{x}_2 - \tilde{x}_2) - \frac{\partial w(\tilde{x}_1)}{\partial x} G(\tilde{x}_2 - \tilde{x}_1). \tag{11}$$

The random boundary formulation is obtained by multiplying each side of the first equation (respectively the second equation) of (11), by the conjugate of the unknown boundary kinematic variable, $\partial w^*(\tilde{x}_1)/\partial x$ (respectively $\partial w^*(\tilde{x}_2)/\partial x$). The expectations with respect to $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_f$ (represented by the symbol $\langle - \rangle$) of the two sides of the equations are then taken into account. One obtains:

$$0 = \left\langle F_0 G(\tilde{x}_1 - \tilde{x}_f) \frac{\partial w^*(\tilde{x}_1)}{\partial x} \right\rangle + \left\langle \frac{\partial w^*(\tilde{x}_1)}{\partial x} \frac{\partial w(\tilde{x}_2)}{\partial x} G(\tilde{x}_1 - \tilde{x}_2) \right\rangle$$
$$- \left\langle \left| \frac{\partial w(\tilde{x}_1)}{\partial x} \right|^2 G(\tilde{x}_1 - \tilde{x}_1) \right\rangle,$$
$$0 = \left\langle F_0 G(\tilde{x}_2 - \tilde{x}_f) \frac{\partial w^*(\tilde{x}_2)}{\partial x} \right\rangle + \left\langle \left| \frac{\partial w(\tilde{x}_2)}{\partial x} \right|^2 G(\tilde{x}_2 - \tilde{x}_2) \right\rangle$$
$$- \left\langle \frac{\partial w(\tilde{x}_1)}{\partial x} \frac{\partial w^*(\tilde{x}_2)}{\partial x} G(\tilde{x}_2 - \tilde{x}_1) \right\rangle. \tag{12}$$

Equations (12) represent the fundamental relations of the random boundary integral formulation. In order to solve the problem, one has to propose some specific rules to separate the different terms appearing between the bracket symbols. This is the aim of the next section.

3.2.1. *Limitation procedure of the number of high order moments*

Before displaying different physical rules in order to limitate the number of the statistical moments, some numerical simplifications can be carried out. Indeed, the expression of the Green kernel $G(\tilde{x}_i, \tilde{x}_i)$, ($i$, j = 1, 2) may be written:

$$G(\tilde{x}_i - \tilde{x}_i) = 1/2\mathrm{i}k. \tag{13}$$

From the relationship (13), one can deduce that $G(\tilde{x}_i - \tilde{x}_i)$ is a deterministic parameter. In other respects, the value of F_0 does not depend on the geometrical parameters. Therefore equations (12) can be expressed:

$$
\begin{aligned}
0 &= F_0\left\langle G(\tilde{x}_1-\tilde{x}_f)\frac{\partial w^*(\tilde{x}_1)}{\partial x}\right\rangle + \left\langle \frac{\partial w^*(\tilde{x}_1)}{\partial x}\frac{\partial w(\tilde{x}_2)}{\partial x}G(\tilde{x}_1-\tilde{x}_2)\right\rangle \\
&\quad - \left\langle \left|\frac{\partial w(\tilde{x}_1)}{\partial x}\right|^2\right\rangle G(\tilde{x}_1-\tilde{x}_1), \\
0 &= F_0\left\langle G(\tilde{x}_2-\tilde{x}_f)\frac{\partial w^*(\tilde{x}_2)}{\partial x}\right\rangle + \left\langle \left|\frac{\partial w(\tilde{x}_2)}{\partial x}\right|^2\right\rangle G(\tilde{x}_2-\tilde{x}_2) \\
&\quad - \left\langle \frac{\partial w(\tilde{x}_1)}{\partial x}\frac{\partial w^*(\tilde{x}_2)}{\partial x}G(\tilde{x}_2-\tilde{x}_1)\right\rangle.
\end{aligned} \tag{14}
$$

Obtaining a finite sequence of equations with a consistent number of unknowns is necessary to solve the set of equations (14). In order to limitate the number of unknowns, some random assumptions are carried out. These assumptions rely on a physical interpretation of the different terms appearing in the different integral representations. Indeed, when considering the integral equations (4, 8), the left sides of these equations are interpreted as the sum of the contributions of different sources located at the boundaries of amplitude, the boundary unknowns (denoted in what follows $A(x_i)$, $i = 1, 2$), and at the loading position of amplitude F_0. The expressions of these contributions are the different terms of the right sides of the integral representations.

In this context, some random hypotheses are introduced governing the stochastic behaviour of the different sources.

Assumption 1: the contribution of a source $A(x_i)G^{(k)}(x_i, x_j)$ with $x_i \neq x_f$ and $A \neq F_0$, ((k) denotes the derivative order appearing in equations (4, 8)) is only correlated with itself and is called the secondary source. On the other hand, if $A = F_0$ or $x_i = x_f$, the source is called a primary source and is assumed to be strongly correlated with the force–displacement unknown $A(x_j)$.

Two main physical reasons lead to the assumption expressed above. First of all, the spatial positions of the different sources as well as the location of the points where the contributions of the sources are evaluated, are statistically independent. In this context, two different contributions (in terms of source location or target position) may be supposed statistically independent. The second reason is that a specific boundary unknown is supposed to be only correlated with the contribution of the loading. Any force–displacement variable of the considered structure naturally depends on the contribution of the loading. On the other hand, the amplitudes of the secondary sources contain the information of the multiple wave reflections on the different boundaries, whose spatial positions are not correlated with the location of the considered boundary unknown. Therefore, the contributions of the secondary sources are not correlated with this boundary unknown.

From this hypothesis one can deduce that $F_0\langle G(\tilde{x}_1-\tilde{x}_f)(\partial w^*(\tilde{x}_1)/\partial x)\rangle$ may not be written as the produce of the two expectations of $G(\tilde{x}_1-\tilde{x}_f)$ and

$(\partial w^*(\tilde{x}_1)/\partial x)$. In other respects, one may state the following relationships:

$$\left\langle \frac{\partial w^*(\tilde{x}_1)}{\partial x}\frac{\partial w(\tilde{x}_2)}{\partial x}G(\tilde{x}_1-\tilde{x}_2)\right\rangle \approx \left\langle \frac{\partial w^*(\tilde{x}_1)}{\partial x}\right\rangle \left\langle \frac{\partial w(\tilde{x}_2)}{\partial x}G(\tilde{x}_1-\tilde{x}_2)\right\rangle,$$
$$\left\langle \frac{\partial w(\tilde{x}_1)}{\partial x}\frac{\partial w^*(\tilde{x}_2)}{\partial x}G(\tilde{x}_2-\tilde{x}_1)\right\rangle \approx \left\langle \frac{\partial w^*(\tilde{x}_2)}{\partial x}\right\rangle \left\langle \frac{\partial w(\tilde{x}_1)}{\partial x}G(\tilde{x}_2-\tilde{x}_1)\right\rangle. \quad (15)$$

Assumption 2: the amplitude of a secondary source located at x and its propagative part $G(x, \xi)$ are not correlated if $x \neq \xi$. Consequently, one can write:

$$\langle(\partial w(\tilde{x}_2)/\partial x)G(\tilde{x}_1-\tilde{x}_2)\rangle \approx \langle[\partial w(\tilde{x}_2)/\partial x]\rangle\langle G(\tilde{x}_1-\tilde{x}_2)\rangle,$$
$$\langle(\partial w(\tilde{x}_1)/\partial x)g(\tilde{x}_2-\tilde{x}_1)\rangle \approx \langle\partial w(\tilde{x}_1)/\partial x\rangle\langle G(\tilde{x}_2-\tilde{x}_1)\rangle. \quad (16)$$

