Journal of Sound and Vibration (1999) **223**(3), 399–412
Article No. jsvi.1998.2129, available online at http://www.idealibrary.com on IDEAL®

NATURAL FREQUENCY AND MODE SHAPE SENSITIVITIES OF DAMPED SYSTEMS: PART I, DISTINCT NATURAL FREQUENCIES

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(*Received* 10 *April* 1996, *and in final form* 3 *December*, 1998)

A procedure for determining the sensitivities of the eigenvalues and eigenvectors of damped vibratory systems with distinct eigenvalues is presented. The eigenpair derivatives of the structural and mechanical damped systems can be obtained consistently by solving algebraic equations with a symmetric coefficient matrix whose order is $(n+1)\times(n+1)$, where n is the number of co-ordinates. The algorithm of the method is very simple and compact. Furthermore, the method can find the exact solutions. As an example of a structural system to verify the proposed method and its possibilities in the case of the proportionally damped system, the finite element model of a cantilever plate is considered, and also a 7-DOF half-car model as a mechanical system in the case of a non-proportionally damped system. The design parameter of the cantilever plate is its thickness, and the design parameter of the car model is a spring. One of the remarkable characteristics of the proposed method is that its numerical stability is established.

1. INTRODUCTION

The dynamic responses of the structural or mechanical systems can be completely identified by obtaining the natural frequencies and mode shapes of the systems. Variations in system parameters lead to changes in these dynamic characteristics and hence in responses. The derivatives of the eigenpairs are useful in design trend studies and for gaining insight into the behavior of physical systems. Using these eigenpair derivatives in large systems can reduce remarkably the cost of reanalyzes. The derivatives of the mode shapes with respect to design parameters are particularly useful in certain analysis and design applications: approximating a new vibration mode shape due to a perturbation in a design parameter, determining the effect of design changes on the dynamic

0022–460X/99/230399 + 14 $30.00/0

behavior of systems [1], and tailoring mode shapes to minimize displacements at certain points on a system [2]. In contrast to computing eigenvalue derivatives where preferred methods exist, there are a number of different methods for calculating mode shape derivatives. The different methods seek to overcome the practical difficulty of solving a singular matrix equation.

Methods for calculating mode shape derivatives include the finite-difference method [3–5], the iterative method [6–9], the modal method [10–18], the modified modal method [19, 20], Nelson's method [21] and Lee and Jung's method [22, 23]. The finite-difference method uses a difference formula to approximate the derivative numerically, which requires calculating the eigenvector at a nominal and at least one perturbed design point. This method is sensitive to round off and truncation errors associated with the step size used. The modal method approximates the mode shape derivatives as a linear combination of mode shapes. This method can be computationally expensive if a large number of modes are needed to represent accurately the mode shape derivative. The modified modal method was developed to reduce the number of modes needed to represent the derivative of mode shapes. Nelson's method is an exact analytical method for calculating mode shape derivatives. This method only requires the knowledge of the eigenvector to be differentiated and is recommended as an efficient solver for calculating the mode shape derivative [18]. However, this method is lengthy and clumsy for programming and is restricted to the eigenvalue problem with only distinct natural frequencies. Nelson's method is extended to the eigenvalue problem with multiple natural frequencies by Dailey [24], however, this method is lengthy and complicated as well. Lee and Jung's method developed recently is an exact analytical method for calculating mode shape derivatives. Furthermore, it is very efficient and simple. For a thorough review of the research in sensitivity methods for finite-dimensional structural problems, the reader may refer to the excellent survey paper by Haftka and Adelman [25].

A number of the prescribed methods can be applied to the damped system; Pomazal and Snyder [26] extended the theory to the complex eigenvalue problem to analyze the effects of adding springs and dampers to viscously damped systems. Hallquist [27] proposed a method for determining the effects of mass modification in viscously damped systems. Recently Zimoch [28] has presented the sensitivity analysis method for determining the dynamic characteristics of mechanical systems to variations in the parameters. The method is applied to conservative as well as non-conservative systems. However, it may be restricted to mechanical systems (lumped systems) with only distinct natural frequencies, and is difficult to apply to systems with multiple natural frequencies.

The proposed method can find the eigenvalue and eigenvector derivatives of the structural and mechanical damped system by solving the algebraic equations with a symmetric coefficient matrix added as a side condition; the proposed algorithm can be applied consistently to the structural and lumped mechanical systems. The algebraic equation may be efficiently solved by the $\mathbf{LDL}^T$ decomposition method [29]. If the derivatives of the stiffness, mass and damping matrices can be found analytically, the proposed method can find the exact eigenpair derivatives. Also the proposed method can be extended to the eigenvalue problem with multiple eigenvalues (refer to Part II).

The second section of this paper presents the sensitivity analysis for finding eigenvalue and eigenvector derivatives of a damped system. The third section presents the numerical stability of the proposed method for the eigenvalue problem with distinct eigenvalues, and the next section presents numerical examples.

