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# NON-LINEAR VIBRATIONS OF A BEAM ON AN ELASTIC FOUNDATION

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The non-linear vibrations of an elastic beam resting on a non-linear tensionless Winkler foundation subjected to a concentrated load at the centre is presented in this paper. Since the foundation is assumed to be tensionless, the beam may lift off the foundation and there exists different regions namely contact and non-contact regions. Since the contact regions are not known in advance, the problem appears as a non-linear one even though there is no non-linear term in the foundation model. In this case, the calculation of the roots of a non-linear equation is needed to obtain contact lengths. The perturbation technique is used to solve the non-linear governing equation associated with the problem. Using this technique, the non-linear problem is reduced to the solution of a set of linearized partial differential equations. The lift-off points and the displacements are obtained in linear and non-linear cases, and the variation of these points with respect to various parameters are presented. It is concluded that the contact length varies with the magnitude of the load because of the non-linearity.



## 1. INTRODUCTION

The vibrations of beams resting on elastic foundation have been investigated by a number of investigators. In these studies it has been mostly assumed that the foundation reacts in tension as well as in compression. Various types of foundation models such as Winkler, Pasternak, Vlasov, Filonenko-Borodich, etc. have been used in the analysis of structures on elastic foundations [1]. Among these, the Winkler model, in which the medium is taken into account as a system composed of infinitely close linear springs, is the simplest and often adopted one. It assumes that the foundation applies only a reaction normal to the beam that is proportional to the beam deflection. Non-linear foundation models like hyperbolic [2], and cubic types [3–6] are also used. In some cases, it is more accurate to consider the foundation as tensionless. In such a case, the problem displays a non-linear

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character and the solution becomes difficult since the contact region is not known in advance due to the tensionless character of the foundation.

Weitsman [7], Villiaggio [8] and Celep [9, 10] studied the static behaviour of beams and plates resting on a tensionless Winkler foundation for various loadings. In these studies, it has been shown that the contact length do not depend on the magnitude of the load. Weitsman [11], Rao [12], Choros and Adams [13] and Lin and Adams [14] analyzed a beam subjected to moving load/loads resting on a tensionless Winkler foundation. Celep *et al*. [15], Celep and Turhan [16] and Güler and Celep [17] studied the dynamic responses of beams and plates subjected to various loadings on such a foundation and obtained contact regions indicating again that these regions are independent of the level of loading. In all these studies, because of the tensionless character of the foundation the problem is non-linear. There have been fewer studies on the non-linear vibrations of beams and plates on non-linear elastic foundation. Raju and Rao [4] studied the effect of a non-linear elastic foundation on the mode shapes of buckling and the free vibration of columns/beams. Eisenberger [18] studied the vibration frequencies of beams resting on variable one and two parameter elastic foundations. Dumir *et al*. [6] studied the vibrations of cylindrically orthotropic thin circular plates resting on Winkler, Pasternak and non-linear Winkler foundations. Bhaskar and Dumir [19] studied the non-linear vibrations of orthotropic thin rectangular plates on elastic foundation with linear and non-linear Winkler parameters. Shih and Blotter [5] studied the non-linear vibrations of arbitrarily laminated thin rectangular plates on elastic foundations.

In this paper, the harmonic vibrations of a finite beam resting on a non-linear tensionless Winkler foundation are investigated. The beam is subjected to a transverse periodic force at the centre, and the non-linearity of the problem arises from the presence of the lift-off and a cubic term in the foundation model. Using the perturbation technique, lift-off points and the vertical displacements are expanded into series in order to obtain the effect of the non-linearity and then arranging them with respect to the powers of the perturbation parameter, the non-linear governing equation is reduced to a set of linearized partial differential equations. These equations are converted to ordinary differential equations by the use of method of separation of variables and then solved analytically by using the boundary and continuity conditions. In the first order, a transcendent equation is obtained to calculate lift-off points. The Newton–Raphson method is used to obtain the solution of the transcendental equation. The influence of some parameters on the lift-off points and the displacements in linear and non-linear cases are obtained.

## 2. PROBLEM FORMULATION

Consider an elastic beam of length $2L$ on a non-linear Winkler foundation subjected to a harmonic load $P = P_0 \cos \Omega t$ at the centre (Figure 1). The foundation reaction $q$ is considered to be of the form $q = k_1 w + k_3 w^3$, where $k_1$

and $k_3$ are the linear and non-linear coefficients of the stiffness. The governing equations of the beam in contact and non-contact regions are, respectively

$$EIw_i^{iv} + \rho A\ddot{w}_i + k_1 w_i + k_3 w_i^3 = 0, \qquad X_{i-1} \leqslant x \leqslant X_i, \qquad i = 1, 3, 5, \ldots \quad (1)$$

$$EIw_j^{iv} + \rho A\ddot{w}_j = 0, \qquad X_i \leqslant x \leqslant X_{i+1}, \qquad j = i + 1 \quad (2)$$

where $EI$ is the flexural rigidity, $w(x, t)$ the vertical displacement, $\rho$ the mass density, $A$ the cross-sectional area of the beam, $X_i$ the lift-off points. The vertical displacements of the beam are zero at the lift-off points. Here, $X_0 = 0$ is not a lift-off point and the displacement is not equal to zero at this point. The Euler–Bernoulli beam is considered and the Bernoulli–Navier hypothesis is also valid. Because of the continuity of the slope of the elastic curve and the symmetry of the beam, the slope of the elastic curve is zero at the centre. The shear force at the centre is $-P/2$ due to the symmetry. At the lift-off points, the displacement, slope, bending moment and shear force must be continuous. Because of the symmetry, only $x \geqslant 0$ region is considered. The boundary and continuity conditions for the problem are as follows:

$$w_1'(0, t) = 0, \qquad EIw_1'''(0, t) = \tfrac{1}{2}P_0 \cos \Omega t, \qquad w_1(X_1, t) = 0 \quad (3)$$

in region 1 and,

$$w_n(0, t) = 0, \qquad w_n'(0, t) = w_{n-1}'(X_{n-1}, t),$$

$$w_n''(0, t) = w_{n-1}''(X_{n-1}, t), \qquad w_n'''(0, t) = w_{n-1}'''(X_{n-1}, t), \qquad n = 2, 3, \ldots, N \quad (4)$$

$$w_N''(\bar{X}_N, t) = 0, \qquad w_N'''(\bar{X}_N, t) = 0,$$

$$w_n(X_n, t) = 0, \qquad n \neq N \quad (4)$$

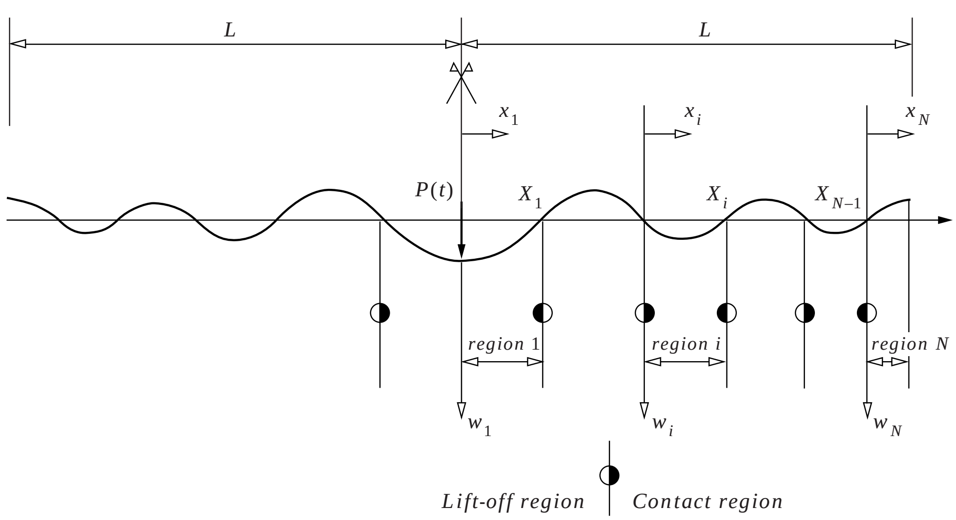


Figure 1. Beam on a non-linear tensionless Winkler foundation.

in region $n$, where $\bar{X}_N = L - X_{N-1}$ and $N$ represents the last region which can either be a contact or a non-contact region. The non-dimensional parameters are introduced as

$$\bar{w} = w/w_0, \quad w_0 = P_0/Lk_1, \quad \beta^4 = k_1/4EI, \quad \xi = \beta x, \quad \xi_i = \beta X_i, \quad \eta = \beta L,$$
$$\varepsilon\bar{k} = 4k_3 w_0^2/k_1, \quad \bar{P} = P_0/EI\beta^3 w_0, \quad s = \Omega t. \tag{5}$$

With the use of these parameters, the governing equations (1) and (2) reduce to the dimensionless forms

$$w_i^{iv} + (4\Omega^2/\omega^2)\ddot{w}_i + 4w_i + \varepsilon\bar{k}w_i^3 = 0, \qquad \xi_{i-1} \leqslant \xi \leqslant \xi_i \quad i = 1, 3, 5, \ldots \tag{6}$$

$$w_j^{iv} + (4\Omega^2/\omega^2)\ddot{w}_j = 0, \qquad \xi_i \leqslant \xi \leqslant \xi_{i+1} \qquad j = i + 1 \tag{7}$$

where $\varepsilon\bar{k}$ indicates the non-linearity parameter and $\omega$ the natural frequency of the system defined as $\omega^2 = k_1/\rho A$.