The physical reasoning behind this new assumption is on the same basis as for the previous one. One can then rewrite equations (14) using the previous hypotheses:

$$0 = F_0\langle G(\tilde{x}_1-\tilde{x}_f)\partial w^*(\tilde{x}_1)/\partial x\rangle$$
$$+\langle\partial w^*(\tilde{x}_1)/\partial x\rangle\langle\partial w(\tilde{x}_2)/\partial x\rangle\langle G(\tilde{x}_1-\tilde{x}_2)\rangle - \langle|\partial w(\tilde{x}_1)/\partial x|^2\rangle G(\tilde{x}_1-\tilde{x}_1),$$
$$0 = F_0\langle G(\tilde{x}_2-\tilde{x}_f)\partial w^*(\tilde{x}_2)/\partial x\rangle$$
$$+\langle|\partial w(\tilde{x}_2)/\partial x|^2\rangle G(\tilde{x}_2-\tilde{x}_2) - \langle\partial w(\tilde{x}_1)/\partial x\rangle\langle\partial w^*(\tilde{x}_2)/\partial x\rangle\langle G(\tilde{x}_2-\tilde{x}_1)\rangle. \quad (17)$$

The expression of $G(\tilde{x}_i-\tilde{x}_j)$ for $i \neq j$ and $x_i \geqslant x_j$ (it is assumed that $\tilde{x}_i \geqslant \tilde{x}_j$) may be written:

$$G(\tilde{x}_i-\tilde{x}_j) = \mathrm{e}^{-\mathrm{i}k_0(x_i-x_j)}\mathrm{e}^{-\mathrm{i}k_0(\varepsilon_i-\varepsilon_j)}\mathrm{e}^{-(\eta/2)k_0(x_i-x_j)}\mathrm{e}^{-(\eta/2)k_0(\varepsilon_i-\varepsilon_j)}, \quad (18)$$

$\eta/2(\varepsilon_i-\varepsilon_j)$ is assumed to be a second order term compared to $\eta/2(x_i-x_j)$ and is neglected. To express the other terms, a Gaussian law has been chosen. The random variables ε_i introduced in equations (10) are defined by their means equal to zero, their standard deviation σ, and their density function f_{ε_i}, $i = 1, 2$ whose expression is:

$$f_{\varepsilon_i}(x) = (1/\sigma\sqrt{2\pi})\exp[-x^2/2\sigma^2] \quad (19)$$

One can also evaluate the joint density function of the random variables, whose expression is in the case of n independent random variables:

$$f_{\varepsilon_1,\dots,\varepsilon_n}(y_1,\dots,y_n) = \prod_{i=1}^{n} f_{\varepsilon_i}(y_i). \quad (20)$$

$G(\tilde{x}_i-\tilde{x}_j)$ is a function of $\tilde{x}_i$ and $\tilde{x}_j$. Therefore, the evaluation of the expectation of G is only carried out with respect to $\tilde{x}_i$ and $\tilde{x}_j$. The explicit expression of

$\langle G(\tilde{x}_i - \tilde{x}_j)\rangle$ for $i \neq j$ is:

$$\langle G(\tilde{x}_i - \tilde{x}_j)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\tilde{x}_i - \tilde{x}_j)\, f_{\varepsilon_i,\varepsilon_j}(\varepsilon_i, \varepsilon_j)\, \mathrm{d}\varepsilon_i\, \mathrm{d}\varepsilon_j$$
$$= (1/2\mathrm{i}k)\mathrm{e}^{-\mathrm{i}k|x_i - x_j|}\mathrm{e}^{-k_0^2\sigma^2}. \quad (21)$$

Equations (17) contain six unknowns:

$$\langle \partial w(\tilde{x}_1)/\partial x\rangle \quad \text{and} \quad \langle \partial w(\tilde{x}_2)/\partial x\rangle; \quad \langle|\partial w(\tilde{x}_1)/\partial x|^2\rangle \quad \text{and} \quad \langle|\partial w(\tilde{x}_2)/\partial x|^2\rangle,$$

$$\langle G(\tilde{x}_1 - \tilde{x}_f)(\partial w^*(\tilde{x}_1)/\partial x)\rangle \quad \text{and} \quad \langle G(\tilde{x}_2 - \tilde{x}_f)(\partial w^*(\tilde{x}_2)/\partial x)\rangle$$

Four more equations must be added to the formulation in order to estimate the six unknowns. Two equations are obtained by considering the expectation of the classical boundary integral equations (11). The same random assumptions are applied to these two equations. The final expression of the different equations have the following form:

$$0 = F_0\langle G(\tilde{x}_1 - \tilde{x}_f)\rangle + \left\langle \frac{\partial w(\tilde{x}_2)}{\partial x}\right\rangle \langle G(\tilde{x}_1 - \tilde{x}_2)\rangle - \left\langle \frac{\partial w(\tilde{x}_1)}{\partial x}\right\rangle G(x_1 - x_1),$$
$$0 = F_0\langle G(\tilde{x}_2 - \tilde{x}_f)\rangle + \left\langle \frac{\partial w(\tilde{x}_2)}{\partial x}\right\rangle G(x_2 - x_2) - \left\langle \frac{\partial w(\tilde{x}_1)}{\partial x}\right\rangle \langle G(\tilde{x}_2 - \tilde{x}_1)\rangle. \quad (22)$$

The last two relationships are obtained by multiplying each side of the first equation (respectively the second equation) of (11) by $G^*(\tilde{x}_1 - \tilde{x}_f)$ (respectively by $G^*(\tilde{x}_2 - \tilde{x}_f)$).

$$0 = F_0\langle|G(\tilde{x}_1 - \tilde{x}_f)|^2\rangle + \left\langle \frac{\partial w(\tilde{x}_2)}{\partial x} G(\tilde{x}_1 - \tilde{x}_2) G^*(\tilde{x}_1 - \tilde{x}_f)\right\rangle$$
$$- \left\langle \frac{\partial w(\tilde{x}_1)}{\partial x} G^*(\tilde{x}_1 - \tilde{x}_f)\right\rangle G(\tilde{x}_1 - \tilde{x}_1),$$
$$0 = F_0\langle|G(\tilde{x}_2 - \tilde{x}_f)|^2\rangle + \left\langle \frac{\partial w(\tilde{x}_2)}{\partial x} G^*(\tilde{x}_2 - \tilde{x}_2)\right\rangle G(\tilde{x}_2 - \tilde{x}_2)$$
$$- \left\langle \frac{\partial w(\tilde{x}_1)}{\partial x} G(\tilde{x}_2 - \tilde{x}_1) G^*(\tilde{x}_2 - \tilde{x}_f)\right\rangle. \quad (23)$$

The first and second assumptions are then utilized to simplify the two terms

$$\langle(\partial w(\tilde{x}_2)/\partial x) G(\tilde{x}_1 - \tilde{x}_2) G^*(\tilde{x}_1 - \tilde{x}_f)\rangle$$

and

$$\langle(\partial w(\tilde{x}_1)/\partial x)G(\tilde{x}_2-\tilde{x}_1)G^*(\tilde{x}_2-\tilde{x}_f)\rangle.$$

