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AN ASYMPTOTIC STUDY OF THE LINEAR VIBRATIONS OF A STRETCHED BEAM WITH CONCENTRATED MASSES AND DISCRETE ELASTIC SUPPORTS

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An asymptotic analysis based on the homogenization technique in the framework of linear dynamics for an arbitrary range of frequencies has been applied to an infinite one-dimensional (1D) system which consists of elastically supported discrete masses, linked by beams. Three scale regions of eigenfrequencies are found. The first one corresponds to the continuum approach, when the system studied can be described as an effectively continuous homogeneous beam and the corrections are of a higher order of magnitude. The second region corresponds to an antiphase mode of neighboring masses vibrating with slowly varying amplitudes. The highest range of frequencies reflects the short beams vibration between neighboring masses, which are immobile in the first term approach. The completeness of the spectrum analysis is shown. Dispersion relations and peculiarities of the corresponding eigenmodes have been discussed. The system studied admits generalizations and may itself serve as an adequate model for various technical applications: civil engineering, ship building etc.

1. INTRODUCTION

A one-dimensional system which consists of the discrete masses linked by a beam with elastic supports is an adequate dynamic mode in a wide field of technical applications: aeronautics, ship-building; civil, nuclear and rocket engineering. Frequently there is a need for analytical estimations of its eigenfrequencies and eigenmodes. Despite the simple geometry of this model, its spectrum may be rather complicated and an exact analytical solution turns out to be impossible. But certain small parameters, as a rule, can be introduced, so that an asymptotic approach becomes feasible.

In this paper an asymptotic analysis of the system, based on the homogenization technique [1–3] in the framework of linear dynamics is performed. Dispersion relations and peculiarities of the corresponding eigenmodes are discussed. A distinctive feature of the proposed approach is in the study of the arbitrary scale of the frequencies, while the conventional

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approaches are restricted by the low frequency region and long wave length approximation.

2. MATHEMATICAL MODEL AND HOMOGENIZATION PROCEDURE

A standard measure of length, l is introduced in order to study the free vibrations of an infinite periodic beam (Figure 1), stretched by a longitudinal force N, supported by the springs of stiffness $c\varepsilon$ at a distance $l\varepsilon$. $M\varepsilon$ is the value of point-like masses and $\rho\varepsilon$ is one-dimensional density, i.e., the mass of unit length of the beam. The transverse displacement is denoted by u and bending rigidity by EI. By assuming that $\varepsilon \ll 1$, an asymptotic analysis in the limit as $\varepsilon \to 0$ is performed, i.e., the number of supports for l, $n = l/\varepsilon$, tends to infinity.

On can suppose that the beam analyzed is an example of a model of a bridge, where l is the length of the bridge and $l\varepsilon$ the distance between supports. One normally has to specify particular boundary conditions at the ends of the bridge. However, this will not be done in the present analysis since the only goal is to make the analysis clearer. As will be discussed later, boundary conditions in the case of a finite system will only produce the low frequency part of the spectrum. Every material constant may be chosen of arbitrary order in comparison with ε. Selection of the orders of magnitude of the material constants makes the spectrum analysis the most wide. For example, ρ (order 0) instead of $\rho\varepsilon$ (order 1) for 1D density results in the absence of the highest region of frequencies, corresponded to the bending vibration of light beams between almost immobile heavy point-like masses.

In order to exclude the length l of order 0 from further analysis one introduces the dimensionless space variable $x' = x/l$. However, the variables and new material parameters will not be redefined so the dimensionless standard length is assumed to be equal to 1, and ε a dimensionless small parameter. The equation for the motion of the beam system (Figure 1) may be written as

$$\left(\varepsilon\rho + \varepsilon M \sum_{j=-\infty}^{\infty} \delta(x - j\varepsilon)\right)\frac{\partial^2 u}{\partial t^2} + EI\frac{\partial^4 u}{\partial x^4} - N\frac{\partial^2 u}{\partial x^2} + c\varepsilon u \sum_{j=-\infty}^{\infty} \delta(x - j\varepsilon) = 0. \tag{2.1}$$

where t is time, x $(-\infty < x < \infty)$ is a one-dimensional space co-ordinate, ε is the distance between neighboring masses (supports), ρ is the density of the beams, M a point-like mass, EI the bending rigidity, N a constant tensile load and c the stiffness of a support.