## 3. SOLUTION

For the solution of the equations (6) and (7), the displacements and the lift-off points are expanded into perturbation series as follows:

$$w_i(\xi, s; \varepsilon) = w_{i,0}(\xi, s) + \varepsilon w_{i,1}(\xi, s) + \ldots$$
$$w_j(\xi, s; \varepsilon) = w_{j,0}(\xi, s) + \varepsilon w_{j,1}(\xi, s) + \ldots$$
$$\xi_n = \xi_{n,0} + \varepsilon\xi_{n,1} + \ldots, \qquad n = 1, 2, \ldots, N-1 \tag{8}$$

where $w_{i,0}$ and $w_{j,0}$ indicate the linear displacements in contact and non-contact regions, $w_{i,1}$ and $w_{j,1}$ are the effects of the first order non-linearity on displacements, $\xi_{n,0}$ are the co-ordinates of lift-off points in linear case and $\xi_{n,1}$ are the effects of the non-linearity on these points. Since the response is harmonic, the method of separation of variables can be used as

$$w_{i,0}(\xi, s) = W_{i,0}(\xi)\cos s, \qquad w_{i,1}(\xi, s) = W_{i,1}(\xi)\cos s + W_{i,2}(\xi)\cos 3s$$
$$w_{j,0}(\xi, s) = W_{j,0}(\xi)\cos s, \qquad w_{j,1}(\xi, s) = W_{j,1}(\xi)\cos s + W_{j,2}(\xi)\cos 3s. \tag{9}$$

Substituting equations (8) and (9) into equations (6) and (7), a set of ordinary differential equations is obtained as:

$$W_{i,0}^{iv} + 4\theta_{10}^4 W_{i,0} = 0 \tag{10}$$

$$W_{j,0}^{iv} - \theta_{20}^4 W_{j,0} = 0 \tag{11}$$

$$W_{i,1}^{iv} + 4\theta_{11}^4 W_{i,1} = -k_{11} W_{i,0}^3 \tag{12}$$

$$W_{j,1}^{iv} - \theta_{21}^4 W_{j,1} = 0 \tag{13}$$

$$W_{i,2}^{iv} + 4\theta_{12}^4 W_{i,2} = -k_{12} W_{i,0}^3 \tag{14}$$

$$W_{j,2}^{iv} - \theta_{22}^4 W_{j,2} = 0 \tag{15}$$

where $k_{11}=3\bar{k}/4$, $k_{12}=\bar{k}/4$, $\bar{\Omega}=\Omega/\omega$, $\theta_{10}^4=\theta_{11}^4=1-\bar{\Omega}^2$, $\theta_{12}^4=1-9\bar{\Omega}^2$, $\theta_{20}^4=\theta_{21}^4=4\bar{\Omega}^2$, $\theta_{22}^4=9\theta_{20}^4$. These equations are solved with respect to the values of frequency ratio $\bar{\Omega}$. There exist 12 unknown integral constants in each region. The system has $N$ regions and the beam lifts off or touchs the foundation in $N-1$ points. In this case, the number of unknowns including the constants and the lift-off points become $N\times 12+(N-1)\times 2=14\times N-2$. The boundary and continuity conditions are expanded into Taylor series to get the necessary equations for obtaining these unknowns. Using (5) and (8) in the equations (3) and (4), the boundary and continuity conditions become

$$W'_{1,0}(0)=0,\quad W'_{1,1}(0)=0,\quad W'_{1,2}(0)=0,\quad W'''_{1,0}(0)=\bar{P}/2,\quad W'''_{1,1}(0)=0$$

$$W'''_{1,2}(0)=0,\quad W_{1,0}(\xi_{1,0})=0,\quad W_{1,2}(\xi_{1,0})=0,\quad \xi_{1,1}=-W_{1,1}(\xi_{1,0})/W'_{1,0}(\xi_{1,0}) \tag{16}$$

in region 1 and,

$$W_{n,0}(0)=0,\qquad W_{n,2}(0)=0,\qquad \xi_{n-1,1}=-W_{n,1}(0)/W'_{n,0}(0)$$

$$W'_{n,0}(0)=W'_{n-1,0}(\xi_{n-1,0}),\qquad W'_{n,1}(0)=W'_{n-1,1}(\xi_{n-1,0}),$$

$$W'_{n,2}(0)=W'_{n-1,2}(\xi_{n-1,0})$$

$$W''_{n,0}(0)=W''_{n-1,0}(\xi_{n-1,0}),\qquad W''_{n,1}(0)=W''_{n-1,1}(\xi_{n-1,0}),$$

$$W''_{n,2}(0)=W''_{n-1,2}(\xi_{n-1,0})$$

$$W'''_{n,0}(0)=W'''_{n-1,0}(\xi_{n-1,0}),\qquad W'''_{n,1}(0)=W'''_{n-1,1}(\xi_{n-1,0}),$$

$$W'''_{n,2}(0)=W'''_{n-1,2}(\xi_{n-1,0})$$

$$W''_{N,0}(\bar{\xi}_N)=0,\quad W''_{N,1}(\bar{\xi}_N)=0,\quad W''_{N,2}(\bar{\xi}_N)=0,\quad W'''_{N,0}(\bar{\xi}_N)=0,\quad W'''_{N,1}(\bar{\xi}_N)=0$$

$$W'''_{n,2}(\bar{\xi}_N)=0,\quad W_{n,1}(0)=W_{n-1,1}(\xi_{n-1,0})\quad n=2,3,\ldots,N$$

$$W_{n,0}(\xi_{n,0})=0,\quad W_{n,2}(\xi_{n,0})=0,\quad \xi_{n,1}=-W_{n,1}(\xi_{n,0})/W'_{n,0}(\xi_{n,0})\quad n\neq N \tag{17}$$

in region $n$, where $\bar{\xi}_N=\eta-(\xi_{N-1,0}+\varepsilon\xi_{N-1,1})$. The evaluation of lift-off point in region 1 is given in Appendix A. As indicated before, with the increase of the number of regions, the number of the unknowns increases rapidly and solution becomes difficult. Only two regions are considered for brevity. Since the solutions of the equations (10)–(15) depend on the value of $\bar{\Omega}$, there exist different cases depending on this parameter. Four cases will be discussed.

### 3.1. SOLUTIONS FOR VARIOUS VALUES OF $\overline{\Omega}$

#### 3.1.1. *The case for* $\overline{\Omega} < 1/3$

In this case, the solutions of equations (10)–(15) are obtained as follows:

$$W_{1,0} = C_1\,e^{m\xi} + C_2\,e^{\bar{m}\xi} + C_4\,e^{-m\xi} \tag{18}$$

$$\begin{aligned} W_{1,1} = {} & E_1\,e^{m\xi} + E_2\,e^{\bar{m}\xi} + E_3\,e^{-\bar{m}\xi} + E_4\,e^{-m\xi} - k_{11}\left\{\frac{C_1^3\,e^{3m\xi}}{(3m)^4 + 4\theta_{11}^4}\right. \\ & + \frac{C_3^3\,e^{-3\bar{m}\xi}}{(3\bar{m})^4 + 4\theta_{11}^4} \\ & + \frac{3C_1^2C_2\,e^{(2m+\bar{m})\xi}}{(2m+\bar{m})^4 + 4\theta_{11}^4} + \frac{3C_1^2C_3\,e^{(2m-\bar{m})\xi}}{(2m-\bar{m})^4 + 4\theta_{11}^4} + \frac{3C_3^2C_1\,e^{(m-2\bar{m})\xi}}{(m-2\bar{m})^4 + 4\theta_{11}^4} \\ & + \frac{3C_3^2C_4\,e^{-(m+2\bar{m})\xi}}{(m+2\bar{m})^4 + 4\theta_{11}^4} \\ & \left. + C.C\right\} - \frac{3k_{11}}{4}\,\xi\left\{\frac{C_1^2C_4\,e^{m\xi}}{m^3} - \frac{C_3^2C_2\,e^{-\bar{m}\xi}}{\bar{m}^3} + \frac{2C_1C_2C_3\,e^{m\xi}}{m^3}\right. \\ & \left. - \frac{2C_1C_3C_4\,e^{-\bar{m}\xi}}{\bar{m}^3} + C.C\right\} \end{aligned} \tag{19}$$

