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ALTERNATIVE FORMULATIONS OF THE CHARACTERISTIC EQUATION OF A BERNOULLI–EULER BEAM TO WHICH SEVERAL VISCOUSLY DAMPED SPRING–MASS SYSTEMS ARE ATTACHED IN-SPAN

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1. INTRODUCTION

The study in reference [1] was considered with the derivation of alternative forms of the frequency equation of a cantilevered Bernoulli–Euler beam (primary system) to which several spring–mass systems (secondary systems) are attached in-span. The present note is in some sense an extension of reference [1] because it is aimed here to derive the characteristic equation for the case in which the attached secondary systems are viscously damped. Two alternative formulations of the characteristic equation of the combined system are given. Both formulations are based on the discretization of the elastic beam by its first n eigenfunctions, according to the assumed modes method. In reference [2], a general method for determining the exact undamped natural frequencies and natural modes of vibration, the orthogonality relation for the natural modes and the response to arbitrary excitation for both damped and undamped combined systems is given. The method is based upon Green functions of the vibrating distributed subsystems. The approach may be quite complicated, as the Green function for the elastic structure needs first to be determined, which can be both tedious and time consuming in certain cases.

The following steps should be taken for finding the characteristic values of the system considered in this work by using the method in reference [2]. After writing down the Green function of the primary system, first the roots of a determinant whose elements are in terms of this function should be found; then the exact eigenfrequencies and mode shapes of the undamped combined system should be obtained. After these steps, the approximate characteristic values of the damped combined system should be obtained by solving a general eigenvalue problem whose matrix dimensions depend on the desired accuracy. While forming the elements of the matrices the eigenvalue problem is based on, one requires definite integrals of the squares of the mode shapes mentioned above. Numerical problems encountered during the study of the example system when using the above method will be mentioned later.

In the first of the methods presented here, characteristic values are obtained by equating a determinant of coefficients to zero, while in the second method they are obtained as the eigenvalues of a matrix. The comparison of the two

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methods clearly shows that finding the eigenvalues of a matrix is by far a numerically less problematic procedure than finding the zeros of a determinant. On the other hand, the fact that the characteristic values obtained by solving the eigenvalue problem are very close to the exact values for an example system indicates the effectiveness of the proposed method.

Although the formulation presented in this study is applied to a cantilever the approach is valid and applicable for any combinations of boundary conditions.

2. FIRST ALTERNATIVE FORM OF THE CHARACTERISTIC EQUATION

It was shown in reference [1] that the kinetic and potential energies of the system in Figure 1 can be expressed as

$$T=\tfrac{1}{2}\sum_{i=1}^{n}\dot{\eta}_i^2+\tfrac{1}{2}\sum_{j=1}^{s}m_{e_j}\dot{z}_j^2,\quad V=\tfrac{1}{2}\sum_{i=1}^{n}\omega_i^2\eta_i^2+\tfrac{1}{2}\sum_{j=1}^{s}k_{e_j}(z_j-\delta_j)^2, \tag{1}$$

where overdots denote derivatives with respect to time t. Here, $\eta_i(t)$ $(i = 1, \ldots, n)$ are the generalized co-ordinates. δ_j denotes the lateral displacement of the attachment point of the jth damped spring–mass system to the beam while z_j represents the displacement of the mass m_{e_j}. Finally, ω_i is the ith bending eigenfrequency of the bare fixed–free Bernoulli–Euler beam.

The Rayleigh dissipation function is

$$F=\tfrac{1}{2}\sum_{j=1}^{s}d_{e_j}(\dot{z}_j-\dot{\delta}_j)^2. \tag{2}$$

In the first alternative formulation the approach of Dowell [3] is used, which was also employed in reference [1]. The approach is essentially based on the assumed modes method in conjunction with the Lagrange multipliers method.

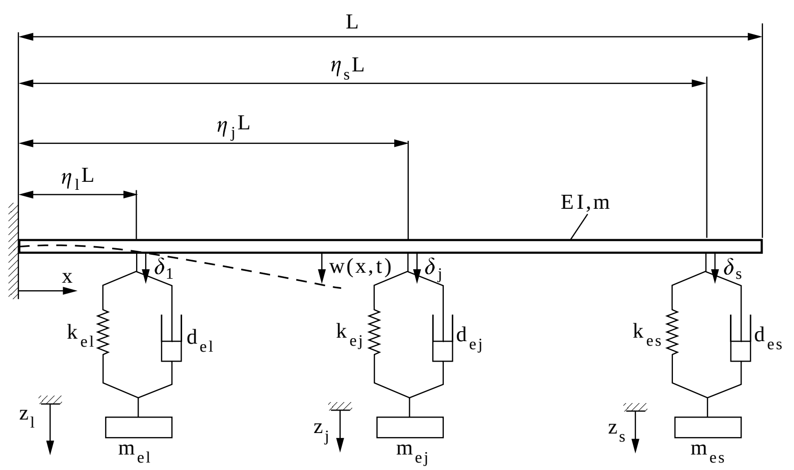

Figure 1. Clamped–free Bernoulli–Euler beam to which s viscously damped spring–mass systems are attached in-span.