As was mentioned in the introduction, complete asymptotic ($\varepsilon \to 0$) analysis of the spectrum is a goal of the paper. The homogenization technique [1, 2] implies

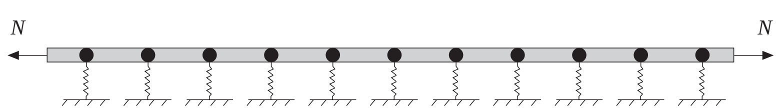

Figure 1. Scheme of the system studied.

the division of the space variable into slow, x, and fast, $y = x/\varepsilon$, ones and substitution into equation (1) the asymptotic series

$$u = \exp\left(\mathrm{i}\,\frac{\omega(\varepsilon)}{\varepsilon^{\alpha}}\,t\right)\sum_{l=0}^{\infty}\varepsilon^{l}u_{l}\left(x, \frac{x}{\varepsilon}\right). \tag{2.2}$$

Because of the limit $\varepsilon \to 0$, the analyzed terms $u_l(x, y)$ should be continuous functions of restricted values. However, usually a stronger assumption of y periodicity is made. Mostly the length of the period Y, $|Y|$, is supposed to be the same as the period of the structure, $|Y| = 1$. A weaker assumption about the integer $|Y|$ will be used. It is convenient further to call the y-periodic function of period $|Y| = P$ as the Py-periodic function.

The purpose of the paper is to study the linear vibration problem. That is why the harmonic time-dependence is assumed. To analyze an arbitrary frequency region, the order α of the frequencies is introduced and the asymptotic series for frequency

$$\omega(\varepsilon) = \sum_{m=0}^{\infty} \omega_m \varepsilon^m \tag{2.3}$$

is assumed. Application to the displacement (2.2) time derivatives will change the magnitude in the following way:

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\omega^2(\varepsilon)}{\varepsilon^{2\alpha}}\,u = -\frac{u}{\varepsilon^{2\alpha}}\sum_{m=0}^{\infty} b_m \varepsilon^m, \qquad b_m = \sum_{i+j=m} \omega_i \omega_j, \tag{2.4}$$

To shorten further the mathematics, it makes sense to introduce the special notation "*inertial*" equations

$$\mathrm{M}_m = M b_m \sum_{j=-\infty}^{\infty} \delta(y - j), \qquad \mathrm{P}_m = \rho b_m; \tag{2.5.1, 2}$$

and "*elastic*" equations

$$A_{-4} = EI\,\partial^4/\partial y^4, \qquad A_{-3} = 4EI\,\partial^4/\partial x\,\partial y^3, \qquad A_{-2} = 6EI\,\partial^4/\partial x^2\,\partial y^2 - N\,\partial^2/\partial y^2, \tag{2.6.1–3}$$

$$A_{-1} = 4EI\frac{\partial^4}{\partial x^3\,\partial y} - 2N\frac{\partial^2}{\partial x\,\partial y}, \qquad A_0 = EI\frac{\partial^4}{\partial x^4} - N\frac{\partial^2}{\partial x^2} + c\sum_{j=-\infty}^{\infty}\delta(y - j) \tag{2.6.4, 5}$$

differential operators in the fast variable y.

After substitution of (2.2) into (2.1) and skipping an oscillator term one obtains a different series of equations for $u_l(x, y)$ which will be dependent on α.

3. ASYMPTOTIC ANALYSIS

Variation of α changes the asymptotic order of the inertial operators and, hence, effects the coupling between inertial (2.5) and elastic (2.6) terms. As will be shown in the Appendix, only three non-empty spectrum domains exist.

3.1. LOW-FREQUENCY REGION ($\alpha = 0$) LONG WAVE MODES

In this region long wave approximations can be applied and in the main term approximation the system analyzed can be observed as an effectively homogeneous beam on a continuous elastic support.