One finally obtains:

$$0=F_0\langle|G(\tilde{x}_1-\tilde{x}_f)|^2\rangle+\left\langle\frac{\partial w(\tilde{x}_2)}{\partial x}\right\rangle\langle G(\tilde{x}_1-\tilde{x}_2)\rangle\langle G^*(\tilde{x}_1-\tilde{x}_f)\rangle$$
$$-\left\langle\frac{\partial w(\tilde{x}_1)}{\partial x}G^*(\tilde{x}_1-\tilde{x}_f)\right\rangle G(\tilde{x}_1-\tilde{x}_1),$$
$$0=F_0\langle|G(\tilde{x}_2-\tilde{x}_f)|^2\rangle+\left\langle\frac{\partial w(\tilde{x}_2)}{\partial x}G^*(\tilde{x}_2-\tilde{x}_f)\right\rangle G(\tilde{x}_2-\tilde{x}_2)$$
$$-\left\langle\frac{\partial w(\tilde{x}_1)}{\partial x}\right\rangle\langle G(\tilde{x}_2-\tilde{x}_1)\rangle\langle G^*(\tilde{x}_2-\tilde{x}_f)\rangle. \quad (24)$$

In order to completely solve the problem, one must finally consider six equations given by the three sets of relationships (17, 22, 24). If a high frequency response is required, the evaluation of the first order moments of the boundary unknowns is not compulsory and there values may be set to zero. Therefore, the two equations (22) may be suppressed within this context.

After the evaluation of the boundary unknowns, it is possible to write an equation giving the expectations of the square unknowns in the whole domain. The spatial position ξ is considered random and the latter assumptions are used. The random assumptions are utilized in the following equation. One obtains:

$$\langle|w(\tilde{\xi})|^2\rangle=F_0^2\langle|G(\tilde{\xi}-\tilde{x}_f)|^2\rangle+\left\langle\left|\frac{\partial w(\tilde{x}_1)}{\partial x}\right|^2\right\rangle\langle|G(\tilde{\xi}-\tilde{x}_1)|^2\rangle$$
$$+\left\langle\left|\frac{\partial w(\tilde{x}_2)}{\partial x}\right|^2\right\rangle\langle|G(\tilde{\xi}-\tilde{x}_2)|^2\rangle$$
$$+2F_0\ \mathrm{Re}\{\langle G(\tilde{\xi}-\tilde{x}_f)\rangle\langle G^*(\tilde{\xi}-\tilde{x}_2)\rangle\langle\partial w^*(\tilde{x}_2)/\partial x\rangle\}$$
$$-2F_0\ \mathrm{Re}\{\langle G^*(\tilde{\xi}-\tilde{x}_1)\rangle\langle G(\tilde{\xi}-\tilde{x}_f)\rangle\langle\partial w^*(\tilde{x}_1)/\partial x\rangle\}$$
$$-2F_0\ \mathrm{Re}\{\langle G^*(\tilde{\xi}-\tilde{x}_1)\rangle\langle G(\tilde{\xi}-\tilde{x}_2)\rangle\langle\partial w(\tilde{x}_2)/\partial x\rangle\langle\partial w^*(\tilde{x}_1)/\partial x\rangle\}. \quad (25)$$

3.3. THE RANDOM FORMULATION FOR ASSEMBLED ONE-DIMENSIONAL STRUCTURES

The different random laws developed for the case of isolated systems are still valid for assembled structures. However, a particular procedure must be carried out to characterize the nature of the sources located at the boundaries. In other terms, some boundary unknowns are identified as primary sources, according to the following statement:

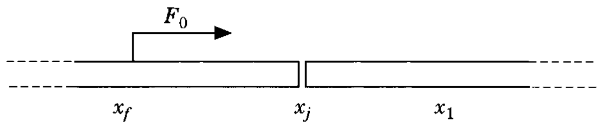

Figure 1. Representation of an infinite rod defined as two assembled semi infinite rods.

Assumption 3: the boundaries connecting two sub-structures, of which one contains a primary source, become a primary source for the other sub-structure. This assumption enables the correlation between an external loading located on one sub-system and a force–displacement unknown of another sub-structure to be expressed.

The relevance of the last assumption is described by a simple example of an infinite rod defined as two assembled semi-infinite rods. The standard notations are used for the mechanical characteristics. The geometrical parameters are defined by Figure 1.

The displacements at the junction location x_j and at the spatial position x_1 are:

$$u(x_j) = F_0 G(x_f, x_j), \quad u(x_1) = F_0 G(x_f, x_1) = u(x_j) G(x_j, x_1). \tag{26}$$

Using the relationships (26), one can deduce that the term $u(x_1)$ is highly correlated with $u(x_j)G(x_j, x_1)$.

In the same way, a structure made of three assembled rods and loaded by a point external loading placed on the first rod is considered (see Figure 2).

According to assumption 3, the boundary x_1 is a primary source for the second sub-structure. On the other hand, this boundary is not a primary source for the first rod. x_2 is a primary source for the third sub-structure, since the second rod contains a primary source located at x_1. The characterisation of the nature of the different sources may be considered as an iterative process. The initiator of this process is always a sub-structure on which an external loading is applied.

When the different sources are well defined, the two assumptions proposed for the isolated structures are applied to each sub-structure.

4. NUMERICAL APPLICATIONS TO DIFFERENT ONE-DIMENSIONAL SYSTEMS

The theoretical results presented in the previous sections have been computed for different one-dimensional structures. The frequency evolutions of the second order moments of the boundary unknowns are given. For the first examples, the calculation of the first order moments of the unknowns evaluated at the

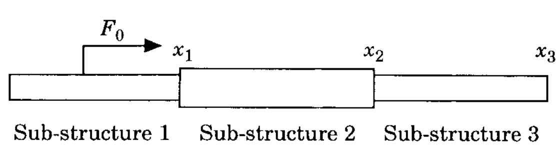

Figure 2. Representation of three assembled rods.

Table 1
Mechanical parameters used for all the structures

ρ (kg/m)	E (N/m^2)	F_0 (N)
7800	$2{\cdot}1 \times 10^{11}$	1

Table 2
Parameters of the clamped/clamped rod

S (m^2)	l (m)	$\lvert x_f - x_1 \rvert$ (m)	η (%)
10^{-4}	6	1·48	4

boundary as well as in the domain, are carried out. Table 1 gives the mechanical characteristics shared by the different structures studied.

4.1. A CLAMPED/CLAMPED ROD

A clamped/clamped rod (represented in Figure 3) loaded by a point loading is considered. Its specific geometrical and mechanical characteristics are given by Table 2.

One can observe the frequency evolution of two first order moment variables in Figures 4 and 5.

The evaluation of these variables provides some information on the behaviour of the structure, only in the low frequency field range for which the random parameters do not greatly disturb the unknowns. When the frequency increases, the expectations with respect to the geometrical random parameters of the variables vanish to zero, as expected. Therefore, a first order moment stochastic formulation may only be used to model a structural complexity (described by the random parameter) in the low frequency range. In order to evaluate the high frequency behaviour of the structures, it is necessary to deal with the second order moments of the different variables. The interest of the second order moment description is shown by Figures 6 and 7.

The expectations of the square displacement variables are in good agreement with the modal description in the low frequency range (the size of the range depends on the value of the standard deviation), and a smooth asymptotic behaviour in the high frequency field is obtained.