The result is a determinantal equation for the characteristic equation of the system. Hence, the characteristic values of the system are obtained by solving this equation numerically.

For a viscously damped system with n degrees of freedom where ν redundant co-ordinates are used, Lagrange's equations in connection with Lagrange's multipliers are [4]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial F}{\partial \dot{q}_k} = \sum_{\ell=1}^{\nu} \lambda_\ell \frac{\partial f_\ell}{\partial q_k}, \quad k = 1, \ldots, n+\nu, \tag{3}$$

with the kinetic potential $L = T\text{-}V$ and ν constraint equations

$$f_\ell(t;\ q_1, \ldots, q_{n+\nu}) = 0, \quad \ell = 1, \ldots, \nu. \tag{4}$$

Here λ_ℓ denotes the ℓth Lagrangian multiplier. In the present case, there are s constraint equations

$$f_j = \sum_{k=1}^{n} w_k(\eta_j L)\eta_k(t) - \delta_j(t) = 0, \quad j = 1, \ldots, s, \tag{5}$$

where $w_k(x)$ represents the kth orthonormalized eigenfunction of the fixed–free Bernoulli–Euler beam.

The evaluation of Lagrange's equations (3) by considering the expressions (1) and (5) results in the equations

$$\ddot{\eta}_k + \omega_k^2 \eta_k = \sum_{l=1}^{s} \lambda_l w_k(\eta_l L), \quad k = 1, \ldots, n, \tag{6}$$

$$d_{e_j}(\dot{z}_j - \dot{\delta}_j) + k_{e_j}(z_j - \delta_j) = \lambda_j, \quad j = 1, \ldots, s, \tag{7}$$

$$m_{e_j}\ddot{z}_j + d_{e_j}(\dot{z}_j - \dot{\delta}_j) + k_{e_j}(z_j - \delta_j) = 0, \quad j = 1, \ldots, s. \tag{8}$$

The substitution of the exponential solutions

$$\eta_k = \bar{\eta}_k\,\mathrm{e}^{\lambda t}, \quad k = 1, \ldots, n, \quad \delta_j = \bar{\delta}_j\,\mathrm{e}^{\lambda t}, \quad z_j = \bar{z}_j\,\mathrm{e}^{\lambda t}, \quad \lambda_j = \bar{\lambda}_j\,\mathrm{e}^{\lambda t}, \quad j = 1, \ldots, s, \tag{9}$$

into equations (6)–(8) and (5), after rearrangement results in

$$\bar{\eta}_k = \sum_{\ell=1}^{s} \bar{\lambda}_\ell w_k(\eta_\ell L)/(\lambda^2 + \omega_k^2), \quad k = 1, \ldots, n, \tag{10}$$

$$d_{e_j}\lambda(\bar{z}_j - \bar{\delta}_j) + k_{e_j}(\bar{z}_j - \bar{\delta}_j) = \bar{\lambda}_j, \quad (m_{e_j}\lambda^2 + d_{e_j}\lambda + k_{e_j})\bar{z}_j = (d_{e_j}\lambda + k_{e_j})\bar{\delta}_j, \quad (11,\ 12)$$

and

$$\sum_{k=1}^{n} w_k(\eta_j L)\bar{\eta}_k - \bar{\delta}_j = 0, \quad j = 1, \ldots, s. \tag{13}$$

From equation (12)

$$\bar{z}_j = -\bar{\lambda}_j / m_{e_j}\lambda^2 \tag{14}$$

can be obtained which, when put into equation (11), yields

$$\bar{\delta}_j = -(m_{e_j}\lambda^2 + d_{e_j}\lambda + k_{e_j})/[m_{e_j}\lambda^2(d_{e_j}\lambda + k_{e_j})]\bar{\lambda}_j, \quad j = 1, \ldots, s. \tag{15}$$

If the above equations and equations (10) are substituted into the constraint equations (13), the following set of s homogeneous equations for $\bar{\lambda}_j$ are obtained:

$$\sum_{k=1}^{n} w_k(\eta_j L) \frac{\left[\sum_{k=1}^{n} \bar{\lambda}_\ell w_k(\eta_\ell L)\right]}{\lambda^2 + \omega_k^2} + \frac{(m_{e_j}\lambda^2 + d_{e_j}\lambda + k_{e_j})}{m_{e_j}\lambda^2(d_{e_j}\lambda + k_{e_j})}\bar{\lambda}_j = 0, \quad j = 1, \ldots, s. \tag{16}$$

