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OSCILLATIONS IN SYSTEMS HAVING VELOCITY DEPENDENT FREQUENCIES

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A vast effort has gone into constructing analytical approximations to oscillatory systems that can be modelled by equations taking the form

$$\ddot{x} + x = \varepsilon F(x, \dot{x}), \tag{1}$$

where the parameter ε is small and F is a polynomial function of its arguments. A summary of many of these techniques is given by Mickens [1]. This paper presents preliminary results on a class of non-linear oscillators for which the frequency is a function of the velocity, i.e.,

$$\ddot{x} + f(\dot{x})x = 0 \tag{2}$$

Note that for the case where f is a positive constant, the angular frequency, ω, is given by $\omega^2 = f$. A search of the research literature indicates that no systematic study has been done on this general class of non-linear oscillators. Several important dynamical systems modelled by equation (2) include nerve conduction [2], rapid directional solidification of a dilute binary alloy [3], and the relativistic harmonic oscillator [4]. The *a priori* absence of a small parameter in equation (2) means that the usual perturbative procedures [1] cannot be applied. However, as will be presented below, the application of methods from the qualitative theory of differential equations [5, 6], i.e., a phase plane analysis, can provide valuable information on the existence of periodic and other types of solutions. In the work to follow, it is assumed that

$$f(0) > 0. \tag{3}$$

The two dimensional system equations for equation (2) are [5, 6]

$$\mathrm{d}x/\mathrm{d}t = y, \qquad \mathrm{d}y/\mathrm{d}t = -f(y)x. \tag{4}$$

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As a consequence, the trajectories in the (x, y) phase space are given by the solutions to the following first order differential equation

$$dy/dx = -f(y)x/y. \tag{5}$$

Examination of equations (4) and (5) leads to three conclusions:
(1) There exists a single fixed point or equilibrium state located at $(\bar{x}, \bar{y}) = (0, 0)$.
(2) Equation (5) is invariant under the transformation

$$T_y : x \rightarrow -x. \tag{6}$$

This means that all trajectories in phase-space have a reflection symmetry in the y-axis.
(3) The differential equation (5) is separable and thus can in principle be integrated. The corresponding first integral [7] is

$$\int_0^y \frac{w\,dw}{f(w)} + \frac{x^2}{2} = E = \text{constant}. \tag{7}$$

This last equation can be interpreted as the energy of a harmonic oscillator for which the kinetic energy is given by

$$KE = \int_0^y \frac{w\,dw}{f(w)} = \frac{y^2}{2f(0)} + O(y^3). \tag{8}$$

In other words, equation (2) represents a non-linear harmonic oscillator where the potential energy is the usual expression [7]

$$PE = x^2/2, \tag{9}$$

but the kinetic energy is modified to the form given in equation (8). Of major significance is the fact that all closed curves of equation (7) correspond to periodic solutions [5, 6, 7].

In addition to the fixed point at $(\bar{x}, \bar{y}) = (0, 0)$, equations (4) have special solutions which are a consequence of $f(y) = 0$ having possible real zeroes. Denote these zeroes by $\{y_k^* : k = 1, 2, \ldots, K\}$, i.e.,

$$f(y^*) = 0. \tag{10}$$

It follows that

$$x_k(t) = y_k^* t + A_k, \qquad y_k(t) = y_k^*, \tag{11a, b}$$

where the A_k are arbitrary constants, are solutions. Thus, the (x, y) phase plane is separated into a number of regions, strips parallel to the x-axis, each of which corresponds to a different type of solution. Since different solutions cannot intersect except at fixed points [8] and since $y(t) = y_k^*$ are solutions which define the boundaries of these regions, it follows that trajectories that start in a particular region are confined to that region. Only the solutions in the region containing the origin are periodic since any periodic solution or closed trajectory must enclose a fixed point [8].

The above rather abstract analysis will now be illustrated by considering a particular example of equation (2),

$$\ddot{x} + (1 + \dot{x})^n x = 0, \tag{12}$$

where n is a positive integer [2, 3]. (The relativistic harmonic oscillator is also of the form given by equation (2); however, its detailed analysis is presented in Mickens [4].) The system of equations are

$$\mathrm{d}x/\mathrm{d}t = y, \qquad \mathrm{d}y/\mathrm{d}t = -(1 + y)^n x, \tag{13}$$

and the trajectories in phase space are solutions to

$$\mathrm{d}y/\mathrm{d}x = -(1 + y)^n x/y. \tag{14}$$

There are two cases to consider, n = even and n = odd. However, first note that equation (13) has the fixed point $(\bar{x}, \bar{y}) = (0, 0)$, and the exact special solution