$$\begin{aligned} W_{1,2} = {} & G_1\,e^{n\xi} + G_2\,e^{\bar{n}\xi} + G_3\,e^{-\bar{n}\xi} + G_4\,e^{-n\xi} - k_{12}\left\{\frac{C_1^3\,e^{3m\xi}}{(3m)^4 + 4\theta_{12}^4}\right. \\ & + \frac{C_3^3\,e^{-3\bar{m}\xi}}{(3\bar{m})^4 + 4\theta_{12}^4} \\ & + \frac{3C_1^2C_2\,e^{(2m+\bar{m})\xi}}{(2m+\bar{m})^4 + 4\theta_{12}^4} + \frac{3C_1^2C_3\,e^{(2m-\bar{m})\xi}}{(2m-\bar{m})^4 + 4\theta_{12}^4} + \frac{3C_3^2C_1\,e^{(m-2\bar{m})\xi}}{(m-2\bar{m})^4 + 4\theta_{12}^4} \\ & + \frac{3C_3^2C_4\,e^{-(m+2\bar{m})\xi}}{(m+2\bar{m})^4 + 4\theta_{12}^4} \\ & + \frac{3C_1^2C_4\,e^{m\xi}}{m^4 + 4\theta_{12}^4} + \frac{3C_3^2C_2\,e^{-\bar{m}\xi}}{\bar{m}^4 + 4\theta_{12}^4} + \frac{6C_1C_2C_3\,e^{m\xi}}{m^4 + 4\theta_{12}^4} \\ & \left. + \frac{6C_1C_3C_4\,e^{-\bar{m}\xi}}{\bar{m}^4 + 4\theta_{12}^4} + C.C\right\} \end{aligned} \tag{20}$$

$$W_{2,0} = D_1\,e^{\theta_{20}\xi} + D_2\,e^{-\theta_{20}\xi} + D_3\,e^{i\theta_{20}\xi} + D_4\,e^{-i\theta_{20}\xi} \tag{21}$$

$$W_{2,1} = F_1\,e^{\theta_{21}\xi} + F_2\,e^{-\theta_{21}\xi} + F_3\,e^{i\theta_{21}\xi} + F_4\,e^{-i\theta_{21}\xi} \tag{22}$$

$$W_{2,2} = H_1\,e^{\theta_{22}\xi} + H_2\,e^{-\theta_{22}\xi} + H_3\,e^{i\theta_{22}\xi} + H_4\,e^{-i\theta_{22}\xi} \tag{23}$$

where $m = (1+i)\theta_{10}$, $\bar{m} = (1-i)\theta_{10}$, $n = (1+i)\theta_{12}$, $\bar{n} = (1-i)\theta_{12}$ and $C.C$ stands for the complex conjugate of the preceding terms. In these solutions $C_i, \ldots, H_i$ are integral constants, and with the lift-off points which are not known yet, the number of unknowns reaches 26. Firstly, the linear case is observed by the use of the equations (18) and (21). Using these equations in the boundary and continuity conditions (16) and (17), and eliminating the unknown integral constants $C_i$ and $D_i$, a transcendental equation is obtained in terms of the unknown $\xi_{1,0}$. Chosing $\bar{\Omega}$, and $\eta$ (the characteristic length) and using the Newton–Raphson technique, the root $\xi_{1,0}$ of this transcendental equation is obtained. Then, the constants $C_i$ and $D_i$ are obtained in terms of the $\bar{P}$ and $\xi_{1,0}$. After the linear solution is completed, the first order solution is obtained by using the equations (19), (20), (22), (23) and the proper boundary and continuity conditions which have not been used up to that point. This solution procedure will be valid also for the cases cited below.

### 3.1.2. *The case for* $1/3 < \bar{\Omega} < 1$

In this case, all the solutions are similar to those in the preceding case except $W_{1,2}$. Defining $\theta_{12}^4 = 4(9\bar{\Omega}^2 - 1)$, one can obtain the solution $W_{1,2}$ as:

$$
\begin{aligned}
W_{1,2} = {} & G_1\,\mathrm{e}^{\theta_{12}\xi} + G_2\,\mathrm{e}^{-\theta_{12}\xi} + G_3\,\mathrm{e}^{\mathrm{i}\theta_{12}\xi} + G_4\,\mathrm{e}^{-\mathrm{i}\theta_{12}\xi} - k_{12}\left\{\frac{C_1^3\,\mathrm{e}^{3m\xi}}{(3m)^4-\theta_{12}^4}\right.\\
& + \frac{C_3^3\,\mathrm{e}^{-3\bar{m}\xi}}{(-3\bar{m})^4-\theta_{12}^4}\\
& + \frac{3C_1^2C_2\,\mathrm{e}^{(2m+\bar{m})\xi}}{(2m+\bar{m})^4-\theta_{12}^4} + \frac{3C_1^2C_3\,\mathrm{e}^{(2m-\bar{m})\xi}}{(2m-\bar{m})^4-\theta_{12}^4} + \frac{3C_3^2C_1\,\mathrm{e}^{(m-2\bar{m})\xi}}{(m-2\bar{m})^4-\theta_{12}^4}\\
& + \frac{3C_3^2C_4\,\mathrm{e}^{-(m+2\bar{m})\xi}}{(m+2\bar{m})^4-\theta_{12}^4}\\
& + \frac{3C_1^2C_4\,\mathrm{e}^{m\xi}}{m^4-\theta_{12}^4} + \frac{3C_3^2C_2\,\mathrm{e}^{-\bar{m}\xi}}{(-\bar{m})^4-\theta_{12}^4} + \frac{6C_1C_2C_3\,\mathrm{e}^{m\xi}}{m^4-\theta_{12}^4}\\
& \left. + \frac{6C_1C_3C_4\,\mathrm{e}^{-\bar{m}\xi}}{(-\bar{m})^4-\theta_{12}^4} + C.C\right\}.
\end{aligned}
\tag{24}
$$

### 3.1.3. *The case for* $\bar{\Omega} = 1$

$$W_{1,0} = C_1 + C_2\xi + C_3\xi^2 + C_4\xi^3 \tag{25}$$

$$
\begin{aligned}
W_{1,1} = {} & E_1 + E_2\xi + E_3\xi^2 + E_4\xi^3 - k_{11}\left\{\frac{C_1^3}{24}\xi^4 + \frac{C_1^2C_3}{120}\xi^6\right.\\
& + \frac{C_1^2C_4}{280}\xi^7 + \frac{C_3^2C_1}{560}\xi^8\\
& \left. + \frac{C_1C_3C_4}{504}\xi^9 + \frac{C_3^3+3C_4^2C_1}{5040}\xi^{10} + \frac{C_3^2C_4}{2640}\xi^{11} + \frac{C_4^2C_3}{3960}\xi^{12} + \frac{C_4^3}{17160}\xi^{13}\right\}
\end{aligned}
\tag{26}
$$

$$W_{1,2} = G_1 \, e^{\theta_{12}\xi} + G_2 \, e^{-\theta_{12}\xi} + G_{31} \cos(\theta_{12}\xi) - G_{32} \sin(\theta_{12}\xi) + \frac{k_{12}}{\theta_{12}^4} \times (m_1 + m_2\xi + m_3\xi^2 + m_4\xi^3 + m_5\xi^4 + m_6\xi^5 + m_7\xi^6 + m_8\xi^7 + m_9\xi^8 + m_{10}\xi^9) \tag{27}$$

$$W_{2,0} = D_1 \, e^{\theta_{20}\xi} + D_2 \, e^{-\theta_{20}\xi} + D_{31} \cos(\theta_{20}\xi) - D_{32} \sin(\theta_{20}\xi) \tag{28}$$

$$W_{2,1} = F_1 \, e^{\theta_{21}\xi} + F_2 \, e^{-\theta_{21}\xi} + F_{31} \cos(\theta_{21}\xi) - F_{32} \sin(\theta_{21}\xi) \tag{29}$$

$$W_{2,2} = H_1 \, e^{\theta_{22}\xi} + H_2 \, e^{-\theta_{22}\xi} + H_{31} \cos(\theta_{22}\xi) - H_{32} \sin(\theta_{22}\xi) \tag{30}$$

where $\theta_{20}^4 = 4\bar{\Omega}^2$, $\theta_{12}^4 = 4(1 - 9\bar{\Omega}^2)$, $\theta_{22}^4 = 36\bar{\Omega}^2$, $\theta_{21}^4 = \theta_{20}^4$ and $m_i$'s are given in Appendix B.