A non-trivial solution of this set is possible if the determinant of the coefficients vanishes. This in turn leads to the following characteristic equation of the mechanical system shown in Figure 1, the equation is written out explicitly in order to reflect the symmetry properties better

$$\begin{vmatrix}
\sum_{k=1}^{n}\left[\frac{w_k^2(\eta_1 L)}{\lambda^2+\omega_k^2} + \frac{(m_{e_1}\lambda^2 + d_{e_1}\lambda + k_{e_1})}{m_{e_1}\lambda^2(d_{e_1}\lambda + k_{e_1})}\right] & \sum_{k=1}^{n}\frac{w_k(\eta_1 L)w_k(\eta_2 L)}{\lambda^2+\omega_k^2} & \cdots & \sum_{k=1}^{n}\frac{w_k(\eta_1 L)w_k(\eta_s L)}{\lambda^2+\omega_k^2} \\
\sum_{k=1}^{n}\frac{w_k(\eta_2 L)w_k(\eta_1 L)}{\lambda^2+\omega_k^2} & \left[\sum_{k=1}^{n}\frac{w_k^2(\eta_2 L)}{\lambda^2+\omega_k^2} + \frac{(m_{e_2}\lambda^2 + d_{e_2}\lambda + k_{e_2})}{m_{e_2}\lambda^2(d_{e_2}\lambda + k_{e_2})}\right] & \cdots & \sum_{k=1}^{n}\frac{w_k(\eta_2 L)w_k(\eta_s L)}{\lambda^2+\omega_k^2} \\
\vdots & \vdots & & \\
\sum_{k=1}^{n}\frac{w_k(\eta_s L)w_k(\eta_1 L)}{\lambda^2+\omega_k^2} & \sum_{k=1}^{n}\frac{w_k(\eta_s L)w_k(\eta_2 L)}{\lambda^2+\omega_k^2} & \cdots & \left[\sum_{k=1}^{n}\frac{w_k^2(\eta_s L)}{\lambda^2+\omega_k^2} + \frac{(m_{e_s}\lambda^2 + d_{e_s}\lambda + k_{e_s})}{m_{e_s}\lambda^2(d_{e_s}\lambda + k_{e_s})}\right]
\end{vmatrix} = 0. \tag{17}$$

For further investigations, it is more suitable to rewrite the characteristic equation above in terms of non-dimensional quantities as

$$\begin{vmatrix} \left[\sum_{k=1}^{n}\frac{a_{k_1}^2}{\lambda^{*2}+\lambda_k}+\frac{\lambda^{*2}+2D_{e_1}\sqrt{(\alpha_{ke_1}/\alpha_{me_1})}\lambda^{*}+\alpha_{ke_1}/\alpha_{me_1}}{\lambda^{*2}(2D_{e_1}\sqrt{\alpha_{ke_1}\alpha_{me_1}}\lambda^{*}+\alpha_{ke_1})}\right] & \sum_{k=1}^{n}\frac{a_{k_2}a_{k_1}}{\lambda^{*2}+\lambda_k} & \sum_{k=1}^{n}\frac{a_{k_1}a_{k_s}}{\lambda^{*2}+\lambda_k} \\ \sum_{k=1}^{n}\frac{a_{k_s}a_{k_2}}{\lambda^{*2}+\lambda_k} & \left[\sum_{k=1}^{n}\frac{a_{k_2}^2}{\lambda^{*2}+\lambda_k}+\frac{\lambda^{*2}+2D_{e_2}\sqrt{(\alpha_{ke_2}/\alpha_{me_2})}\lambda^{*}+\alpha_{ke_2}/\alpha_{me_2}}{\lambda^{*2}(2D_{e_2}\sqrt{\alpha_{ke_2}\alpha_{me_2}}\lambda^{*}+\alpha_{ke_2})}\right] & \sum_{k=1}^{n}\frac{a_{k_2}a_{k_s}}{\lambda^{*2}+\lambda_k} \\ \sum_{k=1}^{n}\frac{a_{k_s}a_{k_1}}{\lambda^{*2}+\lambda_k} & \sum_{k=1}^{n}\frac{a_{k_s}a_{k_2}}{\lambda^{*2}+\lambda_k} & \left[\sum_{k=1}^{n}\frac{a_{k_s}^2}{\lambda^{*2}+\lambda_k}+\frac{\lambda^{*2}+2D_{e_s}\sqrt{(\alpha_{ke_s}/\alpha_{me_s})}\lambda^{*}+\alpha_{ke_s}/\alpha_{me_s}}{\lambda^{*2}(2D_{e_s}\sqrt{\alpha_{ke_s}\alpha_{me_s}}\lambda^{*}+\alpha_{ke_s})}\right] \end{vmatrix} = 0. \quad (18)$$