2. SENSITIVITY ANALYSIS OF DAMPED SYSTEMS

The equation of motion of a damped system can be expressed as

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\ddot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) = \mathbf{f}(t), \tag{1}$$

where **M**, **C** and **K** are the matrices of mass, damping and stiffness, respectively, and these are order n symmetric matrices. **M** is positive definite and **K** is positive definite or semi-positive definite. **f** is the excitation vector and **y** is the response vector. The solution of the free vibration of equation (1) can be assumed as

$$\mathbf{y}(t) = \mathrm{e}^{\lambda t}\phi. \tag{2}$$

Substituting equation (2) into equation (1) gives

$$(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})\phi = \mathbf{0}, \tag{3}$$

where λ and ϕ are the eigenvalue and eigenvector and both are complex values in general. To determine eigenvalues and eigenvectors, one can use the following identity

$$\mathbf{M}\dot{\mathbf{y}}(t) - \mathbf{M}\dot{\mathbf{y}}(t) = \mathbf{0}. \tag{4}$$

Combining equation (3) and equation (4), the $2n$-dimensional eigenvalue problem can be obtained as

$$\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix} = \lambda \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix}, \tag{5}$$

which can be written conveniently as

$$\mathbf{A}\mathbf{z} = \lambda\mathbf{B}\mathbf{z}, \tag{6}$$

where

$$\mathbf{A} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathbf{z} = \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix}. \tag{7}$$

Note that $(2n)\times(2n)$ matrices **A** and **B** are symmetric, but not positive definite. The eigenvalues and eigenvectors of the complex eigenvalue problem $\mathbf{A}\mathbf{z} = \lambda\mathbf{B}\mathbf{z}$ can be found as shown in references [30–32]. The eigenvalues are λ_i and $\bar{\lambda}_j$, and the corresponding eigenvectors are

$$\mathbf{z}_j = \left\{ \begin{array}{c} \phi_j \\ \lambda_j \phi_j \end{array} \right\} \quad \text{and} \quad \bar{\mathbf{z}}_j \left\{ \begin{array}{c} \bar{\phi}_j \\ \bar{\lambda}_j \bar{\phi}_j \end{array} \right\} \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{8}$$

One can normalize the eigenvectors such as

$$\mathbf{z}_j^T \mathbf{A} \mathbf{z}_j = \left\{ \begin{array}{c} \phi_j \\ \lambda_j \phi_j \end{array} \right\}^T \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \left\{ \begin{array}{c} \phi_j \\ \lambda_j \phi_j \end{array} \right\} = \lambda_j, \tag{9}$$

$$\mathbf{z}_j^T \mathbf{B} \mathbf{z}_j = \left\{ \begin{array}{c} \phi_j \\ \lambda_j \phi_j \end{array} \right\}^T \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \left\{ \begin{array}{c} \phi_j \\ \lambda_j \phi_j \end{array} \right\} = 1. \tag{10}$$

Suppose that eigenpairs and matrices $\partial K/\partial p$, $\partial M/\partial p$ and $\partial C/\partial p$ are known where p is a design parameter and all eigenvalues are different, we will present the analysis method for calculating the derivative of eigenvalues and eigenvectors in the rest of this section.

Reconsidering the eigenvalue problem equation (3) for the jth eigenmode,

$$(\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}) \phi_j = \mathbf{0}. \tag{11}$$

To obtain an equation for derivatives of eigenvalue and eigenvector, equation (11) is differentiated with respect to a design parameter p, then

$$(\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}) \frac{\partial \phi_j}{\partial p} = -(2\lambda_j \mathbf{M} + \mathbf{C}) \phi_j \frac{\partial \lambda_j}{\partial p} - \left(\lambda_j^2 \frac{\partial \mathbf{M}}{\partial p} + \lambda_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right) \phi_j. \tag{12}$$

Premultiplying at each side of equation (12) by ϕ_j^T, the eigenvalue derivative can be obtained as

$$\frac{\partial \lambda_j}{\partial p} = -\phi_j^T \left(\lambda_j^2 \frac{\partial \mathbf{M}}{\partial p} + \lambda_j \frac{\partial \mathbf{C}}{\partial p} + \frac{\partial \mathbf{K}}{\partial p} \right) \phi_j. \tag{13}$$

The above equation gives the derivative of the eigenvalue, directly, and now the right side of equation (12) is all known but the eigenvector derivative $\partial \phi_j / \partial p$ cannot be found directly since the matrix $\lambda_j^T \mathbf{M} = \lambda_i \mathbf{C} + \mathbf{K}$ is singular. To overcome this singularity problem and to find the eigenvector derivative, a number of numerical methods have been developed by many researchers: the iterative method [6–9], the algebraic method [6, 22, 23], Nelson's method [21], and the modal method family [10–20]. In this paper the algebraic method for calculating the eigenpair derivatives worked by Lee and Jung [22] is extended for the proportionally and non-proportionally damped systems with distinct eigenvalues and its numerical stability is proved.

The proposed method solves a symmetric linear algebraic equation with side conditions given by differentiating orthonormal conditions. Rewriting the orthonormal condition, equation (10), and arranging it gives

$$\phi_j^t (2\lambda_j \mathbf{M} + \mathbf{C}) \phi_j = 1. \tag{14}$$

Differentiating the normalization condition, equation (14), with respect to the design parameter gives

$$\phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C})\frac{\partial\phi_j}{\partial p}+\frac{1}{2}\phi_j^T\left[2\left(\frac{\partial\lambda_j}{\partial p}\mathbf{M}+\lambda_j\frac{\partial\mathbf{M}}{\partial p}\right)+\frac{\partial\mathbf{C}}{\partial p}\right]\phi_j=0. \tag{15}$$