#### 3.1.4. *The case for* $\bar{\Omega} > 1$

For this frequency, the solution can be obtained if the forcing and the displacements are in different directions. That is, if the forcing is upwards, the contact region will be the same as in the previous cases, otherwise this region will be the non-contact region and the middle of the beam separates from the foundation. In the case of upward loading, the solutions become:

$$W_{1,0} = C_1 \, e^{\theta_{10}\xi} + C_2 \, e^{-\theta_{10}\xi} + C_3 \, e^{i\theta_{10}\xi} + C_4 \, e^{-i\theta_{10}\xi} \tag{31}$$

$$W_{2,0} = D_1 \, e^{\theta_{20}\xi} + D_2 \, e^{-\theta_{20}\xi} + D_3 \, e^{i\theta_{20}\xi} + D_4 \, e^{-i\theta_{20}\xi} \tag{32}$$

$$\begin{aligned} W_{1,1} = {} & E_1 \, e^{\theta_{11}\xi} + E_2 \, e^{-\theta_{11}\xi} + E_3 \, e^{i\theta_{11}\xi} + E_4 \, e^{-i\theta_{11}\xi} - \frac{k_{11}}{80\theta_{10}^4} \{C_1^3 \, e^{3\theta_{10}\xi} + C_2^3 \, e^{-3\theta_{10}\xi} \\ & + C_3^3 \, e^{3i\theta_{10}\xi} + C_4^3 \, e^{-3i\theta_{10}\xi}\} + \frac{3k_{11}}{8\theta_{10}^4(1+3i)} \{C_1^2 C_4 \, e^{(2-i)\theta_{10}\xi} \\ & + C_2^2 C_3 \, e^{-(2+i)\theta_{10}\xi} + C_3^2 C_1 \, e^{(2i+1)\theta_{10}\xi} \\ & + C_4^2 C_2 \, e^{-(2i+1)\theta_{10}\xi} + ((1+3i)/(1-3i)) \\ & \times [C_1^2 C_3 \, e^{(2+i)\theta_{10}\xi} + C_2^2 C_4 \, e^{-(2+i)\theta_{10}\xi} + C_3^2 C_2 \, e^{(2i-1)\theta_{10}\xi} \\ & + C_4^2 C_1 \, e^{(-2i+1)\theta_{10}\xi}]\} \\ & - \frac{3k_{11}}{4\theta_{10}^3} \xi \{(C_1^2 C_2 + 2C_1 C_3 C_4) \, e^{\theta_{10}\xi} - (C_1 C_2^2 + 2C_2 C_3 C_4) \, e^{-\theta_{10}\xi} \\ & + i(C_3^2 C_4 + 2C_1 C_2 C_3) \, e^{i\theta_{10}\xi} - i(C_4^2 C_3 + 2C_1 C_2 C_4) \, e^{-i\theta_{10}\xi}\} \end{aligned} \tag{33}$$

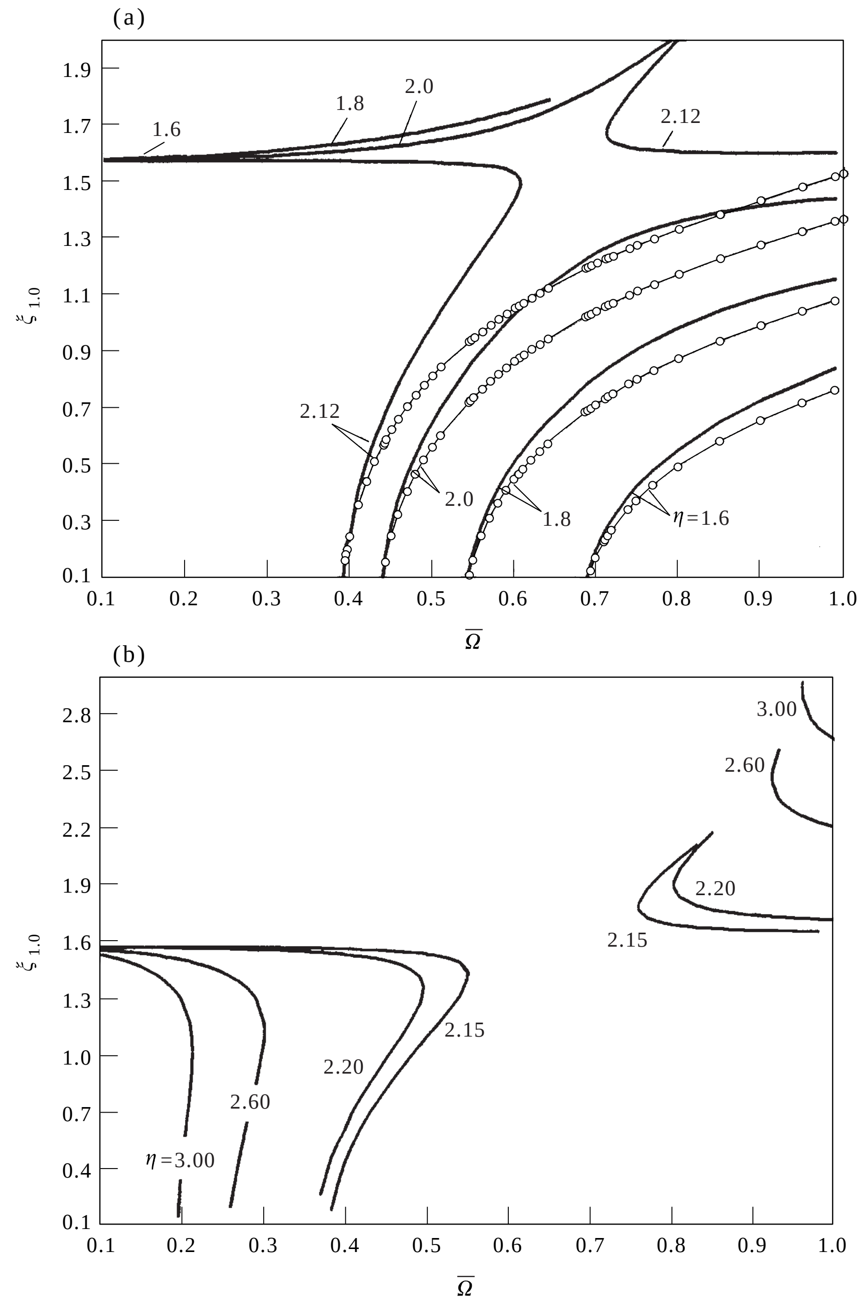


Figure 2. (a) Lift-off points of the beam versus frequency ratio at various beam lengths. (b) Lift-off points of the beam versus frequency ratio at various beam lengths.

$$W_{1,2} = G_1\,e^{\theta_{12}\xi} + G_2\,e^{-\theta_{12}\xi} + G_3\,e^{i\theta_{12}\xi} + G_4\,e^{-i\theta_{12}\xi} - \frac{k_{12}}{81\theta_{10}^4 - \theta_{12}^4}\{C_1^3\,e^{3\theta_{10}\xi}$$
$$+ C_2^3\,e^{-3\theta_{10}\xi} + C_3^3\,e^{3i\theta_{10}\xi} + C_4^3\,e^{-3i\theta_{10}\xi}\} - \frac{3k_{12}}{\theta_{10}^4 - \theta_{12}^4}\{(C_1^2C_2$$
$$+ 2C_1C_3C_4)\,e^{\theta_{10}\xi}$$
$$+ (C_1C_2^2 + 2C_2C_3C_4)\,e^{-\theta_{10}\xi} + (C_3^2C_4 + 2C_1C_2C_3)\,e^{i\theta_{10}\xi}$$
$$+ (C_4^2C_3 + 2C_1C_2C_4)\,e^{-i\theta_{10}\xi}\}$$
$$- \frac{3k_{12}}{\theta_{10}^4(24i + r_1)}\{-C_1^2C_4\,e^{(2-i)\theta_{10}\xi} - C_2^2C_3\,e^{(-2+i)\theta_{10}\xi}$$
$$- C_3^2C_1\,e^{(2i+1)\theta_{10}\xi} - C_4^2C_2\,e^{-(2i+1)\theta_{10}\xi}$$
$$+ ((24i + r_1)/(24i - r_1))[C_1^2C_3\,e^{(2+i)\theta_{10}\xi}$$
$$+ C_2^2C_4\,e^{-(2+i)\theta_{10}\xi} + C_3^2C_2\,e^{(2i-1)\theta_{10}\xi} + C_4^2C_1\,e^{(-2i+1)\theta_{10}\xi}]\} \tag{34}$$

$$W_{2,1} = F_1\,e^{\theta_{21}\xi} + F_2\,e^{-\theta_{21}\xi} + F_3\,e^{i\theta_{21}\xi} + F_4\,e^{-i\theta_{21}\xi} \tag{35}$$

$$W_{2,2} = H_1\,e^{\theta_{22}\xi} + H_2\,e^{-\theta_{22}\xi} + H_3\,e^{i\theta_{22}\xi} + H_4\,e^{-i\theta_{22}\xi}, \tag{36}$$