4.2. A SPECIFIC CASE: DEFINITION OF THE DOMAIN OF VALIDITY OF THE THEORY

The same clamped/clamped rod is considered, submitted to a point loading located near to the boundary of co-ordinate x_2. In this situation, the

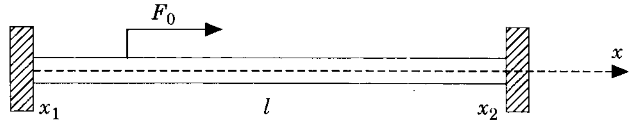

Figure 3. Representation of a clamped/clamped rod excited by a point harmonic loading F_0.

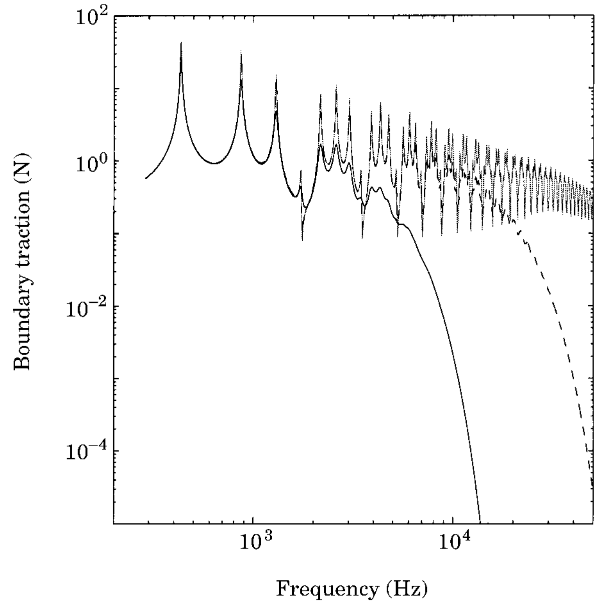

Figure 4. Frequency evolution of the traction at x_2 for the clamped/clamped rod; ....., modulus of the deterministic result; - - - -, modulus of the prediction with $\sigma = 0{\cdot}05$; and ——, $\sigma = 0{\cdot}2$.

relationships (15) are not valid any more. Indeed, one may observe the behaviour of some terms such as $\langle(\partial w^*(\tilde{x}_1)/\partial x)(\partial w(\tilde{x}_2)/\partial x)G(\tilde{x}_1 - \tilde{x}_2)\rangle$. In this example, one can write $x_f \approx x_2$. Consequently, one can assume that

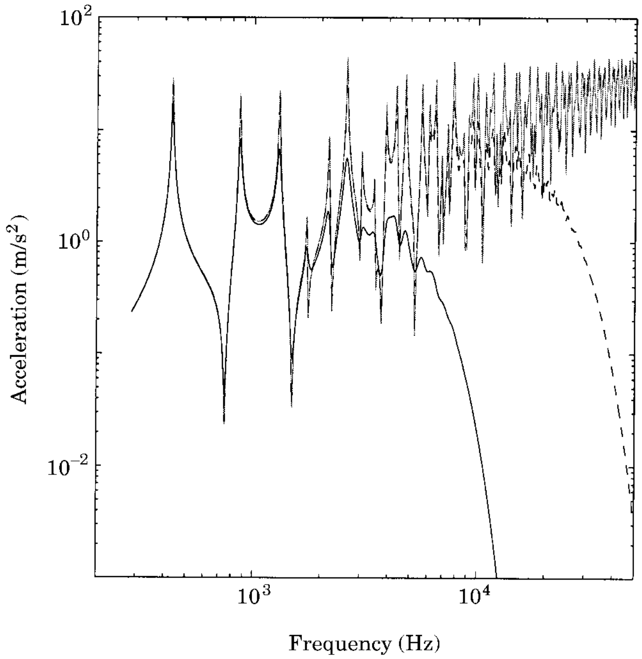

Figure 5. Frequency evolution of the acceleration of the clamped/clamped rod at the point $x = 0{\cdot}6$ m. Key as for figure 4.

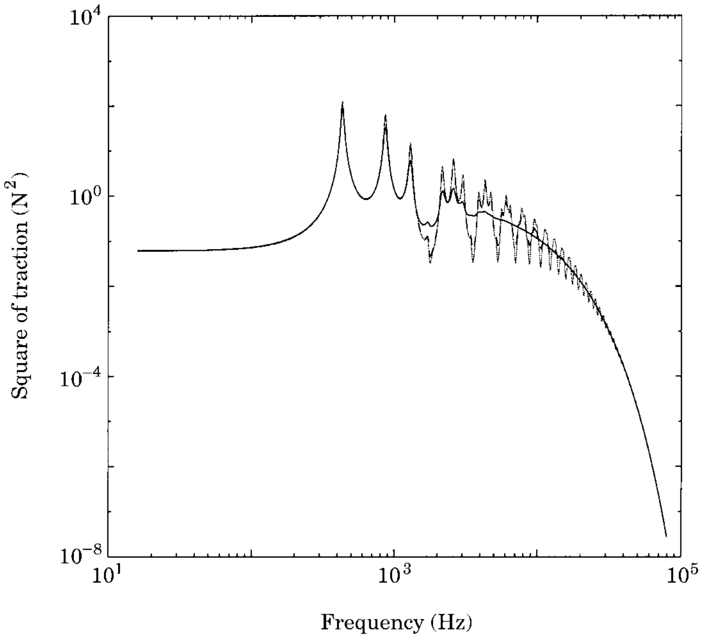

Figure 6. Frequency evolution of the modulus of the square traction at x_2 for the clamped/clamped rod. Key as for Figure 4.

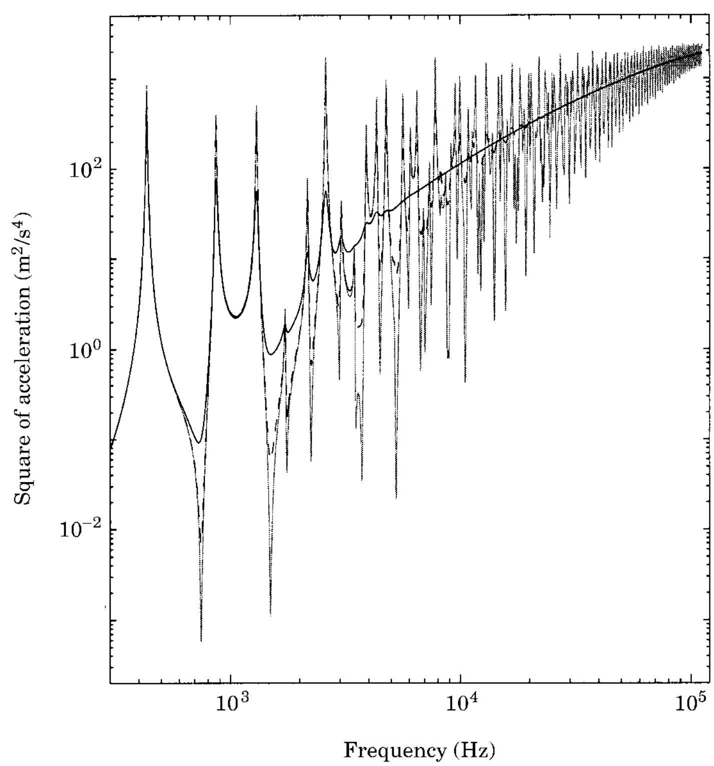

Figure 7. Frequency evolution of the modulus of the square acceleration at $x = 0{\cdot}6$ m for the clamped/clamped rod. Key as for Figure 4.