Equations (12) and (15) may be written as a single matrix equation as

$$\begin{bmatrix}\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K} & (2\lambda_j\mathbf{M}+\mathbf{C})\phi_j\\ \phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C}) & 0\end{bmatrix}\begin{Bmatrix}\dfrac{\partial\phi_j}{\partial p}\\ 0\end{Bmatrix}$$

$$=\begin{Bmatrix}-(2\lambda_j\mathbf{M}+\mathbf{C})\phi_j\dfrac{\partial\lambda_j}{\partial p}-\left(\lambda_j^2\dfrac{\partial\mathbf{M}}{\partial p}+\lambda_i\dfrac{\partial\mathbf{C}}{\partial p}+\dfrac{\partial\mathbf{K}}{\partial p}\right)\phi_j\\ -\dfrac{1}{2}\phi_j^T\left[2\left(\dfrac{\partial\lambda_j}{\partial p}\mathbf{M}+\lambda_j\dfrac{\partial\mathbf{M}}{\partial p}\right)+\dfrac{\partial\mathbf{C}}{\partial p}\right]\phi_j\end{Bmatrix}. \tag{16}$$

Equation (16) is the key idea of the proposed method and the derivative of eigenvector, $\partial\phi_i/\partial p$, can be found directly by solving the algebraic equation. The coefficient matrix on the left side of equation (16) can be decomposed (by means of the $\mathbf{LDL}^T$ decomposition method [29]; $\mathbf{L}$ is a lower triangular matrix, and $\mathbf{D}$ a diagonal matrix) into upper and lower triangular forms and a forward and backward substitution scheme may be used to evaluate the components of $\partial\phi_i/\partial p$.

The procedure of the proposed method in the case of distinct eigenvalues is summarized in Table 1. One can see that the algorithm of the proposed method is very simple and compact. The proposed method has the desirable properties of preserving the band and symmetry of all matrices, and of requiring knowledge of

TABLE 1

The procedure of the proposed method in the case of distinct eigenvalues

(1) Calculate $\dfrac{\partial\lambda_i}{\partial p}=-\phi_j^T\left(\lambda_j^2\dfrac{\partial\mathbf{M}}{\partial p}+\lambda_j\dfrac{\partial\mathbf{C}}{\partial p}+\dfrac{\partial\mathbf{K}}{\partial p}\right)\phi_j$.

(2) Define $\mathbf{A}^*=\begin{bmatrix}\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K} & (2\lambda_j\mathbf{M}+\mathbf{C})\phi_j\\ \phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C}) & 0\end{bmatrix}$.

(3) Compute $\mathbf{f}_j=\begin{Bmatrix}-(2\lambda_j\mathbf{M}+\mathbf{C})\phi_j\dfrac{\partial\lambda_j}{\partial p}-\left(\lambda_j^2\dfrac{\partial\mathbf{M}}{\partial p}+\lambda_j\dfrac{\partial\mathbf{C}}{\partial p}+\dfrac{\partial\mathbf{K}}{\partial p}\right)\phi_j\\ -0{\cdot}5\phi_j^T\left[2\left(\dfrac{\partial\lambda_j}{\partial p}\mathbf{M}+\lambda_j\dfrac{\partial\mathbf{M}}{\partial p}\right)+\dfrac{\partial\mathbf{C}}{\partial p}\right]\phi_j\end{Bmatrix}$.

(4) Compute $\begin{Bmatrix}\dfrac{\partial\phi_j}{\partial p}\\ 0\end{Bmatrix}=[\mathbf{A}^*]^{-1}\mathbf{f}_j$ (using $\mathbf{LDL}^T$ decomposition factorization).

only one eigenpair to be differentiated. Both properties are important in realistic structural problems where the stiffness and mass matrices are of very high-order, since these properties allow the use of efficient storage and solution techniques. The numerical stability of the proposed method in the case of distinct natural frequencies is proved in section 3.

3. NUMERICAL STABILITY OF THE PROPOSED METHOD

To show that the coefficient matrix $\mathbf{A}^*$ is always non-singular, consider another matrix such as $\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}$ where $\mathbf{Y}$ is a nonsingular square matrix of order $(n+1)$. The determinant property, $\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}) = \det(\mathbf{Y}^T)\det(\mathbf{A}^*)\det(\mathbf{Y})$, provides that $\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}) \neq 0$ if and only if $\det(\mathbf{A}^*) \neq 0$ and $\det(\mathbf{Y}) \neq 0$. Therefore, if it is proved that the determinant of $\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}$ is non-zero, then the determinant of matrix $\mathbf{A}^*$ may also be non-zero and $\mathbf{A}^*$ is non-singular.

In this paper, the matrix $\mathbf{Y}$ is assumed as

$$\mathbf{Y} = \begin{bmatrix} \mathbf{\Psi} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{17}$$

where Ψ is a $n \times n$ matrix having arbitrary independent vectors containing the jth eigenvector of the system as its columns, as follows:

$$\mathbf{\Psi} = [\psi_1 \quad \psi_2 \quad \dots \quad \psi_{n-1} \quad \phi_j], \tag{18}$$

where ψ's are arbitrary independent vectors chosen to be independent of ϕ_j. Considering equations (17) and (18), it is clear that the columns of the matrix $\mathbf{Y}$ are all independent vectors. The matrix $\mathbf{Y}$ is non-singular and invertible since it is a set of $(n+1)$ independent vectors. Pre- and post-multiplying $\mathbf{Y}^T$ and $\mathbf{Y}$ to $\mathbf{A}^*$ yields

$$\mathbf{Y}^T\mathbf{A}^*\mathbf{Y} = \begin{bmatrix} \mathbf{\Psi} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}^T \left[\begin{array}{c:c} \lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K} & (2\lambda_j\mathbf{M} + \mathbf{C})\phi_j \\ \hdashline \phi_j^T(2\lambda_j\mathbf{M} + \mathbf{C}) & 0 \end{array}\right] \begin{bmatrix} \mathbf{\Psi} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c:c} \mathbf{\Psi}^T(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{\Psi} & \mathbf{\Psi}^T(2\lambda_j\mathbf{M} + \mathbf{C})\phi_j \\ \hdashline \phi_j^T(2\lambda_j\mathbf{M} + \mathbf{C})\mathbf{\Psi} & 0 \end{array}\right]. \tag{19}$$