Here, the following abbreviations are used:

$$\bar{\beta}_k = \beta_k L, \quad \bar{\beta}_1 = 1{\cdot}875104, \quad \bar{\beta}_2 = 4{\cdot}694091, \quad \bar{\beta}_3 = 7{\cdot}854757, \ldots,$$

$$\lambda_k = \bar{\beta}_k^4, \quad \omega_k^2 = \lambda_k \omega_0^2, \quad \omega_0^2 = EI/mL^4, \quad w_k(\eta_j L) = (1/\sqrt{mL})a_{kj},$$

$$a_{kj} = \text{ch}\,\bar{\beta}_k\eta_j - \cos\bar{\beta}_k\eta_j - \bar{\eta}_k(\text{sh}\,\bar{\beta}_k\eta_j - \sin\bar{\beta}_k\eta_j),$$

$$\bar{\eta}_k = (\text{ch}\,\bar{\beta}_k + \cos\bar{\beta}_k)/(\text{sh}\,\bar{\beta}_k + \sin\bar{\beta}_k), \quad \lambda^* = \lambda/\omega_0, \quad \alpha_{me_j} = m_{e_j}/mL,$$

$$\alpha_{ke_j} = \frac{k_{e_j}}{(EI/L^3)}, \quad \omega_{e_j}^2 = \frac{k_{e_j}}{m_{e_j}}, \quad D_{e_j} = \frac{d_{e_j}}{2m_{e_j}\omega_{e_j}}. \tag{19}$$

The numerical solution of the determinantal equation above with respect to λ^* yields the dimensionless characteristic values of the combined system shown in Figure 1.

3. SECOND ALTERNATIVE FORM OF THE CHARACTERISTIC EQUATION

The second alternative form of the characteristic equation follows directly from the formalism of the Lagrange's equations where the displacements of the attachment points of the secondary systems to the beam are expressed in terms of the generalized co-ordinates [5]. The formalism leads to a standard eigenvalue problem, the solution of which gives the characteristic values of the system.

The kinetic and potential energies of the mechanical system, i.e., expressions (1) can be written in matrix notation as

$$T = \tfrac{1}{2}\dot{\boldsymbol{\eta}}^{\mathrm{T}}\mathbf{I}_n\dot{\boldsymbol{\eta}} + \tfrac{1}{2}\sum_{j=1}^{s} m_{e_j}\dot{z}_j^2, \quad V = \tfrac{1}{2}\boldsymbol{\eta}^{\mathrm{T}}\boldsymbol{\Omega}^2\boldsymbol{\eta} + \tfrac{1}{2}\sum_{j=1}^{s} k_{e_j}(z_j - \delta_j)^2, \tag{20}$$

where

$$\boldsymbol{\eta}^{\mathrm{T}}(t) = [\eta_1(t), \ldots, \eta_n(t)], \quad \boldsymbol{\Omega}^2 = \mathbf{diag}(\omega_i^2), \quad i = 1, \ldots, n,$$

$$\mathbf{I}_n : n \times n \text{ identity matrix.} \tag{21}$$

Upon considering that the bending displacements of the beam $w(x,t)$ are discretized according to

$$w(x,\, t) = \sum_{i=1}^{n} w_i(x)\eta_i(t) = \mathbf{w}^{\mathrm{T}}(x)\boldsymbol{\eta}(t), \tag{22}$$

the displacements of the attachment points of the secondary systems (damped spring–mass) to the beam i.e., $\delta_j(t)$, $j = 1, \ldots, s$, can be expressed in terms of the generalized co-ordinate vector $\boldsymbol{\eta}(t)$ as

$$\delta_j(t) = w(\eta_j L,\, t) = \mathbf{l}_j^{\mathrm{T}}\boldsymbol{\eta}, \quad j = 1, \ldots, s \tag{23}$$

where

$$\mathbf{l}_j = [w_1(\eta_j L), \ldots, w_n(\eta_j L)]^{\mathrm{T}}. \tag{24}$$

By starting with the expressions (2), (20), along with (21) and (23) the following matrix differential equation is obtained, by using the Lagrange's equation formalism:

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{m}_e \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\eta}} \\ \ddot{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} \mathbf{l}\mathbf{d}_e\mathbf{l}^{\mathrm{T}} & -\mathbf{l}\mathbf{d}_e \\ -(\mathbf{l}\mathbf{d}_e)^{\mathrm{T}} & \mathbf{d}_e \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Omega}^2 + \mathbf{l}\mathbf{k}_e\mathbf{l}^{\mathrm{T}} & -\mathbf{l}\mathbf{k}_e \\ -(\mathbf{l}\mathbf{k}_e)^{\mathrm{T}} & \mathbf{k}_e \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{z} \end{bmatrix} = \mathbf{0}. \tag{25}$$

Here, the following matrices and vectors are introduced:

$$\mathbf{l} = [\mathbf{l}_1, \ldots, \mathbf{l}_s], \quad \mathbf{z} = [z_1, \ldots, z_s]^{\mathrm{T}}, \quad \mathbf{k}_e = \mathbf{diag}(k_{e_i}),$$

$$\mathbf{m}_e = \mathbf{diag}(m_{e_i}), \quad \mathbf{d}_e = \mathbf{diag}(d_{e_i}), \quad i = 1, \ldots, s, \tag{26}$$

and $\mathbf{0}$ denotes a zero matrix or vector of appropriate dimensions. In obtaining the above form of the equation of motion, extensive use is made of the formulas regarding the partial derivatives of bilinear forms, quadratic forms and vectors with respect to algebraic vectors [6].