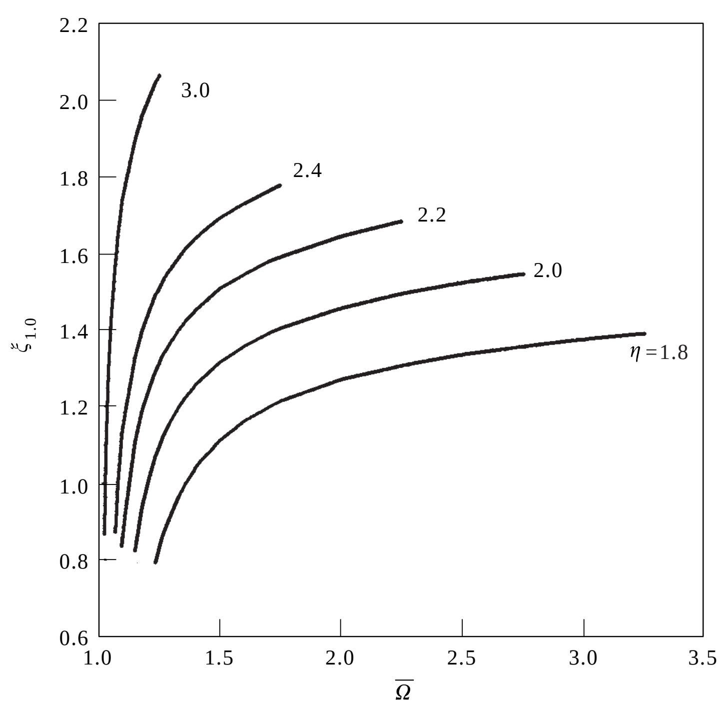


Figure 3. Lift-off points of the beam versus frequency ratio at various beam lengths for downwards loading.

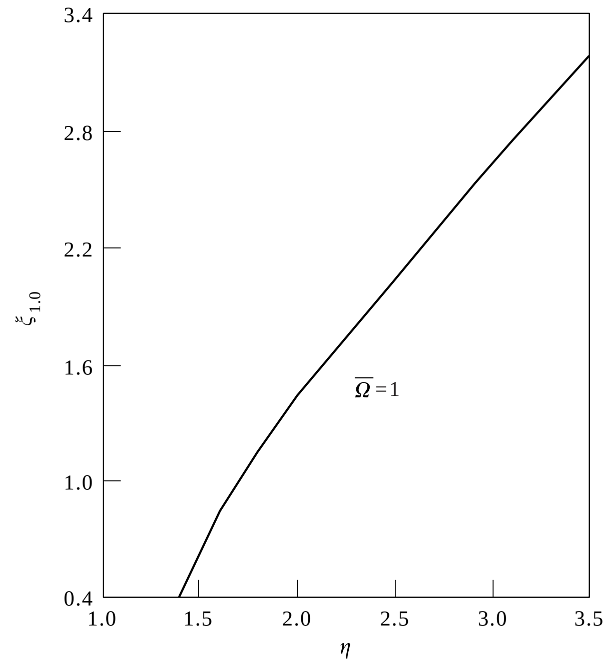


Figure 4. Lift-off points of the beam versus characteristic beam lengths at fixed frequency ratio.

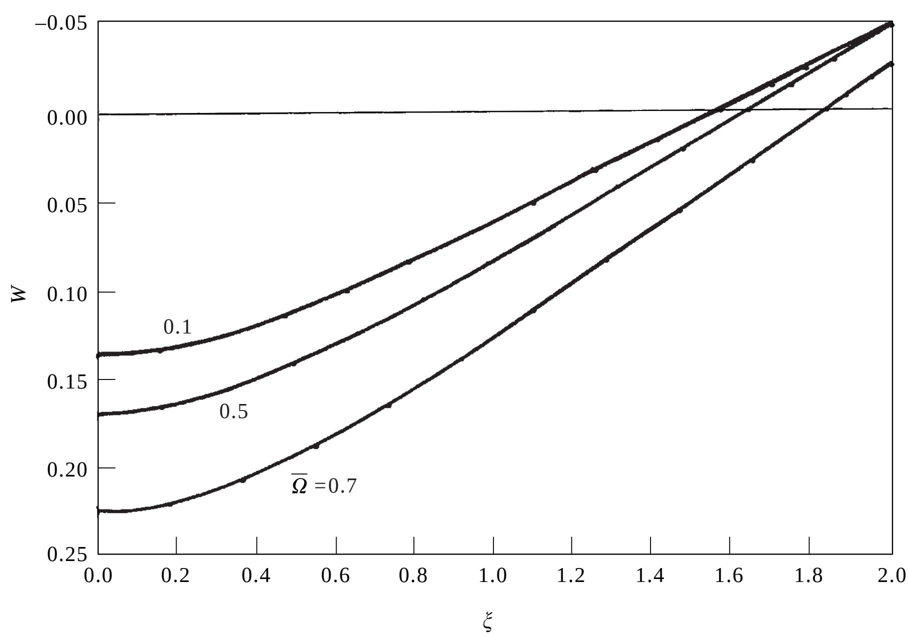


Figure 5. Elastic curves of the beam on tensionless foundation for the first solution.

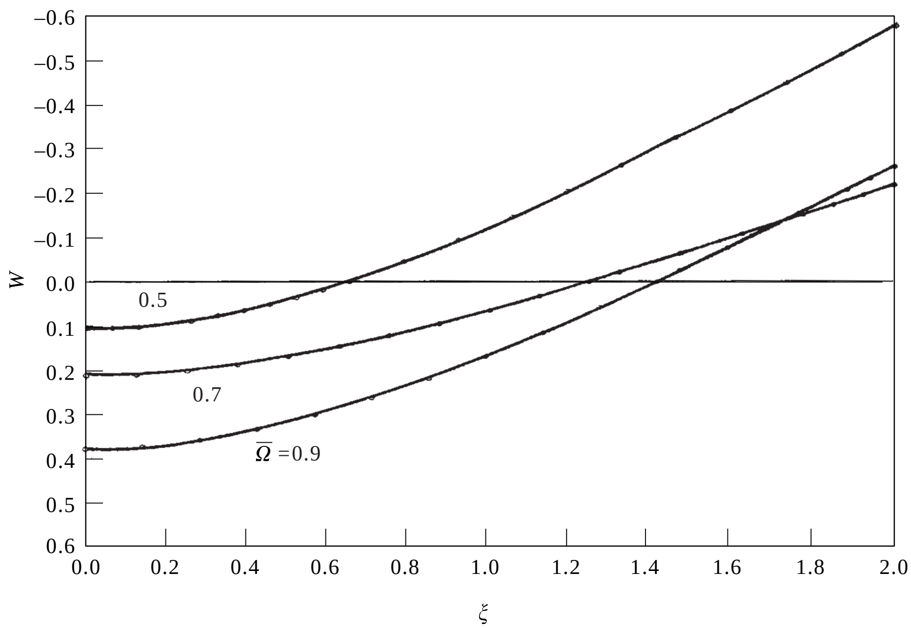


Figure 6. Elastic curves of the beam on tensionless foundation for the second solution.

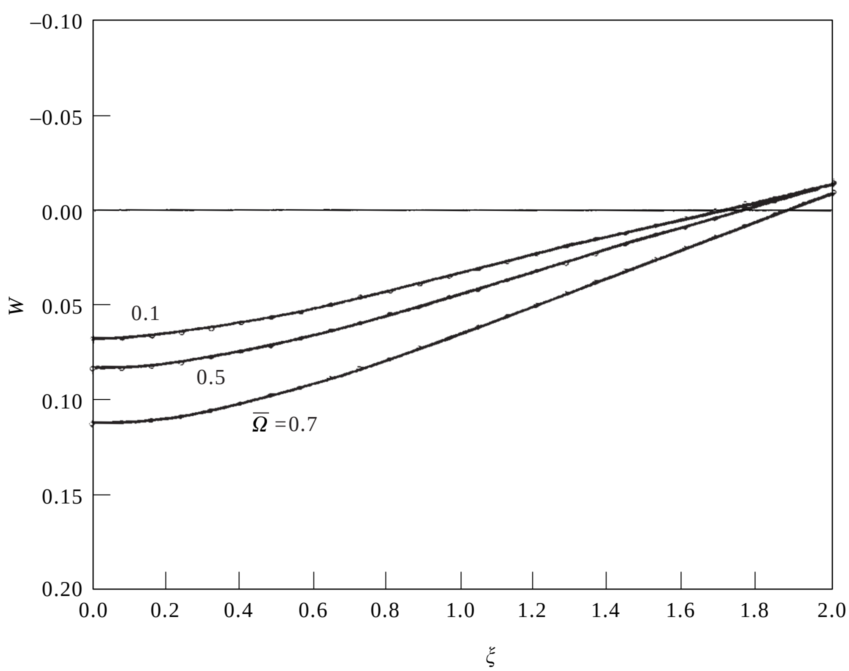


Figure 7. Elastic curves of the beam on foundation which transmits tensile stresses.