It is obvious that the last column and row of the matrix $\mathbf{\Psi}^T(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{\Psi}$ all have zero elements since $(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\phi_j$. That is,

$$\mathbf{\Psi}^T(\lambda_m^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{\Psi} = \begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \tag{20}$$

where $\tilde{\mathbf{A}}$ is a non-zero $(n-1) \times (n-1)$ submatrix. The assumption that λ_i is a distinct eigenvalue of the system provides that the matrices $\lambda_i^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K}$ and $\mathbf{\Psi}^T(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{\Psi}$ of order n have a rank of $n-1$ and they are singular. But $\tilde{\mathbf{A}}$ has a full rank and it is a non-singular matrix, which is given by eliminating the last column and row having all zero elements from $\mathbf{\Psi}^T(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\mathbf{\Psi}$. Therefore, the determinant of $\tilde{\mathbf{A}}$ is non-zero, $\det(\tilde{\mathbf{A}})\neq 0$.

By the normalization condition, the last elements of the column vector $\mathbf{\Psi}^T(2\lambda_j\mathbf{M}+\mathbf{C})\phi_j$ and row vector $\phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{\Psi}$ are unity.

$$\mathbf{\Psi}^T(2\lambda_j\mathbf{M}+\mathbf{C})\phi_j = \begin{Bmatrix}\tilde{\mathbf{b}}\\ 1\end{Bmatrix} \quad \text{and} \quad \phi_j^T(2\lambda_j\mathbf{M}+\mathbf{C})\mathbf{\Psi} = \begin{Bmatrix}\tilde{\mathbf{b}}\\ 1\end{Bmatrix}, \tag{21}$$

where $\tilde{\mathbf{b}}$ is a non-zero vector. Substituting equations (20) and (21) into equation (19) yields

$$\mathbf{Y}^T\mathbf{A}^*\mathbf{Y} = \begin{bmatrix}\tilde{\mathbf{A}} & \mathbf{0} & \tilde{\mathbf{b}}\\ \mathbf{0} & 0 & 1\\ \tilde{\mathbf{b}}^T & 1 & 0\end{bmatrix}. \tag{22}$$

Applying the determinant property of partitioned matrices, the determinant of $\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}$ can be written as follows:

$$\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}) = \det\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\det\left(\tilde{\mathbf{A}} - [\mathbf{0} \quad \tilde{\mathbf{b}}]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0}\\ \tilde{\mathbf{b}}^T\end{bmatrix}\right) \tag{23}$$

or

$$\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y}) = \det(\tilde{\mathbf{A}}) \neq 0. \tag{24}$$

The determinant of $\mathbf{A}^*$ thus is not equal to zero because $\det(\mathbf{Y}^T\mathbf{A}^*\mathbf{Y})\neq 0$. The non-singularity of the matrix $\mathbf{A}^*$ is shown analytically; the numerical stability of the proposed method in the case of distinct natural frequencies is established.

4. NUMERICAL EXAMPLE

To verify the proposed method and its possibilities, two numerical examples are presented. In the first, as an example of a damped system with proportional damping, a cantilever plate is considered, while in the second a half-car modelled 7-DOF is considered to demonstrate an application of the proposed method to a non-proportionally damped system with distinct natural frequencies.

4.1. CANTILEVER PLATE (PROPORTIONALLY DAMPED SYSTEM)

The finite element model of the cantilever plate used in reference [9] is modelled with 36 triangular elements as shown in Figure 1. Each node of the element has three degrees of freedom (z-translation, x-rotation and y-rotation); hence each element has nine degrees of freedom. The number of nodes is 28 and the total degrees of freedom of the structure is 72. For example calculations, Young's modulus is $10{\cdot}5\times 10^5$ N/m^2, the mass density $5{\cdot}88\times 10^{-3}$ kg/m^2 and the Poison's ratio 0·3. The length of the plate is 6 m, width 3 m and thickness 0·01 m.

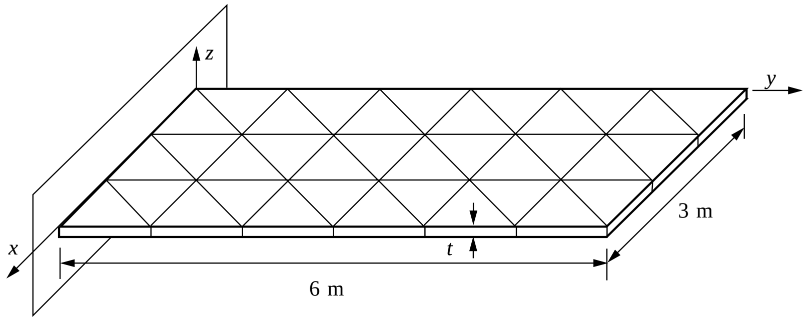

Figure 1. Cantilever plate with the thickness t as the design parameter. Number of nodes: 28; number of elements: 36; number of degrees of freedom: 72; Young's modulus: $E = 10{\cdot}5 \times 10^5$ N/m^2; mass density: $\rho = 5{\cdot}88 \times 10^{-3}$ kg/m^3.