By means of the transformation

$$\begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{T} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{m}_e^{-1/2} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{y} \end{bmatrix}, \tag{27}$$

where $\mathbf{T} = \mathbf{I}_n$ and $\mathbf{y} = [y_1, \ldots, y_s]^{\mathrm{T}}$, the equations of motion (25) can be written as

$$\begin{bmatrix} \ddot{\mathbf{p}} \\ \ddot{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} \mathbf{e}\mathbf{d}_e\mathbf{e}^{\mathrm{T}} & -\mathbf{e}\mathbf{d}_e\mathbf{m}_e^{-1/2} \\ -\mathbf{d}_e\mathbf{m}_e^{-1/2}\mathbf{e}^{\mathrm{T}} & \mathbf{d}_e\mathbf{m}_e^{-1} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Omega}^2 + \mathbf{e}\mathbf{k}_e\mathbf{e}^{\mathrm{T}} & -\mathbf{e}\mathbf{k}_e\mathbf{m}_e^{-1/2} \\ -\mathbf{k}_e\mathbf{m}_e^{-1/2}\mathbf{e}^{\mathrm{T}} & \boldsymbol{\Omega}_e^2 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{y} \end{bmatrix} = \mathbf{0}. \tag{28}$$

Here, the following abbreviations are introduced:

$$\mathbf{e}_j = \mathbf{T}^{\mathrm{T}}\mathbf{l}_j = \mathbf{l}_j, \quad \omega_{e_j}^2 = k_{e_j}/m_{e_j}, \quad j = 1, \ldots, s,$$

$$\boldsymbol{\Omega}_e^2 = \mathbf{diag}(\omega_{e_j}^2), \quad \mathbf{e} = [\mathbf{e}_1, \ldots, \mathbf{e}_s]. \tag{29}$$

Introducing exponential solutions of the form

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{y}} \end{bmatrix} \mathrm{e}^{\lambda t} \tag{30}$$

results in a set of homogeneous equations for the amplitude vectors $\bar{\mathbf{p}}$ and $\bar{\mathbf{y}}$ where $\bar{\mathbf{p}} = [\bar{p}_1, \ldots, \bar{p}_n]^{\mathrm{T}}$ and $\bar{\mathbf{y}} = [\bar{y}_1, \ldots, \bar{y}_s]^{\mathrm{T}}$. A non-trivial solution of this set is possible only if the determinant of the coefficient matrix vanishes. This condition leads to the following form of the characteristic equation of the mechanical system:

$$\begin{vmatrix} \lambda^2 \mathbf{I}_n + \lambda \mathbf{e}\mathbf{d}_e \mathbf{e}^{\mathrm{T}} + \mathbf{\Omega}^2 + \mathbf{e}\mathbf{k}_e \mathbf{e}^{\mathrm{T}} & -(\lambda \mathbf{e}\mathbf{d}_e + \mathbf{e}\mathbf{k}_e)\mathbf{m}_e^{-1/2} \\ -\mathbf{m}_e^{-1/2}(\lambda \mathbf{d}_e \mathbf{e}^{\mathrm{T}} + \mathbf{k}_e \mathbf{e}^{\mathrm{T}}) & \mathbf{C} \end{vmatrix} = 0, \tag{31}$$

where

$$\mathbf{C} = \mathbf{diag}(\lambda^2 + \lambda d_{e_i}/m_{e_i} + \omega_{e_i}^2), \quad i = 1, \ldots, s. \tag{32}$$

The above form is an alternative presentation of the characteristic equation (17). By using non-dimensional quantities defined in equations (19), this equation can be brought into the form

$$\begin{vmatrix} \lambda^{*^2} \mathbf{I}_n + 2\lambda^* \bar{\mathbf{e}} \mathbf{D}_e (\boldsymbol{\alpha}_{k_e} \boldsymbol{\alpha}_{m_e})^{1/2} \bar{\mathbf{e}}^{\mathrm{T}} + \mathbf{\Lambda} + \bar{\mathbf{e}} \boldsymbol{\alpha}_{k_e} \bar{\mathbf{e}}^{\mathrm{T}} & -(2\lambda^* \bar{\mathbf{e}} \mathbf{D}_e \boldsymbol{\alpha}_{k_e}^{1/2} + \bar{\mathbf{e}} \boldsymbol{\alpha}_{k_e} \boldsymbol{\alpha}_{m_e}^{-1/2}) \\ -(2\lambda^* \bar{\mathbf{e}} \mathbf{D}_e \boldsymbol{\alpha}_{k_e}^{1/2} + \bar{\mathbf{e}} \boldsymbol{\alpha}_{k_e} \boldsymbol{\alpha}_{m_e}^{-1/2})^{\mathrm{T}} & \mathbf{C}^* \end{vmatrix} = 0, \tag{33}$$