Substituting (2.2) into (2.1), using series (2.3) and notations (2.5), (2.6) one obtains the following system of equations:

$$\varepsilon^{-4}:\quad A_{-4}u_0 = 0, \qquad \varepsilon^{-3}:\quad A_{-3}u_0 + A_{-4}u_1 = 0, \qquad \varepsilon^{-2}:\quad A_{-2}u_0 + A_{-3}u_1 + A_{-4}u_2 = 0, \tag{3.1.1–3}$$

$$\varepsilon^{-1}:\quad A_{-1}u_0 + A_{-2}u_1 + A_{-3}u_2 + A_{-4}u_3 = 0, \tag{3.1.4}$$

$$\varepsilon^{0}:\quad A_0u_0 + A_{-1}u_1 + A_{-2}u_2 + A_{-3}u_3 + A_{-4}u_4 - \mathrm{M}_0u_0 = 0, \tag{3.1.5}$$

$$\varepsilon^{1}:\quad A_0u_1 + A_{-1}u_2 + A_{-2}u_3 + A_{-3}u_4 + A_{-4}u_5 - (\mathrm{M}_0u_1 + \mathrm{M}_1u_0) - \mathrm{P}_0u_0 = 0, \quad \vdots \tag{3.1.6}$$

and so on.

The structure of the operator A_{-4} (see (2.6.1)), the requirement of finite values for u_0 and equation (3.1.1) implies that u_0 is independent of y. This result indicates that one can conclude the same for u_1, using equation (3.1.2) (see (2.5.2)) and sequentially for u_2, u_3, using equations (3.1.3), (3.1.4) and the notations (2.5.3), (2.5.4).

By taking into account the y independence of u_0, u_1, u_2 and u_3 one can regard (3.1.5) as an equation for u_4 with the right side dependent on u_0:

$$EI\frac{\partial^4 u_4}{\partial y^4} = -\left[EI\frac{\partial^4 u_0}{\partial x^4} - N\frac{\partial^2 u_0}{\partial x^2} + (c - M\omega_0^2)\sum_{j=-\infty}^{\infty}\delta(y-j)u_0\right]. \tag{3.2}$$

Obviously, the function u_4 has finite values if and only if its third derivative satisfies the same condition. In turn, this condition is equivalent to the equation for $u_0(x)$:

$$EI\,\partial^4 u_0/\partial x^4 - N\,\partial^2 u_0/\partial x^2 + (c - M\omega_0^2)u_0 = 0. \tag{3.3}$$

Equation (3.3) means that at low frequencies the system analyzed is equivalent to an effectively homogeneous beam of bending rigidity EI, loaded by longitudinal force N on the uniformly distributed transverse elastic support of the stiffness c per unit of length and of distributed mass per unit of the length equal to M (remember that ρ is of lower order of magnitude). The longitudinal load is assumed to be positive, i.e., $N > 0$. With such an assumption equation (3.3) has a finite solution if and only if

$$\omega_0 = \sqrt{b_0} > \sqrt{c/M}. \tag{3.4}$$

So, there is a frequency gap in the long wave range of the spectrum. The corresponding dispersion relation is determined by the following formula

$$\Omega \approx \omega_0 = \sqrt{(EIk^4 + Nk^2 + c)/M}. \tag{3.5}$$

Taking into account (3.3), equation (3.2) can be represented in the form:

$$EI \frac{\partial^4 u_4}{\partial y^4} = (Mb_0 - c)\left(\sum_{j=-\infty}^{\infty} \delta(y-j) - 1\right) u_0(x). \tag{3.6}$$

Using the y-periodic solution $\chi(y)$ of the equation $\mathrm{d}^4\chi/\mathrm{d}y^4 = \delta(y) - 1$, which has a break in the third derivative,

$$\chi(y) = -y^2(1-y)^2/24, \quad \text{at} \quad y \in [0, 1], \tag{3.7}$$

one can represent u_4 as

$$u_4(x, y) = (Mb_0 - c)u_0(x)\chi(y) + v_4(x), \tag{3.8}$$

where v_4 is a y independent unknown function.

Subsequent application of the same chain of arguments to equations (3.1.6), (3.1.7) . . . , will give the next terms of series (2.2) $u_1(x)$, $u_5(x, y)$ . . . , and corrections ω_1 . . . , (series (2.3)) to the frequencies. In particular,

$$\omega_1 = - \sim \omega_0 \rho/2M. \tag{3.9}$$

3.2. MEDIUM FREQUENCY REGION ($\alpha = 2$) TOOTH-LIKE WAVE MODES

In a given frequency scale interaction between "inertial" (2.5) and "elastic" (2.6) terms starts from the first equation of asymptotic series:

$$\varepsilon^{-4}: \quad A_{-4}u_0 - \mathrm{M}_0 u_0 = 0, \qquad \varepsilon^{-3}: \quad A_{-3}u_0 + A_{-4}u_1 - (\mathrm{M}_0 u_1 + \mathrm{M}_1 u_0) - \mathrm{P}_0 u_0 = 0, \tag{3.10.1, 2}$$

$$\varepsilon^{-2}: \quad A_{-2}u_0 + A_{-3}u_1 + A_{-4}u_2 - \sum_{p+q=2} \mathrm{M}_p u_q - \sum_{p+q=1} \mathrm{P}_p u_q = 0, \tag{3.10.3}$$

$$\varepsilon^{-1}: \quad A_{-1}u_0 + A_{-2}u_1 + A_{-3}u_2 + A_{-4}u_3 - \sum_{p+q=3} \mathrm{M}_p u_q - \sum_{p+q=2} \mathrm{P}_p u_q = 0, \tag{3.10.4}$$

$$\varepsilon^{0}: \quad A_0 u_0 + A_{-1}u_1 + A_{-2}u_2 + A_{-3}u_3 + A_{-4}u_4 - \sum_{p+q=4} \mathrm{M}_p u_q - \sum_{p+q=3} \mathrm{P}_p u_q = 0, \tag{3.10.5}$$

$$\varepsilon^{1}: \quad A_0 u_1 + A_{-1}u_2 + A_{-2}u_3 + A_{-3}u_4 + A_{-4}u_5 - \sum_{p+q=5} \mathrm{M}_p u_q - \sum_{p+q=4} \mathrm{P}_p u_q = 0,$$

$$\vdots \tag{3.10.6}$$

and so on.

Obviously, in the family of $1y$-periodic functions equation (3.10.1) has only a trivial solution. However, $1y$-periodicity is not a necessary condition: finite values of u_0 will exist if

$$\left| \sum_{j \geqslant y-a}^{j \leqslant y+a} u_0(j) \right| \leqslant \text{const} \qquad \text{at} \qquad a \to \infty \tag{3.11}$$

uniformly with respect to y. In turn, the condition (3.11) is true if u_0 is a Py-periodic function, $P > 1$, and $u_0(1) + u_0(2) + \cdots + u_0(\mathrm{P}) = 0$. Restricting consideration to $2y$-periodic functions we will introduce the basic one:

$$\psi(y) = \left\{ \begin{array}{ccc} 3(\frac{1}{2} + y) - 4(\frac{1}{2} + y)^3 & \text{at} & -1 \leqslant y \leqslant 0; \\ 3(\frac{1}{2} - y) - 4(\frac{1}{2} - y)^3 & \text{at} & 0 \leqslant y \leqslant 1; \\ \text{periodically extended (with period 2)} & \text{at other} & y; \end{array} \right\}. \tag{3.12}$$

The function $\psi(y)$ satisfies the following conditions: (1) it is $2y$-periodic; (2) $\psi(y)$ and its derivatives up to second order are continuous functions; (3) $\psi(2n) = 1$, $\psi(2n+1) = -1$; (4) $\Delta\psi'''(2n) = 48$, $\Delta\psi'''(2n+1) = -48$ ($\Delta\psi'''$ is the notation for a jump in a third derivative); (5) $\mathrm{d}^4/\mathrm{d}y^4 - 0$ for a non-integer y. Co-ordination between function values and jumps in the third derivative is governed by the main frequency value

$$\omega_0 = \sqrt{48EI/M}. \tag{3.13}$$

So

$$u_0(x, y) = v_0(x)\psi(y) \tag{3.14}$$

is a solution of equation (3.10.1) at arbitrary $v_0(x)$ if and only if (3.13) is true.†

By denoting $\psi_k(y)$ a $2y$-periodic integral of $\psi(y)$ of zero area:

$$\frac{\mathrm{d}^k\psi_k}{\mathrm{d}y^k} = \psi, \qquad \int_Y \psi_k(y)\,\mathrm{d}y = 0 \tag{3.15}$$

and $\psi_{k0} = \psi_k(0)$, $\psi_{k0} \neq 0$ for even k only. For example, $\psi_0(y) = \psi(y)$, $\psi_{00} = 1$,

$$\psi_1(y) = \left\{ \begin{array}{lcc} \frac{3}{2}(\frac{1}{2} + y)^2 - (\frac{1}{2} + y)^4 - \frac{5}{16} & \text{at} & -1 \leqslant y \leqslant 0; \\ -\frac{3}{2}(\frac{1}{2} - y)^2 + (\frac{1}{2} - y)^4 + \frac{5}{16} & \text{at} & 0 \leqslant y \leqslant 1; \end{array} \right\}, \qquad \psi_{10} = 0.$$