Assume that the damping matrix is a linear combination of the stiffness and mass matrices as

$$\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M} \tag{25}$$

where α and β are the Rayleigh coefficients and $\alpha = \beta = 0{\cdot}01$. The design parameter is the plate thickness t. The stiffness and mass matrices of the structure are proportional to t^3 and t, respectively. The derivatives of the stiffness and mass matrices can be immediately obtained by differentiating them with respect to plate thickness t, and the derivative of the damping matrix by combining them.

Some sensitivity results are represented in Table 2. The lowest 20 natural frequencies and their derivatives of the initial cantilever plate are listed in the second and third columns of Table 2. The fourth and fifth columns represent actual natural frequencies and approximated natural frequencies of a changed system of which the thickness is thicker than the initial plate and the ratio of thickness change to initial thickness is $\Delta t/t = 0{\cdot}01$. The approximated frequencies or eigenvalues, $\bar{\lambda}_{changed}$, are computed by

$$\bar{\lambda}_{changed} = \lambda_{initial} + \frac{\partial\lambda}{\partial t}\Delta t, \tag{26}$$

and the approximated eigenvectors can be computed in the same way. The variations of exact natural frequencies and eigenvectors, which are calculated by $|\lambda_{initial} - \lambda_{changed}|/|\lambda_{initial}|$ and $||\phi_{initial} - \phi_{changed}||_2/||\phi_{initial}||_2$ respectively, are shown in the next two columns. The last two columns are errors of approximations calculated by $|\lambda_{changed} - \bar{\lambda}_{changed}|/|\lambda_{changed}|$ and $||\phi_{changed} - \bar{\phi}_{changed}||_2/||\phi_{changed}||_2$, respectively. Considering the amount of variations of the eigenpair between initial and changed system, the errors of approximated eigenpair computed by using derivatives of the eigenpair given by the proposed method are relatively quite small. Hence, one can say that the proposed method gives very good results.

TABLE 2

The natural frequencies of the initial and changed cantilever plate system, and results of the sensitivity analysis