where, additional to those given in equations (19), the following abbreviations are introduced:

$$\begin{aligned} &\mathbf{\Lambda} = \mathbf{diag}(\lambda_i), \quad i = 1, \ldots, n, \quad \bar{\mathbf{e}}_i = [a_{1i}, \ldots, a_{ni}]^{\mathrm{T}} \\ &\bar{\mathbf{e}} = [\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n], \quad \boldsymbol{\alpha}_{k_e} = \mathbf{diag}(\alpha_{k_{e_j}}), \quad \boldsymbol{\alpha}_{m_e} = \mathbf{diag}(\alpha_{m_{e_j}}), \\ &\mathbf{C}^* = \mathbf{diag}(\lambda^{*2} + 2\lambda^* D_{e_j} \sqrt{(\alpha_{k_{e_j}}/\alpha_{m_{e_j}})} + (\alpha_{k_{e_j}}/\alpha_{m_{e_j}})) \\ &\mathbf{D}_e = \mathbf{diag}(D_{e_j}), \quad j = 1, \ldots, s. \end{aligned} \tag{34}$$

Equation (33) represents now an alternative form of the characteristic equation (18). This last form enables one to obtain λ^*, i.e., the dimensionless characteristic values as the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{n+s} & \mathbf{I}_{n+s} \\ -\mathbf{C} & -\mathbf{B} \end{bmatrix}, \tag{35}$$

where

$$\mathbf{B} = 2 \begin{bmatrix} \bar{\mathbf{e}} \mathbf{D}_e (\boldsymbol{\alpha}_{k_e} \boldsymbol{\alpha}_{m_e})^{1/2} \bar{\mathbf{e}}^{\mathrm{T}} & -\bar{\mathbf{e}} \mathbf{D}_e \boldsymbol{\alpha}_{k_e}^{1/2} \\ -(\bar{\mathbf{e}} \mathbf{D}_e \boldsymbol{\alpha}_{k_e}^{1/2})^{\mathrm{T}} & \mathbf{D}_e \boldsymbol{\alpha}_{k_e}^{1/2} \boldsymbol{\alpha}_{m_e}^{-1/2} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{\Lambda} + \mathbf{e} \boldsymbol{\alpha}_{k_e} \bar{\mathbf{e}}^{\mathrm{T}} & -\bar{\mathbf{e}} \boldsymbol{\alpha}_{k_e} \boldsymbol{\alpha}_{m_e}^{-1/2} \\ -(\bar{\mathbf{e}} \boldsymbol{\alpha}_{k_e} \boldsymbol{\alpha}_{m_e}^{-1/2})^{\mathrm{T}} & \boldsymbol{\alpha}_{k_e} \boldsymbol{\alpha}_{m_e}^{-1} \end{bmatrix}. \tag{36}$$

The suitability of the above mentioned two alternatives of the characteristic equation from the point of view of numerical calculations will be comparatively considered in the following section.

4. NUMERICAL RESULTS

This section is devoted to the numerical evaluation of the expressions established in the preceding sections. For the numerical applications, the following

TABLE 1

Non-dimensional characteristic values λ^* *of the system in Figure* 1, *with* $s = 1$; $\eta_1 = \alpha_{k_{e_1}} = 0{\cdot}5$, $\alpha_{m_{e_1}} = 1$, $D_e = 0{\cdot}05$ *are chosen*

From equation (A7)	From equation (18)	From matrix (35)
−0·033857 ± 0·698847i	−0·033882 ± 0·698868i	−0·033858 ± 0·698847i
−0·017576 ± 3·549540i	−0·017567 ± 3·549540i	−0·017576 ± 3·549541i
−0·072233 ± 22·057189i	−0·072233 ± 22·057187i	−0·072233 ± 22·057188i
−0·000050 ± 61.697221i	−0·000055 ± 61·697214i	−0·000055 ± 61·697221i
−0·069477 ± 120·905906i	−0·070818 ± 120.905889i	−0·070728 ± 120.905910i

values are chosen for the physical data of the mechanical system in Figure 1. $s = 1$ is chosen, i.e., only one secondary system is considered. $\eta_1 = \alpha_{k_{e_1}} = 0{\cdot}5$, $\alpha_{m_{e_1}} = 1$, $D_e = 0{\cdot}05$. The number of modes n in the expansion (22) is chosen as 10.

The first five pairs of dimensionless eigenvalues λ^* of the system (arranged with respect to the magnitude of the imaginary parts) are given in Table 1. The complex numbers in the first column represent the "exact" characteristic values λ^*, which are obtained from the solution of equation (A7) of the Appendix with MATLAB. The complex numbers in the second column are obtained from the solution of equation (18) with MATLAB. Finally, the numbers in the third column represent simply the eigenvalues of the matrix **A** which are obtained also with MATLAB.