Substitution of (3.14) into equation (3.10.1) determines an equation for u_1.

$$EI\frac{\partial^4 u_1}{\partial y^4} = Mb_0 \sum_{j=-\infty}^{\infty} \delta(y-j)u_1 + Mb_1 \sum_{j=-\infty}^{\infty} \delta(y-j)v_0\psi(y)$$
$$+ \rho b_0 v_0(x)\psi(y) - 4EIv_0'\psi'''(y). \tag{3.16}$$

† It should be noted that the value (3.13) of the main frequency is caused by assuming $2y$-periodicity. Analysis based on Py- or quasiperiodic functions will give a different ω_0 value.

The $2y$-periodic solution of equation (3.16) can be represented in the form:

$$u_1(x, y) = v_1(x)\psi(y) + (\rho b_0 v_0(x)/EI)\psi_4(y) - 4v_0'(x)\psi_1(y), \tag{3.17}$$

with arbitrary $v_1(x)$. Co-ordination between the values of the right sides of (3.16) in break points and the jump in the third derivative of $u_1(x, y)$ is the condition which determines the first correction ω_1 to the main frequency:

$$\omega_1 = -0.5\omega_0\psi_{40}\rho/EI. \tag{3.18}$$

Substitution of (3.14) and (3.17) into the third equation of asymptotic series (3.10) form the equation for $u_2(x, y)$ similarly to (3.16)

$$EI\frac{\partial^4 u_2}{\partial y^4} = M\sum_{j=-\infty}^{\infty}\delta(y)b_0u_2 + M\sum_{j=-\infty}^{\infty}\delta(y)f_2(x, y) + g_2(x, y), \tag{3.19}$$

where continuous $2y$-periodic functions $f_2(x, y)$ and $g_2(x, y)$ are determined by $v_0(x)$, $v_1(x)$, $\psi(y)$ and their derivatives:

$$\begin{aligned} f_2(x, y) &= b_1v_1\psi + (\rho b_0 b_1/EI)v_0\psi_4 - 4b_1v_0'\psi_1 + b_2v_0\psi, \\ g_2(x, y) &= (\rho^2 b_0^2/EI)v_0\psi_4 - 8\rho b_0 v_0'\psi_1 + \rho(b_0v_1 + b_1v_0)\psi \\ &\quad + (10EIv_0'' + Nv_0)\psi'' - 4EIv_1'\psi'''. \end{aligned} \tag{3.20}$$

Similarly to equation (3.16), the $2y$-periodic solution of equation (3.19)

$$\begin{aligned} u_2(x, y) &= v_2(x)\psi + (\rho^2 b_0^2/EI^2)v_0\psi_8 - 8(\rho b_0/EI)v_0'\psi_5 \\ &\quad + (\rho/EI)(b_0v_1 + b_1v_0)\psi_4 + (1/EI)(10EIv_0'' + Nv_0)\psi_2 - 4v_1'\psi_1 \end{aligned} \tag{3.21}$$

exists if and only if

$$v_0'' + \beta v_0 = 0, \tag{3.22}$$

with

$$\beta = (\rho^2 b_0^2/10EI^2)(\psi_{80}/\psi_{20}) + (2\rho b_1/10EI)(\psi_{40}/\psi_{20}) + N/10EI + (1/10b_0\psi_{20})b_2, \tag{3.23}$$

which determines long-wave modulations $v_0(x)$ of the tooth-like term $\psi(x/\varepsilon)$. Requirement of finite values of $v_0(x)$, $\beta > 0$, determines the region of the second correction to the frequencies ω_2. The next order modulations $v_1, v_2, \ldots$ and the next corrections to the spectrum should be determined from the next asymptotic equations from the series (3.10).