Mode number	Initial system: Eigenvalue	Initial system: Eigenvalue derivative	Changed system: Eigenvalue	Changed system: Approximated eigenvalue	Variation of eigenpair: Eigenvalue	Variation of eigenpair: Eigenvector	Error of approximation: Eigenvalue	Error of approximation: Eigenvector
1, 2	$-1{\cdot}5367\times10^{-2}$ $\pm j5{\cdot}4528\times10^{0}$	$-2{\cdot}9734\times10^{0}$ $\pm j5{\cdot}4528\times10^{2}$	$-1{\cdot}5666\times10^{-2}$ $\pm j5{\cdot}5073\times10^{0}$	$-1{\cdot}5664\times10^{-2}$ $\pm j5{\cdot}5073\times10^{0}$	$1{\cdot}0000\times10^{-2}$	$9{\cdot}9010\times10^{-3}$	$2{\cdot}7299\times10^{-7}$	$9{\cdot}9980\times10^{-5}$
3, 4	$-2{\cdot}3671\times10^{-1}$ $\pm j2{\cdot}1734\times10^{1}$	$-4{\cdot}7242\times10$ $\pm j2{\cdot}1731\times10^{3}$	$-2{\cdot}4146\times10^{-1}$ $\pm j2{\cdot}1951\times10^{1}$	$-2{\cdot}4143\times10^{-1}$ $\pm j2{\cdot}1951\times10^{1}$	$1{\cdot}0000\times10^{-2}$	$9{\cdot}9006\times10^{-3}$	$1{\cdot}1406\times10^{-6}$	$9{\cdot}9810\times10^{-5}$
5, 6	$-5{\cdot}4802\times10^{-1}$ $\pm j3{\cdot}3087\times10^{1}$	$-1{\cdot}0950\times10^{2}$ $\pm 3{\cdot}3078\times10^{3}$	$-5{\cdot}5902\times10^{-1}$ $\pm j3{\cdot}3418\times10^{1}$	$-5{\cdot}5897\times10^{-1}$ $\pm j3{\cdot}3418\times10^{1}$	$1{\cdot}0001\times10^{-2}$	$9{\cdot}8998\times10^{-3}$	$1{\cdot}6590\times10^{-6}$	$9{\cdot}9870\times10^{-5}$
7, 8	$-2{\cdot}3450\times10^{0}$ $\pm j6{\cdot}8436\times10^{1}$	$-4{\cdot}6890\times10^{2}$ $\pm j6{\cdot}8356\times10^{3}$	$-2{\cdot}3921\times10^{0}$ $\pm j6{\cdot}9120\times10^{1}$	$-2{\cdot}3919\times10^{0}$ $\pm j6{\cdot}9120\times10^{1}$	$1{\cdot}0005\times10^{-2}$	$9{\cdot}8953\times10^{-3}$	$3{\cdot}4309\times10^{-6}$	$9{\cdot}9779\times10^{-5}$
9, 10	$-4{\cdot}2682\times10^{0}$ $\pm 9{\cdot}2288\times10^{1}$	$-8{\cdot}5354\times10^{2}$ $\pm j9{\cdot}2091\times10^{3}$	$-4{\cdot}3540\times10^{0}$ $\pm j9{\cdot}3209\times10^{1}$	$-4{\cdot}3535\times10^{0}$ $\pm j9{\cdot}3209\times10^{1}$	$1{\cdot}0010\times10^{-2}$	$9{\cdot}8905\times10^{-3}$	$4{\cdot}6119\times10^{-6}$	$9{\cdot}9829\times10^{-5}$
11, 12	$-7{\cdot}0234\times10^{0}$ $\pm j1{\cdot}1831\times10^{2}$	$-1{\cdot}4046\times10^{3}$ $\pm j1{\cdot}1789\times10^{4}$	$-7{\cdot}1646\times10^{0}$ $\pm j1{\cdot}1949\times10^{2}$	$-7{\cdot}1639\times10^{0}$ $\pm j1{\cdot}1949\times10^{2}$	$1{\cdot}0017\times10^{-2}$	$9{\cdot}8838\times10^{-3}$	$5{\cdot}9542\times10^{-6}$	$9{\cdot}9561\times10^{-5}$
13, 14	$-8{\cdot}6930\times10^{0}$ $\pm j1{\cdot}3157\times10^{2}$	$-1{\cdot}7385\times10^{3}$ $\pm j1{\cdot}3099\times10^{4}$	$-8{\cdot}8677\times10^{0}$ $\pm j1{\cdot}3287\times10^{2}$	$-8{\cdot}8668\times10^{0}$ $\pm j1{\cdot}3288\times10^{2}$	$1{\cdot}0021\times10^{-2}$	$9{\cdot}8794\times10^{-3}$	$6{\cdot}5905\times10^{-6}$	$9{\cdot}9794\times10^{-5}$
15, 16	$-1{\cdot}1866\times10^{1}$ $\pm j1{\cdot}5359\times10^{2}$	$-2{\cdot}3731\times10^{3}$ $\pm j1{\cdot}5267\times10^{4}$	$-1{\cdot}2104\times10^{1}$ $\pm j1{\cdot}5512\times10^{2}$	$-1{\cdot}2103\times10^{1}$ $\pm j1{\cdot}5512\times10^{2}$	$1{\cdot}0029\times10^{-2}$	$9{\cdot}8715\times10^{-3}$	$7{\cdot}7292\times10^{-6}$	$9{\cdot}9655\times10^{-5}$
17, 18	$-1{\cdot}5690\times10^{1}$ $\pm j1{\cdot}7644\times10^{2}$	$-3{\cdot}1379\times10^{3}$ $\pm j1{\cdot}7505\times10^{4}$	$-1{\cdot}6005\times10^{1}$ $\pm j1{\cdot}7819\times10^{2}$	$-1{\cdot}6004\times10^{1}$ $\pm j1{\cdot}7819\times10^{2}$	$1{\cdot}0039\times10^{-2}$	$9{\cdot}8615\times10^{-3}$	$8{\cdot}8519\times10^{-6}$	$9{\cdot}9999\times10^{-5}$
19, 20	$-2{\cdot}1958\times10^{1}$ $\pm j2{\cdot}0841\times10^{2}$	$-4{\cdot}3915\times10^{3}$ $\pm j2{\cdot}0609\times10^{4}$	$-2{\cdot}2399\times10^{1}$ $\pm j2{\cdot}1047\times10^{2}$	$-2{\cdot}2397\times10^{1}$ $\pm j2{\cdot}1047\times10^{2}$	$1{\cdot}0055\times10^{-2}$	$9{\cdot}8450\times10^{-3}$	$1{\cdot}0501\times10^{-5}$	$1{\cdot}0046\times10^{-4}$

4.2. 7-DOF HALF-CAR MODEL (NON-PROPORTIONALLV DAMPED SYSTEM)

A simple model of a truck used in reference [28] is considered in this second example problem for the non-proportionally damped system and is shown in Figure 2. The truck is modelled as the lumped system with 7-DOF. Only the vibrations in the vertical plane are considered; all the horizontal, rolling and yawing degrees of freedom are suppressed. The components of the mass matrix **M** of the system m_{ij}'s are

$$m_{11} = m_1;\ m_{22} = m_2;\ m_{33} = m_3;\ m_{44} = m_4;\ m_{55} = m_5;\ m_{66} = m_6;$$
$$m_{77} = m_5; \quad \text{and} \quad m_{ij} = 0, \quad \text{if} \quad i \neq j.$$

The components of the stiffness matrix k_{ij}'s are given as

$$k_{11} = k_0 + k_1; \quad k_{14} = -k_1; \quad k_{15} = k_1 z_4; \quad k_{12} = k_{13} = k_{16} = k_{17} = 0;$$
$$k_{22} = k_0 + \tfrac{1}{4}k_2; \quad k_{23} = \tfrac{1}{4}k_2; \quad k_{24} = -\tfrac{1}{2}k_2; \quad k_{25} = -\tfrac{1}{2}k_2(L - z_4);$$
$$k_{26} = k_{27} = 0; \quad k_{33} = k_0 + \tfrac{1}{4}k_2; \quad k_{34} = -\tfrac{1}{2}k_2; \quad k_{33} = -\tfrac{1}{2}k_2(L - z_4);$$

$$k_{36} = k_{37} = 0; \quad k_{44} = k_1 + k_2 + 5k_3; \quad k_{45} = -k_1 z_4 + k_2(L - z_4) + \sum_{t=1}^{5}(z_1^* - z_4)k_3;$$