Before comparing the complex numbers in the three columns of Table 1, it is in order to mention the numerical problems encountered during the application of the method in reference [2].

Because no results are available in the literature for the characteristic values of the system in Figure 1 ($s = 1$ case), to the knowledge of the author, it was intended to make comparisons using the results of the method in reference [2]. With the notation used in the above reference, N is selected as 11 and using MATHCAD some results are obtained in which α_6 and α_8, i.e., the sixth and eighth exact dimensionless frequency parameters of the undamped system were not accurate enough. Running the MATLAB programs was successful with accurate results except the difficulties in obtaining α_8. The later stages of the computations were realized in both MATLAB and MATHEMATICA, and the real parts of the characteristic values were found way too small except for the first characteristic value, though the imaginary parts were quite accurately found. Apparently, the example investigated represents a numerically ill-conditioned case. Therefore, the alternative of establishing the exact characteristic equations given in the Appendix was chosen and solved numerically.

The comparison of the complex numbers from the second and third columns indicate clearly that the results of both alternative forms of the characteristic equation are identical. This is nothing else but the numerical justification of the fact, that both alternative forms are identical indeed. On the other hand, the comparison of these values with those from the first column reveals clearly that the two alternative forms of the characteristic equation yield very good approximations to the "exact" characteristic values of the mechanical system in Figure 1.

Although it is well known that generally problems are encountered in finding the roots of transcendental equations, no significant problems were encountered in obtaining the roots of equations (A7) and (18) for selected numerical values. However, it must be stated that this is, to a great extent, due to the fact that the eigenvalues of the matrix **A** can be used as precise starting values. Considering that the solution of an eigenvalue problem gives all eigenvalues of the corresponding matrix simultaneously, the following conclusion can be drawn by inspection of Table 1. When especially the first couple of the characteristic values of the system in Figure 1 are needed, the most practical way to follow is to solve the eigenvalue problem of matrix **A**.

5. CONCLUSIONS

The present study deals with the establishment of two alternative forms of the characteristic equation of a combined system consisting of a clamped–free Bernoulli–Euler beam to which several viscously damped spring–masses are attached in-span. Both formulations are based on the discretization of the elastic beam by its first n eigenfunctions, according to the assumed modes method. One of the alternatives enables one to determine the characteristic values as the roots of a determinantal equation, whereas the second alternative yields the characteristic values as the eigenvalues of a special matrix. Although the formulation presented in this study is applied to a cantilever the approach is valid and applicable for any combinations of boundary conditions.

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APPENDIX

Derivation of the exact characteristic equation of the system in Figure 1 for $s = 1$, i.e., only one secondary system, proceeds as follows. The equations of motion are

$$EIw_i^{IV}(x, t) + m\ddot{w}_i(x, t) = 0, \quad i = 1, 2, \tag{A1}$$

where $w_1(x, t)$ and $w_2(x, t)$ denote the bending displacements within the regions to the left and right of the attachment point of the damped spring–mass to the beam. The corresponding boundary and matching conditions are

$$w_1(0, t) = 0, \quad w_1'(0, t) = 0, \quad w_1(\eta L, t) = w_2(\eta L, t),$$

$$w_1'(\eta L, t) = w_2'(\eta L, t), \quad w_1''(\eta L, t) = w_2''(\eta L, t),$$

$$EIw_1'''(\eta L, t) - EIw_2'''(\eta L, t) - m_e\ddot{z}_1 = 0,$$

$$m_e\ddot{z}_1 - k_e[w_1(\eta L, t) - z_1] - d_e[\dot{w}_1(\eta L, t) - \dot{z}_1] = 0,$$

$$EIw_2''(L, t) = 0, \quad EIw_2'''(L, t) = 0, \tag{A2}$$

where $z_1(t)$ denotes the displacement of the mass m_e, and the subscripts 1 on the system parameters are omitted for simplicity.

Assuming exponential solutions of the form

$$w_j(x, t) = W_j(x)\mathrm{e}^{\lambda t}, \quad z_1(t) = Z_1\,\mathrm{e}^{\lambda t}, \tag{A3}$$

where λ denotes the unknown characteristic value, which is a complex number in general, one obtains, from equations (A1),

$$W_j^{IV}(x) - \beta^4 W_j(x) = 0, \quad j = 1, 2, \tag{A4}$$

where

$$\beta^4 = -m\lambda^2/EI. \tag{A5}$$

The general solutions of the differential equations (A4) are

$$W_1(x) = C_1\,\mathrm{e}^{\beta x} + C_2\,\mathrm{e}^{-\beta x} + C_3\,\mathrm{e}^{\mathrm{i}\beta x} + C_4\,\mathrm{e}^{-\mathrm{i}\beta x},$$