TABLE 3
Parameters of the second clamped/clamped rod

S (m^2)	l (m)	$\lvert x_f - x_1 \rvert$ (m)	η (%)
10^{-3}	6	5·95	4

$(\partial w(\tilde{x}_2)/\partial x)G(\tilde{x}_1 - \tilde{x}_2)$ becomes the contribution at the spatial location x_1 of a primary source located at x_2. Using the rule defining the dependency of a boundary unknown and the contribution of the primary sources (referring to assumption 1), one finally states that: $(\partial w(\tilde{x}_2)/\partial x)G(\tilde{x}_1 - \tilde{x}_2)$ and $(\partial w^*(\tilde{x}_1)/\partial x)$ are correlated.

The example of a clamped/clamped rod whose geometrical and mechanical characteristics are given in Table 3 is chosen in order to illustrate the limitations of the assumptions proposed before.

Figure 8 clearly shows that the high frequency asymptotic trend of the deterministic result is not obtained any more, when using the previous assumptions.

This example is of great interest since it defines the limit of the validity of the random hypothesis. Indeed, the assumptions proposed in section 3.2.1 are valid as far as the location of the secondary and primary sources are distinct.

4.3. THE FLEXURAL BEAM

The example of a clamped/clamped beam (represented in Figure 9), is treated. The system is subjected to a point lateral loading.

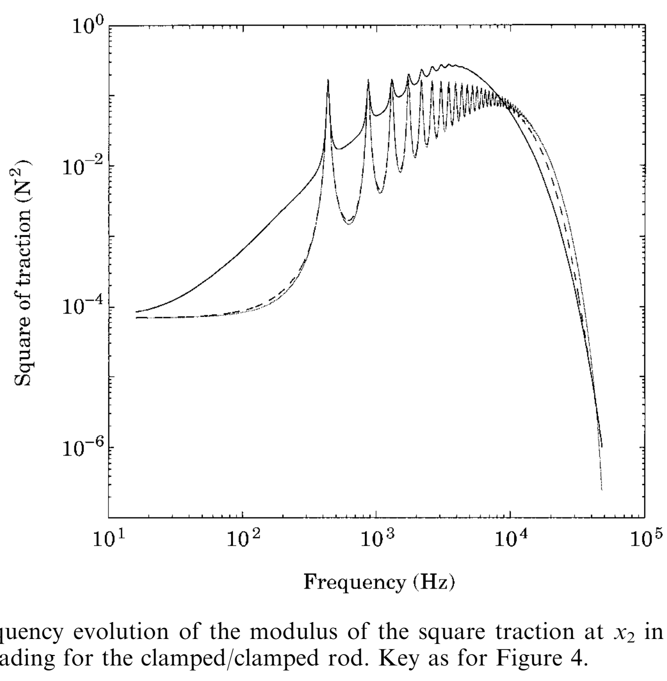

Figure 8. Frequency evolution of the modulus of the square traction at x_2 in the case of close boundary and loading for the clamped/clamped rod. Key as for Figure 4.

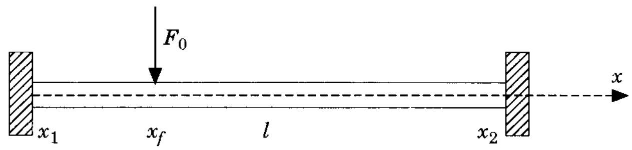

Figure 9. Representation of a clamped/clamped beam excited by a point harmonic loading F_0.

The boundary equations formulation on the expectations with respect to the boundary and input loading locations of the square boundary variables is obtained using equations (8) and following a similar procedure as for the rod. The analytical calculations lead to the determination of sixteen unknowns, which are:

the first order moments:

$$\langle \partial^3 w(\tilde{x}_1)\partial x^3\rangle, \quad \langle \partial^2 w(\tilde{x}_1)/\partial x^2\rangle, \quad \langle \partial^3 w(\tilde{x}_2)/\partial x^3\rangle \quad \langle \partial^2 w(\tilde{x}_2)/\partial x^2\rangle;$$

the second order moments:

$$\langle |\partial^3 w(\tilde{x}_1)/\partial x^3|^2\rangle, \quad \langle |\partial^3 w(\tilde{x}_2)/\partial x^3|^2\rangle, \quad \langle |\partial^2 w(\tilde{x}_1)/\partial x^2|^2\rangle, \quad \langle |\partial^2 w(\tilde{x}_2)/\partial x^2|^2\rangle,$$

$$\langle (\partial^3 w(\tilde{x}_1)/\partial x^3)(\partial^2 w^*(\tilde{x}_1)/\partial x^2)\rangle, \quad \langle (\partial^2 w(\tilde{x}_1)/\partial x^2)(\partial^3 w^*(\tilde{x}_1)/\partial x^3)\rangle,$$

$$\langle (\partial^3 w(\tilde{x}_2)/\partial x^3)(\partial^2 w^*(\tilde{x}_2)/\partial x^2)\rangle, \quad \langle (\partial^2 w(\tilde{x}_2)/\partial x^2)(\partial^3 w^*(\tilde{x}_2)/\partial x^3)\rangle;$$

the other second order moments:

$$\langle G^*(\tilde{x}_1, \tilde{x}_f)(\partial^3 w(\tilde{x}_1)/\partial x^3)\rangle, \quad \langle G^*(\tilde{x}_1, \tilde{x}_f)(\partial^2 w(\tilde{x}_1)/\partial x^2)\rangle,$$

$$\langle G^*(\tilde{x}_2, \tilde{x}_f)(\partial^3 w(\tilde{x}_2)/\partial x^3)\rangle, \quad \langle G^*(\tilde{x}_2, \tilde{x}_f)(\partial^2 w(\tilde{x}_2)/\partial x^2)\rangle,$$

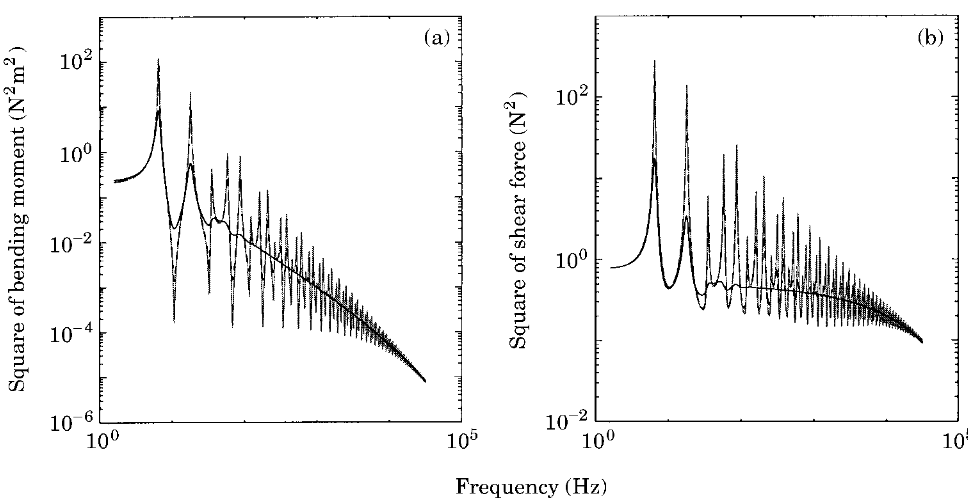

Figure 10. Frequency evolution of (a) modulus of the square bending moment at x_1; (b) modulus of the square shear force at x_1 for the clamped/clamped beam: key as for Figure 4.