$$k_{46} = -5k_3; \quad k_{47} = -2k_3\left[\sum_{i=1}^{5}(z_i^* - z_5)\right];$$

$$k_{55} = \left[k_1 z_4^2 + k_2(L - z_4)^2 + \sum_{i=1}^{5}(z_i^* - z_1)^2 k_3\right];$$

$$k_{56} = -k_3\left[\sum_{i=1}^{5}(z_i^* - z_4)\right]; \quad k_{57} = -k_3\left[\sum_{i=1}^{5}(z_i^* - z_4)(z_i^* - z_5)\right]; \quad k_{66} = 5k_3;$$

$$k_{67} = k_3\left[\sum_{i=1}^{5}(z_i^* - z_5)\right]; \quad k_{77} = k_3\sum_{i=1}^{5}(z_i^* - z_5)^2 \quad \text{and} \quad k_{ij} - k_{ij}.$$

The damping matrix **C** has an analogous form to the stiffness matrix: e.g.,

$$c_{11} = c_0 + c_1; \quad c_{14} = -c_1; \quad c_{15} = c_1 z_4; \quad c_{12} = c_{13} = c_{16} = c_{17} = 0;$$
$$c_{22} = c_0 + \tfrac{1}{4}c_2, \quad \text{etc.}$$

The design parameter selected in this example is m_5 which is mass of the container.

Some sensitivity results of the 7-DOF half car model are represented in Table 3. The lowest 14 natural frequencies and their derivatives of the initial cantilever plate are listed in the second and third columns of Table 3. The fourth and fifth columns of the table represent actual natural frequencies and approximated natural frequencies of a changed system of which the mass of the container is more massive than that of the initial model and the ratio of mass change to

TABLE 3

The natural frequencies of the initial and changed half-car system, and results of the sensitivity analysis

Mode number	Initial system: Eigenvalue	Initial system: Eigenvalue derivative	Changed system: Eigenvalue	Changed system: Approximated eigenvalue	Variation of eigenpair: Eigenvalue	Variation of eigenpair: Eigenvector	Error of approximation: Eigenvalue	Error of approximation: Eigenvector
1, 2	$-6{\cdot}9831 \times 10^{-3}$ $\pm j1{\cdot}3424 \times 10^{0}$	$1{\cdot}2809 \times 10^{-6}$ $\mp j1{\cdot}8533 \times 10^{-4}$	$-6{\cdot}9601 \times 10^{-3}$ $\pm j1{\cdot}3390 \times 10^{0}$	$-6{\cdot}9601 \times 10^{-3}$ $\pm j1{\cdot}3390 \times 10^{0}$	$2{\cdot}4773 \times 10^{-3}$	$1{\cdot}2777 \times 10^{-3}$	$7{\cdot}8769 \times 10^{-6}$	$5{\cdot}3031 \times 10^{-6}$
3, 4	$-1{\cdot}9389 \times 10^{-2}$ $\pm j1{\cdot}9168 \times 10^{0}$	$7{\cdot}7061 \times 10^{-7}$ $\mp j5{\cdot}9212 \times 10^{-5}$	$-1{\cdot}9375 \times 10^{-2}$ $\pm j1{\cdot}9157 \times 10^{0}$	$-1{\cdot}9375 \times 10^{-2}$ $-1{\cdot}9157 \times 10^{0}$	$5{\cdot}5644 \times 10^{-4}$	$1{\cdot}6435 \times 10^{-3}$	$3{\cdot}8106 \times 10^{-7}$	$5{\cdot}8904 \times 10^{-6}$
5, 6	$-1{\cdot}3702 \times 10^{-1}$ $\pm j3{\cdot}8022 \times 10^{0}$	$5{\cdot}4075 \times 10^{-5}$ $\mp j6{\cdot}8017 \times 10^{-4}$	$-1{\cdot}3606 \times 10^{-1}$ $\pm j3{\cdot}7900 \times 10^{0}$	$-1{\cdot}3605 \times 10^{-1}$ $\pm j3{\cdot}7899 \times 10^{0}$	$3{\cdot}2023 \times 10^{-3}$	$5{\cdot}2593 \times 10^{-3}$	$2{\cdot}5847 \times 10^{-5}$	$3{\cdot}1281 \times 10^{-5}$
7, 8	$-3{\cdot}8500 \times 10^{-1}$ $\pm j6{\cdot}2351 \times 10^{0}$	$-1{\cdot}6484 \times 10^{-4}$ $\mp j1{\cdot}2880 \times 10^{-3}$	$-3{\cdot}8207 \times 10^{-1}$ $\pm j6{\cdot}2121 \times 10^{0}$	$-3{\cdot}8204 \times 10^{-1}$ $\pm j6{\cdot}2119 \times 10^{0}$	$3{\cdot}7097 \times 10^{-3}$	$5{\cdot}1720 \times 10^{-3}$	$3{\cdot}1903 \times 10^{-5}$	$2{\cdot}3434 \times 10^{-5}$
9, 10	$-7{\cdot}5000 \times 10^{-1}$ $\pm 1{\cdot}2224 \times 10^{1}$	$0{\cdot}0000 \times 10^{0}$ $\mp j0{\cdot}0000 \times 10^{0}$	$-7{\cdot}5000 \times 10^{-1}$ $\pm j1{\cdot}2224 \times 10^{1}$	$-7{\cdot}5000 \times 10^{-1}$ $\pm j1{\cdot}2224 \times 10^{1}$	$0{\cdot}0000 \times 10^{0}$	$0{\cdot}0000 \times 10^{0}$	$0{\cdot}0000 \times 10^{0}$	$0{\cdot}0000 \times 10^{0}$
11, 12	$-9{\cdot}0807 \times 10^{-1}$ $\pm 1{\cdot}4490 \times 10^{1}$	$8{\cdot}1395 \times 10^{-8}$ $\mp j1{\cdot}8114 \times 10^{-7}$	$-9{\cdot}0807 \times 10^{-1}$ $\pm j1{\cdot}4490 \times 10^{1}$	$-9{\cdot}0807 \times 10^{-1}$ $\pm j1{\cdot}4490 \times 10^{1}$	$2{\cdot}4339 \times 10^{-7}$	$1{\cdot}9756 \times 10^{-5}$	$2{\cdot}8206 \times 10^{-9}$	$2{\cdot}3154 \times 10^{-7}$
13, 14	$-1{\cdot}8271 \times 10^{0}$ $\pm j1{\cdot}8661 \times 10^{1}$	$1{\cdot}3796 \times 10^{-8}$ $\mp j2{\cdot}0553 \times 10^{-8}$	$-1{\cdot}8271 \times 10^{0}$ $\pm j1{\cdot}8661 \times 10^{1}$	$-1{\cdot}8271 \times 10^{0}$ $\pm j1{\cdot}8661 \times 10^{1}$	$2{\cdot}3508 \times 10^{-8}$	$6{\cdot}0194 \times 10^{-6}$	$2{\cdot}5593 \times 10^{-10}$	$6{\cdot}5643 \times 10^{-8}$