$$x(t) = -t + A, \qquad y(t) = -1, \tag{15}$$

where A is an arbitrary constant. Thus, the phase plane is separated into two regions: $y > -1$ and $y < -1$, with $y = -1$ giving the boundary. Since the fixed point lies in the region $y > -1$, it is only possible for periodic solutions to exist there. This can be easily demonstrated using the concept of null clines [5, 6]. Null clines are curves along which the slope $\mathrm{d}y/\mathrm{d}x$, of trajectories in phase space, is either zero or unbounded. For equation (14), they are given by

$$\mathrm{d}y/\mathrm{d}x = 0 : x = 0 \text{ or the } y\text{-axis, and } y = -1; \tag{16a}$$

$$\mathrm{d}y/\mathrm{d}x = \infty : y = 0 \text{ or the } x\text{-axis.} \tag{16b}$$

Figure 1 shows the null clines and the sign of $\mathrm{d}y/\mathrm{d}x$ in the six regions that the phase space is divided by the null clines. For n = even, the sign of the derivative is the same for $y > -1$, however, they differ for $y < -1$. Using the same arguments as Mickens and Semwogerere [9], it can be shown, for any positive integer n, that all trajectories that initiate in the phase-space region, $y > -1$, are

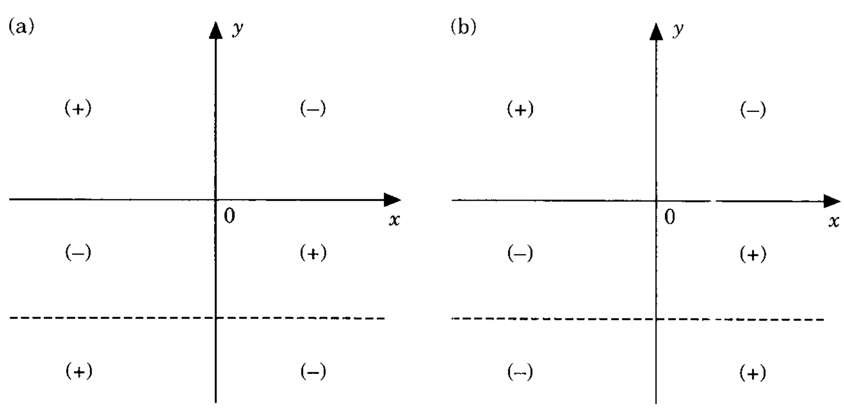

Figure 1. The sign of $\mathrm{d}y/\mathrm{d}x$ in the six regions that the phase plane is divided by the null clines:

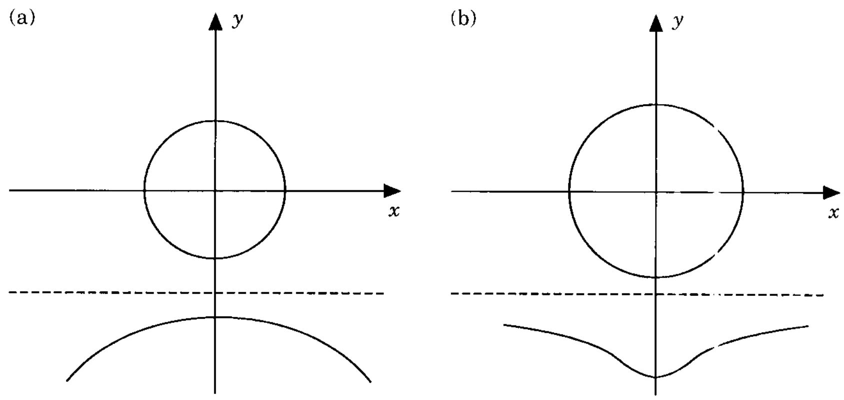

Figure 2. Typical phase plane trajectories for the two regions: $y > -1$ and $y < -1$; (a) n = odd, (b) n = even.

periodic, while those that lie in $y < -1$ are non-periodic and may become unbounded. Figure 2 gives typical trajectories near the origin for the two cases n = even and n = odd. Trajectories starting far from the origin, in the region $y > -1$, are distorted to a pear-like shape.

The general difficulty in considering particular cases of equation (2), for example equation (12), is the lack of methods to calculate good analytical approximations to the periodic solutions. The major problem is the lack of a small parameter; thus, the usual perturbation procedure cannot be used [1, 10]. It is also of interest to note that the application of the method of harmonic balance (at least in the rational formalism [5, 11]) does not lead to a consistent result for an approximate solution to equation (12) with $n = 1$. Future work will consist of finding additional oscillatory systems having velocity dependent frequencies and devising analytical procedures for calculating analytical approximations to the corresponding periodic solutions.

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