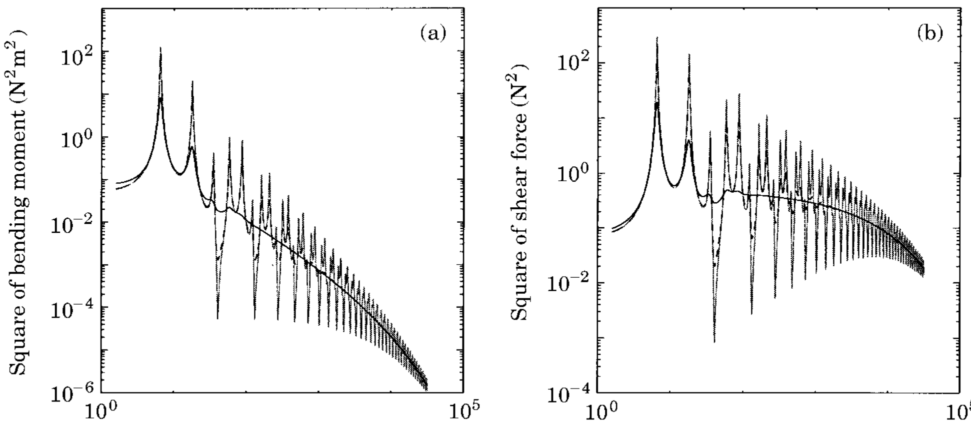

Figure 11. Frequency evolution of (a) the modulus of the square of bending moment at x_2; (b) modulus of the square of shear force at x_2 for the clamped/clamped beam. Key as for Figure 4.

$$\langle(\partial G^*(\tilde{x}_1, \tilde{x}_f)/\partial\xi)(\partial^3 w(\tilde{x}_1)/\partial x^3)\rangle, \quad \langle(\partial G^*(\tilde{x}_1, \tilde{x}_f)/\partial\xi)(\partial^2 w(\tilde{x}_1)/\partial x^2)\rangle,$$

$$\langle(\partial G^*(\tilde{x}_2, \tilde{x}_f)/\partial\xi)(\partial^3 w(\tilde{x}_2)/\partial x^3)\rangle, \quad \langle(\partial G^*(\tilde{x}_2, \tilde{x}_f)/\partial\xi)(\partial^2 w(\tilde{x}_2)/\partial x^2)\rangle.$$

The square values at the spatial position $\tilde{x}_1$ of the bending moment and the shear force are given in Figures 10 and 11 for different values of the standard deviation. The mechanical and geometrical characteristics of the beam are summarised in Table 4.

The same observations as for the case of the rod can be made. The low frequency response is reached accurately while a smooth behaviour corresponding to the general trend of the deterministic result is given in the high frequency domain.

4.4. COUPLING BETWEEN A ROD AND A MASS/SPRING/DASHPOT SYSTEM

After the study of isolated structures, one must deal with assembled systems. A simple case of this is a one-dimensional element coupled to an n-degrees-of-freedom system. This type of coupling is of great industrial interest. Indeed, complex assembled structures are often made of classical mechanical systems such as stiffeners, plates and shells, to which small equipment is attached. It has been shown that this small equipment may induce important perturbations of the response of the whole system and may also be used to reduce vibration levels

TABLE 4

Parameters of the clamped/clamped beam

S (m^2)	l (m)	$\lvert x_f - x_1\rvert$ (m)	I (m^4)	η (%)
10^{-4}	3	1	10^{-9}	3

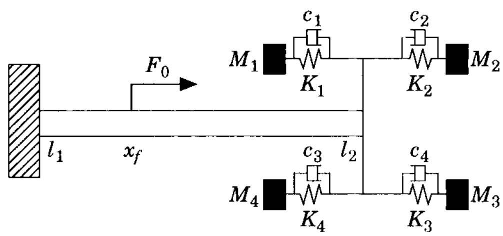

Figure 12. Representation of the clamped/free rod submitted to a point loading and a four-degrees-of-freedom system.

[18]. Generally, this small equipment is defined using an *n*-degrees-of-freedom system such as combinations of springs, point masses and dashpots.

Thus, a clamped/free rod is considered, on which is connected four-one-degree-of-freedom systems composed each of a spring, a dashpot and a mass. The whole structure is represented by Figure 12.

It is usual practice to consider an equivalent impedance describing the effect of the four-degrees-of-freedom system. This impedance is denoted Z and can be expressed as:

$$Z = \sum_{n=1}^{4} \frac{\omega^2 M_n K_n (1 - \mathrm{i} c_n)}{\omega^2 M_n - K_n (1 - \mathrm{i} c_n)}. \tag{27}$$

The integral formulation for this specific junction case has the following

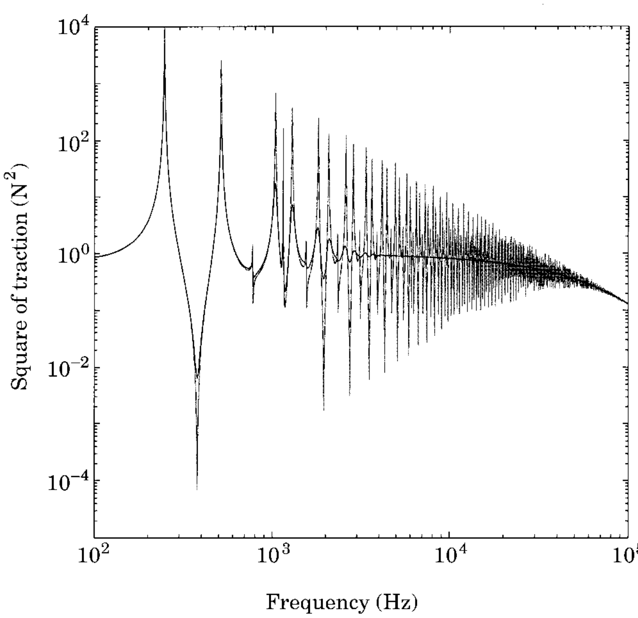

Figure 13. Frequency evolution of the modulus of the square of the traction at x_1 for the clamped/free rod submitted to a point four-degrees-of-freedom system at x_2. Key as for Figure 4.