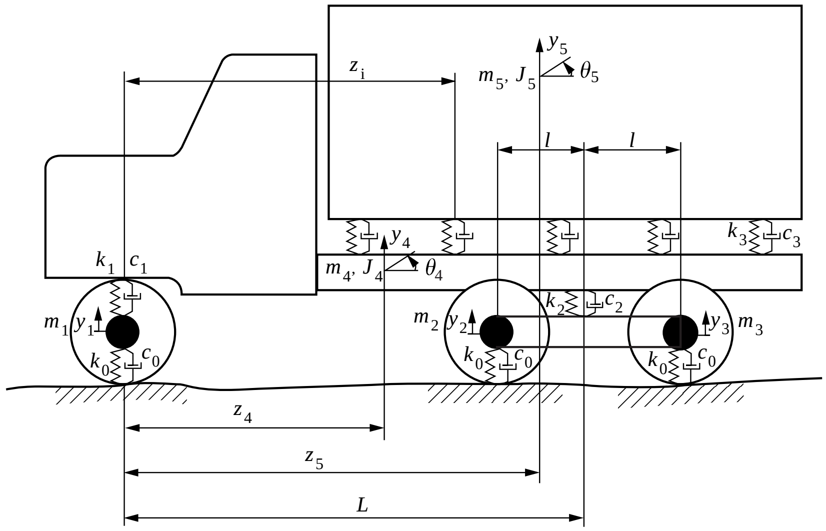

Figure 2. 7-DOF half-car model as a non-proportionally damped system. $z_1^* = 1{\cdot}8$ m, $z_2^* = 2{\cdot}3$ m, $z_3^* = 3{\cdot}8$ m, $z_4^* = 4{\cdot}3$ m, $z_5^* = 5{\cdot}0$ m, $z_4 = 2{\cdot}0$ m, $z_5 = 3{\cdot}0$ m, $L = 3{\cdot}5$, $l = 0{\cdot}85$ m, $m_1 = 75$ kg, $m_2 = m_3 = 80$ kg, $m_4 = 3500$ kg, $m_5 = 1800$ kg, $c_0 = 120$ Ns/m, $c_1 = 150$ Ns/m, $c_2 = 50$ Ns/m, $c_3 = 80$ Ns/m, $k_0 = 12\,000$ N/m, $k_1 = 14\,000$ N/m, $k_2 = 9500$ N/m, $k_3 = 4000$ N/m.

initial mass is $\Delta m_5/m_5 = 0{\cdot}01$. The variations of exact natural frequencies and eigenvectors are shown in the next two columns. The last two columns are errors of approximations. Considering the variations of the eigenpair between the initial and the changed system, the errors of the approximated eigenpair computed by using derivatives of the eigenpair given by the proposed sensitivity analysis method are relatively quite small. Thus, one can say that the proposed method gives very good results.

The theory of the proposed method and its possibilities are demonstrated through the examples. The proposed method can be applied very well to both the proportionally and non-proportionally damped systems.

5. CONCLUSIONS

This paper proposes an efficient numerical method whose stability is proved for calculation of the derivatives of natural frequencies and the corresponding mode shapes of the structural and mechanical damped system with distinct natural frequencies. The proposed method can find the derivatives of eigenvalues and eigenvectors in both proportionally and non-proportionally damped systems. The method has the desirable properties of preserving the band and symmetry of the system matrices and of requiring knowledge of only one eigenpair to be differentiated. The algorithm of the method can be added easily to the commercial FEM code because its numerical stability is guaranteed and gives exact solutions.

ACKNOWLEDGMENTS

This research was supported by the Korea Science and Engineering Foundation, and Korea Industrial Co. Ltd. The support of the Korea Science and Engineering Foundation (203-014-2), and Korea Industrial Co. Ltd are greatly appreciated.

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