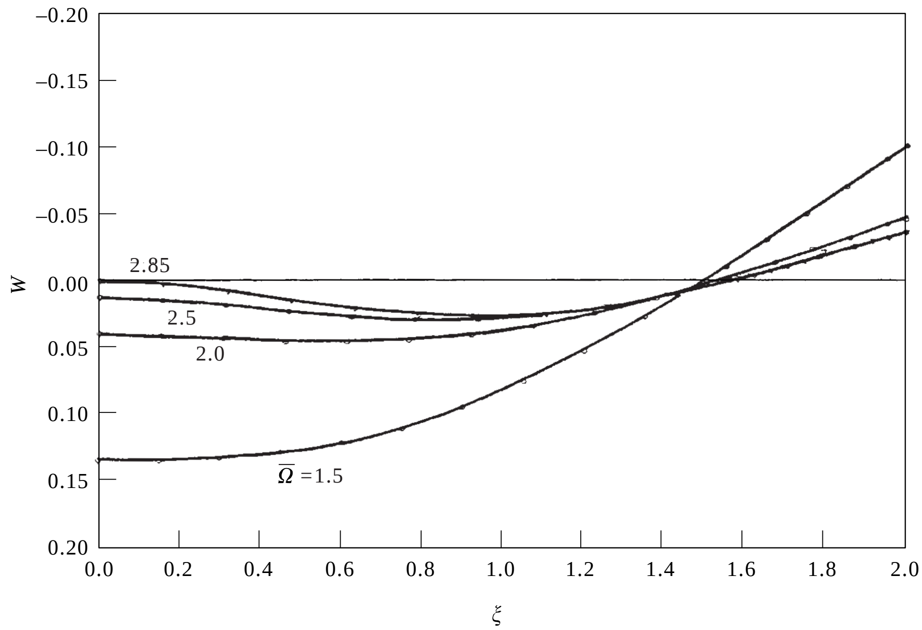


Figure 8. Elastic curves of the beam when forcing is upwards.

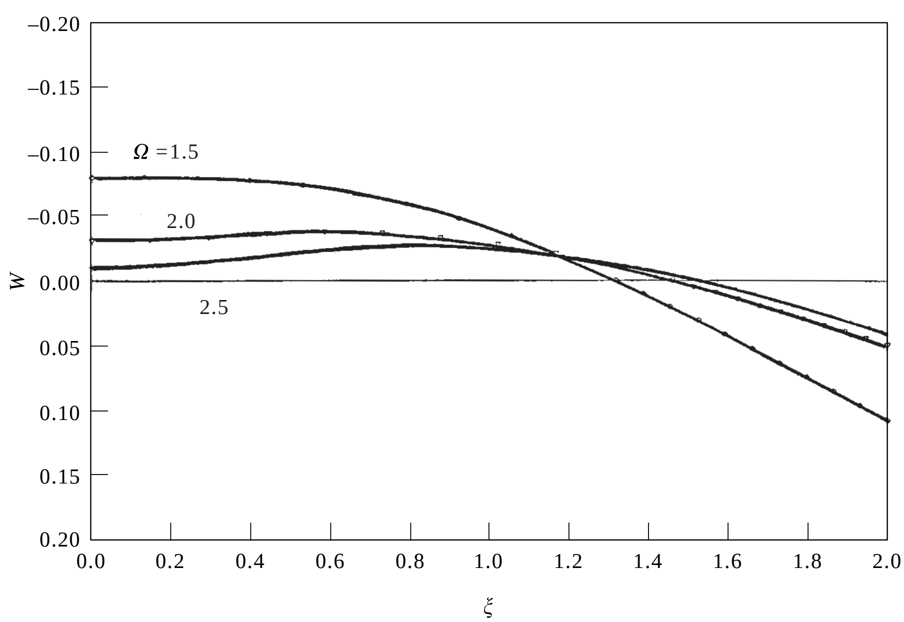


Figure 9. Elastic curves of the beam when forcing is downwards.

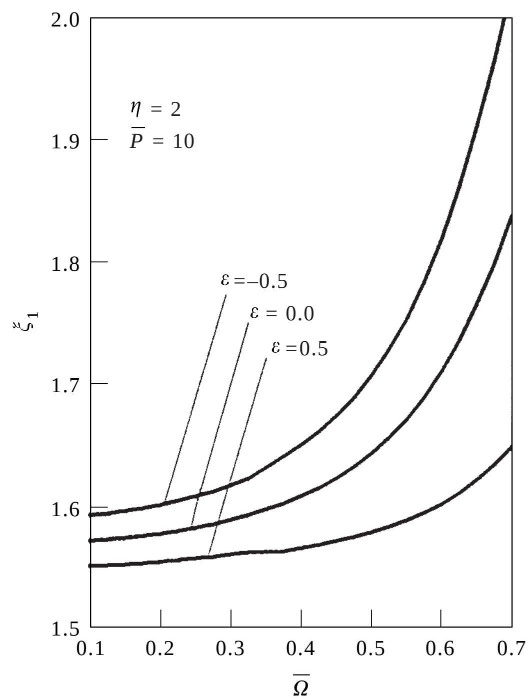


Figure 10. Lift-off points of the beam versus frequency ratio for non-linear case.

where $\theta_{10}^4 = 4(\bar{\Omega}^2 - 1)$, $\theta_{20}^4 = 4\bar{\Omega}^2$, $\theta_{11}^4 = \theta_{10}^4$, $\theta_{21}^4 = \theta_{20}^4$, $\theta_{12}^4 = 4(9\bar{\Omega}^2 - 1)$, $\theta_{22}^4 = 36\bar{\Omega}^2$, $r_1 = 7 + \theta_{12}^4/\theta_{10}^4$.

In the case of downwards loading, the cubic term appears in the solutions $W_{2,1}$ and $W_{2,2}$, and solutions become:

$$W_{1,0} = C_1\, e^{\theta_{10}\xi} + C_2\, e^{-\theta_{10}\xi} + C_3\, e^{i\theta_{10}\xi} + C_4\, e^{-i\theta_{10}\xi} \tag{37}$$

$$W_{2,0} = D_1\, e^{\theta_{20}\xi} + D_2\, e^{-\theta_{20}\xi} + D_3\, e^{i\theta_{20}\xi} + D_4\, e^{-i\theta_{20}\xi} \tag{38}$$

$$W_{1,1} = E_1\, e^{\theta_{11}\xi} + E_2\, e^{-\theta_{11}\xi} + E_3\, e^{i\theta_{11}\xi} + E_4\, e^{-i\theta_{11}\xi} \tag{39}$$

$$W_{1,2} = G_1\, e^{\theta_{12}\xi} + G_2\, e^{-\theta_{12}\xi} + G_3\, e^{i\theta_{12}\xi} + G_4\, e^{-i\theta_{12}\xi} \tag{40}$$

$$\begin{aligned} W_{2,1} = {} & F_1\, e^{\theta_{21}\xi} + F_2\, e^{-\theta_{21}\xi} + F_3\, e^{i\theta_{21}\xi} + F_4\, e^{-i\theta_{21}\xi} \\ & - \frac{k_{11}}{80\theta_{20}^4}\{D_1^3\, e^{3\theta_{20}\xi} + D_2^3\, e^{-3\theta_{20}\xi} \\ & + D_3^3\, e^{3i\theta_{20}\xi} + D_4^3\, e^{-3i\theta_{20}\xi}\} + \frac{3k_{11}}{8\theta_{20}^4(1+3i)}\{D_1^2 D_4\, e^{(2-i)\theta_{20}\xi} \\ & + D_2^2 D_3\, e^{(-2+i)\theta_{20}\xi} + D_3^2 D_1\, e^{(2i+1)\theta_{20}\xi} + D_4^2 D_2\, e^{-(2i+1)\theta_{20}\xi} \\ & + ((1+3i)/(1-3i))[D_1^2 D_3\, e^{(2+i)\theta_{20}\xi} \\ & + D_2^2 D_4\, e^{-(2+i)\theta_{20}\xi} + D_3^2 D_2\, e^{(2i-1)\theta_{20}\xi} + D_4^2 D_1\, e^{(-2i+1)\theta_{20}\xi}]\} \\ & - \frac{3k_{11}}{4\theta_{20}^3}\xi\{(D_1^2 D_2 + 2D_1 D_3 D_4)\, e^{\theta_{20}\xi} - (D_1 D_2^2 + 2D_2 D_3 D_4)\, e^{-\theta_{20}\xi} \\ & + i(D_3^2 D_4 + 2D_1 D_2 D_3)\, e^{i\theta_{20}\xi} - i(D_4^2 D_3 + 2D_1 D_2 D_4)\, e^{-i\theta_{20}\xi}\} \end{aligned} \tag{41}$$