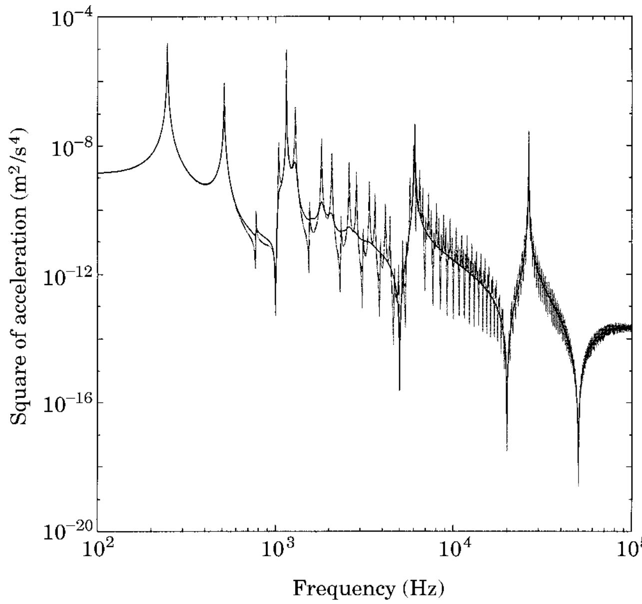

Figure 14. Frequency evolution of the modulus of the square acceleration at x_2 for the clamped/free rod submitted to a point four-degrees-of-freedom system at x_2. Key as for Figure 4.

expression:

$$w(\tilde{\xi}) = -w(\tilde{x}_2)\left\{\frac{Z}{ES}G(\tilde{\xi}, \tilde{x}_2) + \frac{\partial G(\tilde{\xi}, \tilde{x}_2)}{\partial x}\right\} - \frac{\partial w(\tilde{x}_1)}{\partial x}G(\tilde{\xi}, \tilde{x}_1) + \frac{F_0}{ES}G(\tilde{\xi}, \tilde{x}_f). \tag{28}$$

From this equation, one can easily deduce the stochastic formulation using the random assumptions. Figures 13 and 14 represent the expectation of the modulus of the square traction at the clamped boundary and the expectation of the modulus of the square acceleration at the location of the four-degrees-of-freedom system. The characteristics of the rod and the point system are given in Tables 5 and 6.

The observation of Figure 13 shows the same trends as for the previously studied structures. The low frequency response is accurately described whilst the high frequency evolution of the stochastic response gives a smooth trend of the deterministic result. In other respects, the observation of Figure 14 highlights a new phenomena which is of great interest. At first, a global observation shows that the random response has the same behaviour considering the boundary

TABLE 5

Parameters of the four-degrees-of-freedom spring/dashpot/mass system

K_1 (N/m)	K_2 (N/m)	K_3 (N/m)	K_4 (N/m)	M_1, M_2, M_3, M_4 (kg)	c_1, c_2, c_3, c_4 (%)
$3{\cdot}95 \times 10^8$	$9{\cdot}87 \times 10^7$	$1{\cdot}58 \times 10^{10}$	$9{\cdot}87 \times 10^{10}$	3·9	1

TABLE 6

Parameters of the clamped free rod coupled with a four-degrees-of-freedom-system

S (m^2)	l	$\|x_f - x_1\|$	η (%)
10^{-4}	10	3·37	0·5

point x_1 (Figure 13) or the boundary point x_2 (Figure 14). On the other hand, the stochastic formulation detects accurately four eigenfrequencies each corresponding to the eigenfrequency of a coupled one-degree-of-freedom system. Indeed, the different spring/mass/dashpot systems are described as deterministic systems. Therefore, their behaviour is obtained without any uncertainty.

4.5. COUPLING BETWEEN ONE-DIMENSIONAL STRUCTURES

4.5.1. *Coupling between rods*

Two types of coupling concerning rods are proposed in order to validate the random formulation for coupled one-dimensional structures. The first example is the coupling of two rods with different geometrical and mechanical characteristics. The system is described in Figure 15.

The deterministic integral representation for two coupled rods may be written:

$$w(\xi_1) = F_0 G_1(x_f, \xi_1) + (\partial w(x_2)/\partial x) G_1(x_2, \xi_1) - (\partial w(x_1)/\partial x) G_1(x_1, \xi_1) + w(x_1)\partial G_1(x_1, \xi_1)/\partial x,$$

$$w(\xi_2) = \frac{\partial w(x_2)}{\partial x} G_2(x_2, \xi_2) - \frac{\partial w(x_3)}{\partial x} G_2(x_3, \xi_2) + w(x_3)\frac{\partial G_2(x_3, \xi_2)}{\partial x}. \quad (29)$$

G_1 (respectively G_2) represents the Green kernel of the rod bounded by x_1 and x_2 (respectively bounded by x_1 and x_3). ξ_1 (respectively ξ_2) represents the co-ordinate of an indefinite point of the rod between x_1 and x_2 (respectively defined by x_1 and x_3). One can then easily obtain the equations corresponding to the random formulation by multiplying equations (29) by the displacement–force variables, in the same manner as for the simple clamped/clamped rod. Using the random procedure defined previously, one obtains the following unknowns for the stochastic formulation.

$$\langle(\partial w(x_1)/\partial x) w^*(x_2) G_2(x_1, x_2)\rangle, \quad \langle w(x_1) w^*(x_2)(\partial G_2(x_1, x_2)/\partial x)\rangle, \quad \langle|w(x_1)|^2\rangle,$$

$$\langle|\partial w(x_1)/\partial x|^2\rangle, \quad \langle w^*(x_1)\partial w(x_1)/\partial x\rangle, \quad \langle w(x_1)\partial w^*(x_1)/\partial x\rangle, \quad \langle|\partial w(x_2)/\partial x|^2\rangle,$$

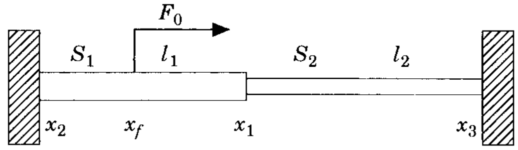

Figure 15. Representation of two coupled rods submitted to a point loading.

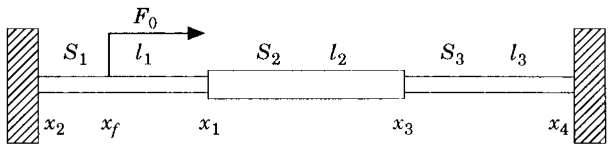

Figure 16. Representation of three coupled rods submitted to a point loading.

TABLE 7

Parameters of the two coupled rods

S_1 (m^2)	S_2 (m^2)	l_1 (m)	l_2 (m)	$\|x_f - x_2\|$ (m)	η (%)
10^{-4}	10^{-5}	1·75	1·5	1	5

TABLE 8

Parameters of the three coupled rods

S_1 (m^2)	S_2 (m^2)	S_3 (m^2)	l_1 (m)	l_2 (m)	l_3 (m)	$\|x_f - x_2\|$ (m)	η (%)
10^{-4}	10^{-5}	10^{-4}	1·89	1·5	1·25	1·12	5

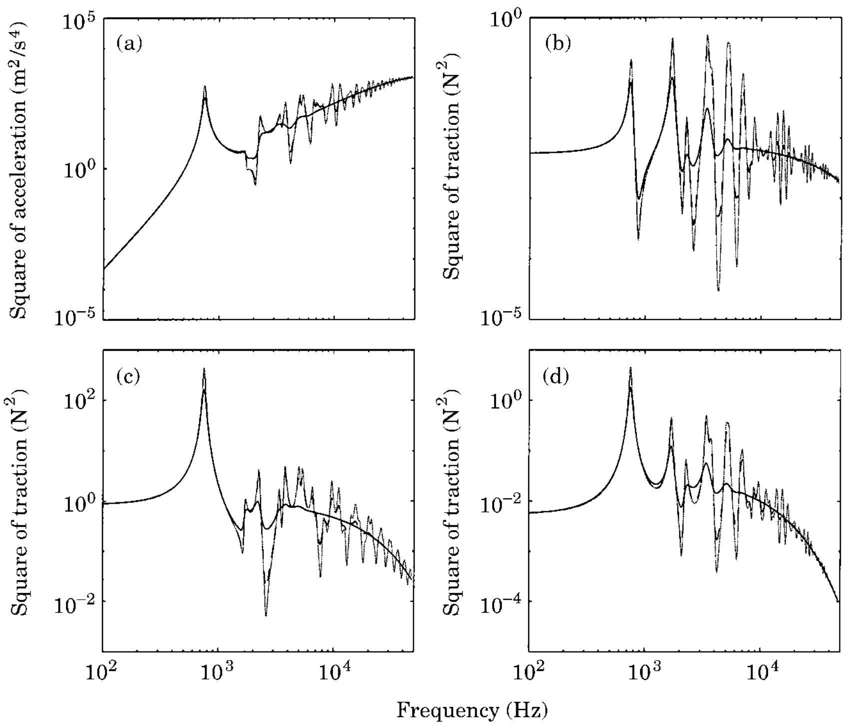

Figure 17. Frequency evolution of two coupled rods: (a) the modulus of the square of acceleration at x_1; (b) the modulus of the square of traction at x_1; (c) the modulus of the square traction at x_2; (d) the modulus of the square traction at x_3. Key as for Figure 4.