So, the modulated tooth-like mode, based on the family of $2y$-periodic function is found. The length, λ, of the tooth-like wave is equal to: $\lambda = 2\varepsilon$. This means that the wave number is equal to $k = 2\pi/\lambda = \pi/\varepsilon$. Using (3.13) and (3.18) for the frequencies, one can estimate the dispersion relation for the short waves as

$$\Omega \approx \omega_0/\varepsilon^2 + \omega_1/\varepsilon = \pi k^2\sqrt{48EI/M}(1 - 24\pi\psi_{40}/kM). \tag{3.24}$$

Returning to the asymptotic series (2.2), it is convenient to divide the time dependence into the production of high frequency, $\exp(\mathrm{i}\Omega t)$, and normal frequency, $\exp(\mathrm{i}\omega_2 t)$, terms as was done for $u_0(x, y)$ with respect to the space variable:

$$u(x, x/\varepsilon, t) \approx \exp(\mathrm{i}([\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2]/\varepsilon^2)t)v_0(x)\psi(x/\varepsilon)$$
$$= [\exp(\mathrm{i}\Omega t)\psi(x/\varepsilon)][\exp(\mathrm{i}\omega_2 t)v_0(x)]. \tag{3.25}$$

The representation (3.25) makes it possible to consider $\exp(\mathrm{i}\omega_2 t)v_0(x)$ as an amplitude of a tooth-like vibration $\exp(\mathrm{i}\Omega t)\psi(x/\varepsilon)$, which, in turn is also a wave, but of a long (zero order) wave length, $\lambda_0 = 2\pi/k_0 = 2\pi/\sqrt{\beta}$ (see (3.22, 23)) and normal (zero order) frequency ω_2.

3.3. HIGH FREQUENCY REGION ($\alpha = 2.5$) VIBRATIONS OF THE BEAM BETWEEN IMMOBILE HEAVY MASSES

Obviously, an increase in α leads to an increase in the order of inertial terms. In the case analyzed ε^{-5} is the highest order of the asymptotic series of equations:

$$\varepsilon^{-5}\colon\quad -\mathrm{M}_0 u_0 = 0, \qquad \varepsilon^{-4}\colon\quad A_{-4}u_0 - (\mathrm{M}_0 u_1 + \mathrm{M}_1 u_0) - \mathrm{P}_0 u_0 = 0, \tag{3.26.1, 2}$$

$$\varepsilon^{-3}\colon\quad A_{-3}u_0 + A_{-4}u_1 - \sum_{i+j=2} \mathrm{M}_i u_j - \sum_{i+j=1} \mathrm{P}_i u_j = 0, \tag{3.26.3}$$

$$\varepsilon^{-2}\colon\quad A_{-2}u_0 + A_{-3}u_1 + A_{-4}u_2 - \sum_{i+j=3} \mathrm{M}_i u_j - \sum_{i+j=2} \mathrm{P}_i u_j = 0, \tag{3.26.4}$$

$$\varepsilon^{-1}\colon\quad A_{-1}u_0 + A_{-2}u_1 + A_{-3}u_2 + A_{-4}u_3 - \sum_{i+j=4} \mathrm{M}_i u_j - \sum_{i+j=3} \mathrm{P}_i u_j = 0, \tag{3.26.5}$$

$$\varepsilon^{0}\colon\quad A_0 u_0 + A_{-1}u_1 + A_{-2}u_2 + A_{-3}u_3 + A_{-4}u_4 - \sum_{i+j=5} \mathrm{M}_i u_j - \sum_{i+j=4} \mathrm{P}_i u_j = 0. \tag{3.26.6}$$

Clearly, if one analyzes the frequency range $\alpha = 2{\cdot}5$, one should keep in mind that $\omega_0 \neq 0$, otherwise it reduces to a range already analyzed. Thereby equation (3.26.1) implies the fixed masses in the main order term:

$$u_0\big|_{y=n} = 0. \tag{3.27}$$

At every segment $Y_n = [n-1, n]$ equation (3.26.2) together with (3.27) form a homogeneous boundary value problem. Its solution, $u_0^{(n)}(y) = u_0(y)\big|_{y\in Y_n}$, is:

$$u_0^{(n)}(y) = a_n(\sin(2\lambda y - 2\lambda n + \lambda) - (\sin\lambda/\sinh\lambda)\sinh(2\lambda y - 2\lambda n + \lambda))$$
$$+ b_n(\cos(2\lambda y - 2\lambda n + \lambda) - (\cos\lambda/\cosh\lambda)\cosh(2\lambda y - 2\lambda n + \lambda)), \tag{3.28}$$