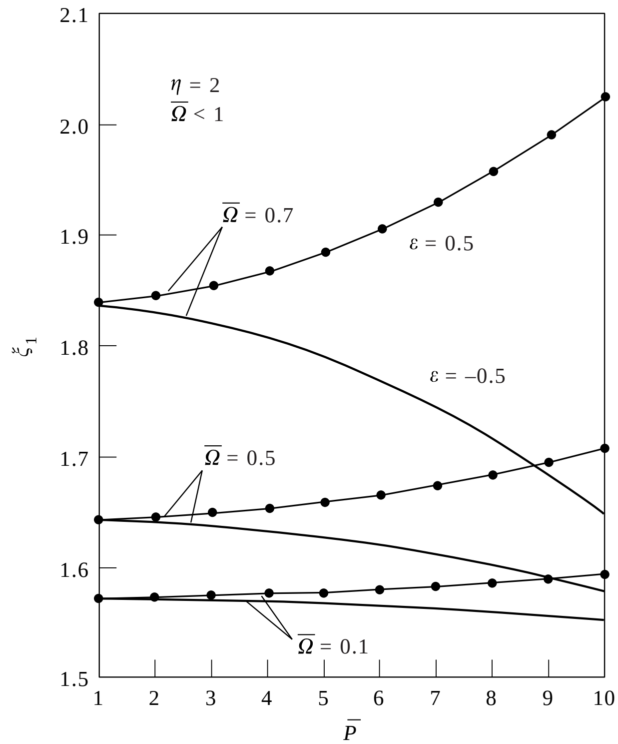


Figure 11. Lift-off points of the beam versus load for non-linear case at various frequency ratios.

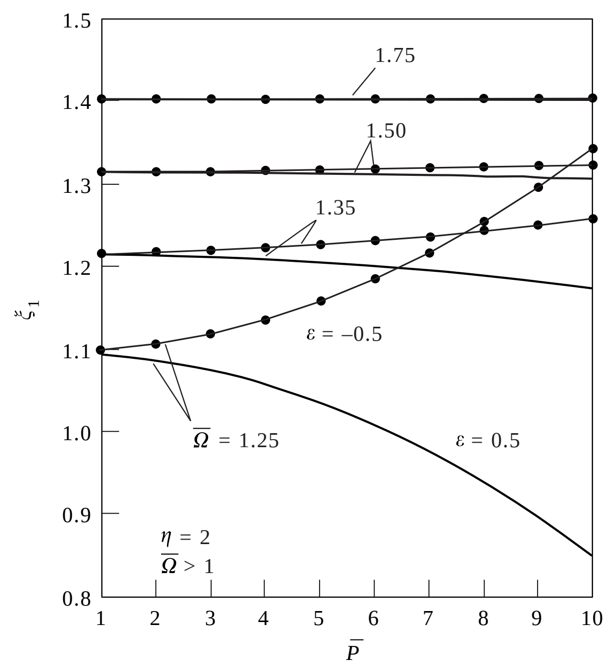


Figure 12. Lift-off points of the beam versus load for non-linear case at various frequency ratios.

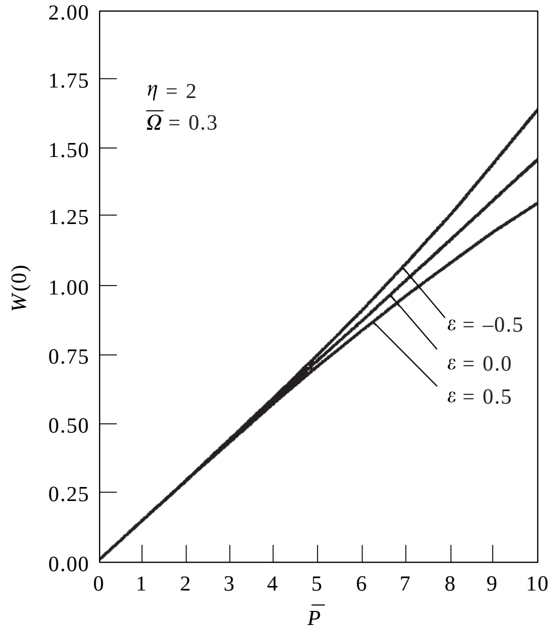


Figure 13. Deflection in the beam centre versus load.

$$
\begin{aligned}
W_{2,2} = {} & H_1\,\mathrm{e}^{\theta_{22}\xi} + H_2\,\mathrm{e}^{-\theta_{22}\xi} + H_3\,\mathrm{e}^{\mathrm{i}\theta_{22}\xi} + H_4\,\mathrm{e}^{-\mathrm{i}\theta_{22}\xi} - \frac{k_{12}}{81\theta_{20}^4 - \theta_{22}^4}\{D_1^3\,\mathrm{e}^{3\theta_{20}\xi} \\
& + D_2^3\,\mathrm{e}^{-3\theta_{20}\xi} + D_3^3\,\mathrm{e}^{3\mathrm{i}\theta_{20}\xi} + D_4^3\,\mathrm{e}^{-3\mathrm{i}\theta_{20}\xi}\} - \frac{3k_{12}}{\theta_{20}^4 - \theta_{22}^4}\{(D_1^2 D_2 \\
& + 2D_1 D_3 D_4)\,\mathrm{e}^{\theta_{20}\xi} + (D_1 D_2^2 \\
& + 2D_2 D_3 D_4)\,\mathrm{e}^{-\theta_{20}\xi} + (D_3^2 D_4 + 2D_1 D_2 D_3)\,\mathrm{e}^{\mathrm{i}\theta_{20}\xi} \\
& + (D_4^2 D_3 + 2D_1 D_2 D_4)\,\mathrm{e}^{-\mathrm{i}\theta_{20}\xi}\} \\
& - \frac{3k_{12}}{\theta_{20}^4(24\mathrm{i} + r_1)}\{-D_1^2 D_4\,\mathrm{e}^{(2-\mathrm{i})\theta_{20}\xi} \\
& - D_2^2 D_3\,\mathrm{e}^{(-2+\mathrm{i})\theta_{20}\xi} - D_3^2 D_1\,\mathrm{e}^{(2\mathrm{i}+1)\theta_{20}\xi} \\
& - D_4^2 D_2\,\mathrm{e}^{-(2\mathrm{i}+1)\theta_{20}\xi} + ((24\mathrm{i} + r_1)/(24\mathrm{i} - r_1))[D_1^2 D_3\,\mathrm{e}^{(2+\mathrm{i})\theta_{20}\xi} \\
& + D_2^2 D_4\,\mathrm{e}^{-(2+\mathrm{i})\theta_{20}\xi} + D_3^2 D_2\,\mathrm{e}^{(2\mathrm{i}-1)\theta_{20}\xi} + D_4^2 D_1\,\mathrm{e}^{(-2\mathrm{i}+1)\theta_{20}\xi}]\}
\end{aligned}
\tag{42}
$$

where $\theta_{10}^4 = 4\bar{\Omega}^2$, $\theta_{20}^4 = 4(\bar{\Omega}^2 - 1)$, $\theta_{11}^4 = \theta_{10}^4$, $\theta_{21}^4 = \theta_{20}^4$, $\theta_{12}^4 = 36\bar{\Omega}^2$, $\theta_{22}^4 = 4(9\bar{\Omega}^2 - 1)$, $r_1 = 7 + \theta_{22}^4/\theta_{20}^4$.