$$\langle |\partial w(x_3)/\partial x|^2\rangle, \quad \langle (\partial w^*(x_1)/\partial x)G_1(x_1, x_f)\rangle, \quad \langle w^*(x_1)(\partial G_1(x_1, x_f)/\partial x)\rangle,$$

$$\langle (\partial w^*(x_2)/\partial x)G_1(x_2, x_f)\rangle.$$

The second example is concerned with the coupling between three rods (Figure 16), the results are given without any further developments.

For this case, one has to consider a total of twenty-five unknowns, composed of six first order moments and nineteen second order moments. The mechanical and geometrical properties of two rods (respectively three rods) are given in Table 7 (respectively 8).

The results for two and three rods are given in Figures 17 and 18. The random formulation simulations describe the same behaviour as for the previous examples. That is to say, the low frequency response is accurately approached while the high frequency trend of the deterministic result is smoothly approached.

4.5.2. *Beam coupling*

An example of coupling between two beams submitted to flexural loading (shown in Figure 19) is proposed.

The formulations are not developed, the stochastic assumptions and the formulation of the equations are quite similar to those developed in the previous sections. Forty-six unknowns are required to solve the problem. The mechanical and geometrical properties of the two beams are given in Table 9.

The frequency evolution of two boundary unknowns are illustrated by Figure 20.

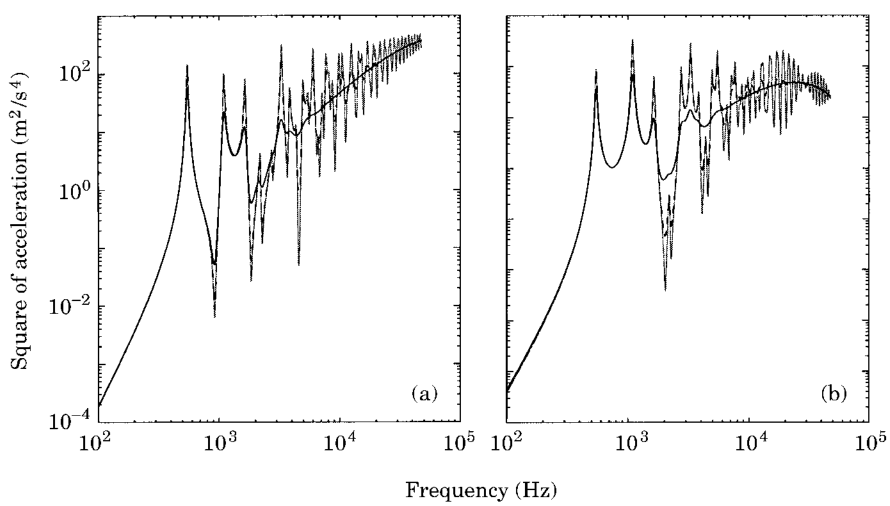

Figure 18. Frequency evolution of the modulus of the square of acceleration of three coupled rods at: (a) x_2; (b) x_3. Key as for Figure 4.

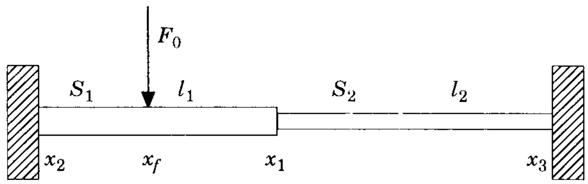

Figure 19. Representation of two coupled beams submitted to a point loading.

TABLE 9
Parameters of the two coupled beams

S_1 (m^2)	S_2 (m^2)	l_1 (m)	l_2 (m)	$\lvert x_f - x_2 \rvert$ (m)	I (m^4)	η (%)
10^{-4}	10^{-4}	1	0·75	0·5	10^{-7}	3

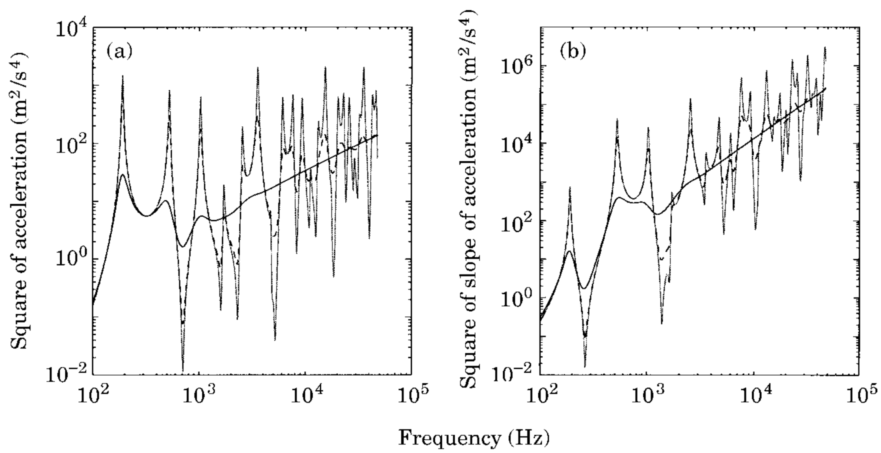

Figure 20. Frequency evolution of two coupled beams: (a) the modulus of the square of acceleration at x_1; (b) the modulus of the square of the slope of acceleration at x_1. Key as for Figure 4.

5. CONCLUSION

A new boundary formulation is presented in this paper, in order to study the behaviour of one-dimensional structures submitted to large frequency field excitations. These new developments have been applied to numerous examples of beams, rods and coupled systems. The results highlight a smooth transition from the exact low frequency modal description to the high frequency non-modal domain. One of the main drawbacks of the formulation is the increasing number of equations appearing to solve the problem, however this default is counterbalanced by the very low frequency sampling required in the high frequency range. This formulation gives local results and is still valid close to the boundaries and singularities present in the structures, which is not the case for energy methods such as the SEM. The lack of confidence inherent in the SEA highlighted by authors like Fahy [12], disappears with the boundary equation formulation. Indeed, the results obtained by this new formulation are rigorously identified in terms of the stochastic expectations with respect to the spatial positions of the boundaries and input force of the square displacement variables, described as functions of random variables.

The study of this new formulation is the beginning of a large research program including developments concerning bi-dimensional systems. For these types of structures the notion of random boundary must be extended. However, these future results will be reported in a future article.

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