$$W_2(x) = C_5\,\mathrm{e}^{\beta x} + C_6\,\mathrm{e}^{-\beta x} + C_7\,\mathrm{e}^{\mathrm{i}\beta x} + C_8\,\mathrm{e}^{-\mathrm{i}\beta x}, \tag{A6}$$

where C_1–C_8 represent eight integration constants to be determined and $\mathrm{i} = \sqrt{-1}$. Introduction of equations (A6) into the corresponding boundary conditions (A2) yields a set of nine homogeneous equations for $C_1, \ldots, C_8$ and Z_1. For non-trivial solutions to exist, the determinant of the coefficients must vanish, which in turn results in, after simple rearrangements, the characteristic equation for the dimensionless parameter $\bar{\beta}$,

$$\det \bar{\mathbf{A}} = 0, \tag{A7}$$

where the elements of the 9×9 matrix $\bar{\mathbf{A}}$ are defined as

$$
\begin{aligned}
&a_{11} = \mathrm{e}^{-\eta\bar{\beta}}, \quad a_{12} = \mathrm{e}^{\eta\bar{\beta}}, \quad a_{13} = \mathrm{e}^{-\mathrm{i}\eta\bar{\beta}}, \quad a_{14} = \mathrm{e}^{\mathrm{i}\eta\bar{\beta}},\\
&a_{21} = \mathrm{e}^{-\eta\bar{\beta}}, \quad a_{22} = -\mathrm{e}^{\eta\bar{\beta}}, \quad a_{23} = \mathrm{i}\mathrm{e}^{-\mathrm{i}\eta\bar{\beta}}, \quad a_{24} = -\mathrm{i}\mathrm{e}^{\mathrm{i}\eta\bar{\beta}},\\
&a_{31} = a_{32} = a_{33} = a_{34} = 1, \quad a_{35} = a_{36} = a_{37} = a_{38} = -1,\\
&a_{41} = 1, \quad a_{42} = -1, \quad a_{43} = \mathrm{i}, \quad a_{44} = -\mathrm{i}, \quad a_{45} = -1,\\
&a_{46} = 1, \quad a_{47} = -\mathrm{i}, \quad a_{48} = \mathrm{i},\\
&a_{51} = a_{52} = 1, \quad a_{53} = a_{54} = a_{55} = a_{56} = -1, \quad a_{57} = a_{58} = 1,\\
&a_{61} = 1, \quad a_{62} = -1, \quad a_{63} = -\mathrm{i}, \quad a_{64} = \mathrm{i}, \quad a_{65} = -1, \quad a_{66} = 1,\\
&a_{67} = \mathrm{i}, \quad a_{68} = -\mathrm{i}, \quad a_{69} = a,\\
&a_{71} = a_{72} = a_{73} = a_{74} = a_1 + a_2, \quad a_{79} = -(1 + a_1 + a_2),\\
&a_{85} = \mathrm{e}^{(1-\eta)\bar{\beta}}, \quad a_{86} = \mathrm{e}^{-(1-\eta)\bar{\beta}}, \quad a_{87} = -\mathrm{e}^{\mathrm{i}(1-\eta)\bar{\beta}}, \quad a_{88} = -\mathrm{e}^{-\mathrm{i}(1-\eta)\bar{\beta}},\\
&a_{95} = \mathrm{e}^{(1-\eta)\bar{\beta}}, \quad a_{96} = -\mathrm{e}^{-(1-\eta)\bar{\beta}}, \quad a_{97} = -\mathrm{i}\mathrm{e}^{\mathrm{i}(1-\eta)\bar{\beta}}, \quad a_{98} = \mathrm{i}\mathrm{e}^{-\mathrm{i}(1-\eta)\bar{\beta}}. \qquad \text{(A8)}
\end{aligned}
$$

All other elements are zero.

In the above expressions the following abbreviations are introduced,

$$
\begin{aligned}
&\bar{\beta} = \beta L, \quad \alpha_{m_e} = m_e/mL, \quad \alpha_{k_e} = k_e/(EI/L^3),\\
&D_e = d_e/2m_e\omega_e, \quad \omega_e^2 = k_e/m_e, \quad \omega_0^2 = EI/mL^4,\\
&a_1 = -\alpha_{k_e}/\alpha_{m_e}\bar{\beta}^4, \quad a_2 = \pm(2D_e\mathrm{i}/\bar{\beta}^2)\sqrt{\alpha_{k_e}/\alpha_{m_e}}, \quad a = -\alpha_{m_e}\bar{\beta}. \qquad \text{(A9)}
\end{aligned}
$$

The solution of equation (A7) with respect to $\bar{\beta}$ yields, via

$$\lambda^* = \pm\mathrm{i}\bar{\beta}^2 \qquad \text{(A10)}$$

the "exact" values of the unknown non-dimensional complex characteristic values of the system in Figure 1 for $s = 1$.