## 4. NUMERICAL RESULTS AND DISCUSSION

The first terms in the perturbation expansions for the vertical displacements and the lift-off points denote the linear behaviour of the beam, and the other terms display the effect of the non-linearity. Firstly, the behaviour of the beam is investigated taking into account the linear foundation. In this case, even though the governing equations are linear, due to a transcendent equation obtained from the boundary and continuity conditions which gives us the lift-off points, the behaviour of the beam is not linear. More than one root can be obtained from this equation for the lift-off point for some frequency domains and beam lengths. As a result of this, multiple solutions are obtained in some cases. The lift-off points are dependent on the foundation modulus, physical and geometrical properties of the beam and the forcing frequency. The variation of the position of the lift-off points with respect to the frequency ratio is presented in Figures 2(a) and (b) for some characteristic lengths. In Figure 2(a), when $\eta = 2$ and $\bar{P} = 1$, for instance, a unique solution is obtained in the interval $0 \leqslant \bar{\Omega} \leqslant 0{\cdot}43$, two solutions in the interval $0{\cdot}43 \leqslant \bar{\Omega} \leqslant 0{\cdot}8$ and a unique solution again is obtained in the interval $0{\cdot}8 \leqslant \bar{\Omega} \leqslant 1$. As it is seen in Figure 2(b), the solution domains separate from each other considerably in case of larger characteristic lengths. In addition to this, the marked lines below the solutions represent the resonance case for $0{\cdot}4 \leqslant \bar{\Omega} \leqslant 1$, in Figure 2(a). The vertical displacements around these lines increase a great deal. When $\bar{\Omega} > 1$, solutions can be obtained only if the load and displacement are in opposite directions. The variation of the position of the lift-off points with respect to the frequency ratio when loading is downwards, is presented in Figure 3. For all lengths, as it is seen from the figure, the position of the lift-off points increase as $\bar{\Omega}$ increases. In this case only one solution is obtained and the frequency domain becomes small as $\eta$ increases.

The coordinates of the lift-off points are independent of the magnitude of the load for a fixed $\bar{\Omega}$ and the vertical displacements are directly proportional to the magnitude of the load as it is clear from the boundary conditions, similar to the case of static solution [7, 15]. But for a fixed load, the position of the lift-off points vary depending on the values of $\bar{\Omega}$.

The variation of the position of the lift-off points with respect to the characteristic length $\eta$ when $\bar{\Omega} = 1$, is presented in Figure 4. As it is seen from the figure, the contact length increases as the characteristic length $\eta$ increases.

As it was mentioned before, for a characteristic length, more than one solution (lift-off point) can be obtained for a fixed frequency ratio. The elastic curves of the beam are presented in Figures 5, 6 and 7 for the parameters $\eta = 2$ and $\bar{\Omega} < 1$. Figures 5 and 6 show the elastic curves of the beam for the first and second solutions, respectively, and the foundation is taken to be tensionless. Figure 7 shows the elastic curves of the beam on the foundation that transmits tensile stresses. In these figures, it is observed that the lift-off points and the vertical displacements increase with the increase of $\bar{\Omega}$. The vertical displacements obtained for the tensionless case are larger than for the foundation which transmits tensile stresses. This result is expected, because, in case of the foundation that transmits tensile stresses, the contact region becomes larger than that of the tensionless foundation. In this case, the foundation reaction (i.e., the vertical displacement) of tensionless case becomes larger for the vertical equilibrium of the beam.

The elastic curves of the beam are presented in Figures 8 and 9 for $\eta = 2$ and $\bar{\Omega} > 1$. In Figure 8, the loading is upwards and as the values of $\bar{\Omega}$ increase, the beam deflections decrease. If the values of $\bar{\Omega}$ increase a little more, the centre of the beam lifts off the foundation and the contact and the non-contact regions interchange. When downwards loading is taken into account as in Figure 9, the vertical displacements decrease as the values of $\bar{\Omega}$ increase. If the values of $\bar{\Omega}$ incrase a little more, the centre of the beam touches on the foundation and the regions change once more. The lengths of the contact regions increase for both cases.

The effect of non-linearity on the lift-off points is presented in Figure 10. Here, the first order terms in the perturbation expansions are taken into account. In the case of positive non-linearity, the contact region decreases while it increases for the negative one with respect to the linear case. This is because when the positive sign is used, the foundation becomes stiffer, and the foundation becomes softer when the negative sign is used.

In Figures 11 and 12, the effect of the loading on the position of the lift-off points for positive and negative non-linearities when $\bar{\Omega} < 1$ and $\bar{\Omega} > 1$, respectively, is presented. The position of the lift-off points varies with the magnitude of the load for both cases. The variation of the contact region is proportional to the square of the load $\bar{P}$. This is because the term $\xi_{1,1} = -W_{1,1}(0)/W'_{1,0}(0)$ includes the cubic power of $\bar{P}$ in the numerator and $\bar{P}$ in the denominator.

The effect of $\bar{P}$ and the sign of $\varepsilon$ on the vertical displacements can be observed in Figure 13. Using the linear solution $W_{1,0}$ and the non-linear solutions $W_{1,1}$ and $W_{1,2}$, the vertical displacement of the beam centre is calculated as $w_{1,0}(0) = \{[W_{1,0}(0) + \varepsilon W_{1,1}(0)]^2 + [\varepsilon W_{1,2}(0)]^2\}^{1/2}$. While the vertical displacements are directly proportional to the magnitude of the load for the linear case, in the non-linear case, the displacements are not exactly proportional to the load. The displacements decrease when the positive non-linearity is used and increase when the negative one is used with respect to the linear case. For some detailed and additional results refer to Coskun [20].

## 5. CONCLUSIONS

The response of a finite beam on a non-linear tensionless Winkler foundation subjected to a concentrated dynamic load has been studied. The perturbation technique and the method of separation of variables have been used in the solution for the evaluation of a set of ordinary differential equations instead of the nonlinear partial governing equations. Using the Newton–Raphson method, the coordinate
of the lift-off points and the vertical displacements of the beam in the linear and non-linear cases have been obtained. It is concluded that the position of the lift-off points and the vertical displacements change with the parameter $\bar{\Omega}$, both in the linear and non-linear cases. In contrast to the linear case, the position of the lift-off points change depending on the magnitude of the load and the vertical displacements change with the square of the load in the nonlinear case. The dynamic effect in the linear case, and the dynamic effect and the non-linearity arising from the foundation modulus in the non-linear case affect the variation of the contact lengths and the vertical displacements of the beam.

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## APPENDIX A

The vertical displacement where the beam separates from the foundation is zero at the right side of the first region.

$$w_1(X_1) = 0 \rightarrow w_{1,0}(\xi_1 = \xi_{1,0} + \varepsilon\xi_{1,1}) + \varepsilon w_{1,1}(\xi_1 = \xi_{1,0} + \varepsilon\xi_{1,1}) = 0. \quad \text{(A1)}$$

Using equations (9) and the Taylor expansion, the above equation become:

$$\{W_{1,0}(\xi_{1,0}) + \varepsilon\xi_{1,1} W'_{1,0}(\xi_{1,0}) + \dots\} \cos \Omega t + \varepsilon\{\{W_{1,1}(\xi_{1,0}) + \varepsilon W'_{1,1}(\xi_{1,0}) + \dots\} \cos \Omega t + \{W_{1,2}(\xi_{1,0}) + \varepsilon\xi_{1,1} W'_{1,2}(\xi_{1,0}) + \dots\} \cos 3\Omega t\} = 0. \quad \text{(A2)}$$

Arranging this equation with respect to the perturbation parameter $\varepsilon$ and neglecting the second order terms, the lift-off point is obtained as

$$\xi_{1,1} = -W_{1,1}(\xi_{1,0})/W'_{1,1}(\xi_{1,0}). \quad \text{(A3)}$$

## APPENDIX B

The values of $m_i$ which appear in equation (27) are given as follows:

$$m_1 = C_1^3 + \frac{72}{\theta_{12}^4} C_3^2 C_1 + \frac{120960}{\theta_{12}^8} C_4^2 C_3 \quad \text{(B1)}$$

$$m_2 = \frac{720}{\theta_{12}^4} C_1 C_3 C_4 + \frac{362880}{\theta_{12}^8} C_4^3 \quad \text{(B2)}$$

$$m_3 = 3C_1^2 C_3 + \frac{1080}{\theta_{12}^4} C_4^2 C_1 + \frac{360}{\theta_{12}^4} C_3^3 \quad \text{(B3)}$$

$$m_4 = 3C_1^2 C_4 + \frac{2520}{\theta_{12}^4} C_3^2 C_4 \quad \text{(B4)}$$

$$m_5 = 3C_3^2 C_1 + \frac{5040}{\theta_{12}^4} C_4^2 C_3 \quad \text{(B5)}$$

$$m_6 = 6C_1 C_3 C_4 + \frac{3024}{\theta_{12}^4} C_4^3 \quad \text{(B6)}$$

$$m_7 = C_3^3 + 3C_4^2 C_1 \quad \text{(B7)}$$

$$m_8 = C_3^2 C_4 \quad \text{(B8)}$$

$$m_9 = 3C_4^2 C_3 \quad \text{(B9)}$$

$$m_{10} = C_4^3. \quad \text{(B10)}$$