$(2\lambda)^4 = \rho b_0/EI$, in terms of local variable $z_n = y - (2n-1)/2$ of Y_n. Jumps in the third derivatives of u_0 at integer y can be illuminated by an additive

$$\mathrm{M}_1 u_1(y) = Mb_1 \sum_{j=-\infty}^{\infty} u_1(j)\delta(y-i)$$

in the right side of (3.26.2). However, first and second derivatives should be continuous. These requirements couple the coefficients a_n, b_n of (3.28) by means a matrix $\mathbf{C}$:

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \mathbf{C}^{n-1} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \tag{3.29}$$

where

$$\mathbf{C} = \frac{1}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \begin{bmatrix} \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21} & -2\alpha_{12}\alpha_{22} \\ -2\alpha_{11}\alpha_{21} & \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21} \end{bmatrix},$$

$$\alpha_{11} = \cos\lambda - \frac{\sin\lambda}{\sinh\lambda}\cosh\lambda, \qquad \alpha_{12} = \sin\lambda + \frac{\cos\lambda}{\cosh\lambda}\sinh\lambda,$$

$$\alpha_{21} = \sin\lambda, \qquad \alpha_{22} = -\cos\lambda. \tag{3.30}$$

Requirement of $u_0(x, y)$ finite values is equivalent to the same for the coefficients (3.29). Necessary analysis is possible, but rather complicated. It will lead to certain corrections, but not to any conceptional conclusions. That is why consideration is restricted to the y-periodic case, as in the previous sections. Because of the representation (3.28), relations (3.29) and expression (3.30) for matrix $\mathbf{C}$, y-periodicity of u_0 implies the existence of two series of eigenmodes. The first one is:

$$u_{0,m}(y) = a_m(x)\sin(\lambda_m(2y - 2n + 1)), \qquad \text{at} \qquad y \in Y_n, \tag{3.31}$$

where $\lambda_m = \pi m$ are the roots of the equation $\alpha_{21} = 0$. The second,

$$u_{0,m}(y) = b_m(x)(\cos(\eta_m(2y - 2n + 1)) + \sin\eta_m \cosh(\eta_m(2y - 2n + 1))) \tag{3.31}$$

corresponds to the solutions η_m of the equation $\alpha_{12} = 0$.

Requirements of u_0, $2y$-, $3y$-, . . . , or quasi y-periodicity will give the new series of representations of the main asymptotic term. Everyone of them has a structure of high-frequency vibration of the beam between neighbouring immobile supports, modulated by the normal slowly changing amplitude $a_m(x)$, $b_m(x)$, etc. Equations for amplitudes and asymptotic values of frequencies can be found by substitution of (3.31.1, 2), into the nest of equations of the asymptotic series (3.26). It makes no sense to expand the text by including unnecessary technical details, which are similar to the techniques presented in the previous sections. That is why such substitutions will not be performed in the framework of the present paper. However, it is interesting to note that there are no non-interacting localized vibrations of the different beam sections between immobile masses M as would be expected. The reason is that M is really large in comparison with ρ, but masses are supposed to be point-like and, hence can "twist", without inertia. If one introduces high inertia moment, independent vibration will occur.

4. CONCLUSIONS

The results of the asymptotic analysis performed give evidence of the existence of three characteristic spectrum regions for the system studied.

4.1. LOW FREQUENCY (LONG-LENGTH WAVE) REGION: LONG WAVE MODE ($\omega \propto \varepsilon^0$)

For the accepted orders of geometrical, inertial and stiffness parameters the main term of the asymptotic series describes the dynamics of a homogeneous continuum beam on an elastic foundation, the inertial properties of which are determined by the values of the discrete masses only. The next three corrections are also of a long-length wave type and account for the inertial properties of the beam itself. Only the fourth term of asymptotic series is sensitive to the discrete nature of the system. It is important to underline that not only displacements, but also bending moments and forces are determined (in the main term) by a continuum approximation. The sketch of this long-wave mode is represented on Figure 2(a).

4.2. INTERMEDIATE FREQUENCY REGION: TOOTH-LIKE MODE ($\omega \propto \varepsilon^2$)

In this region of frequencies the main term of the displacements is a tooth-like antiphase vibration, modulated by a long-wave amplitude (Figure 2(b)). The

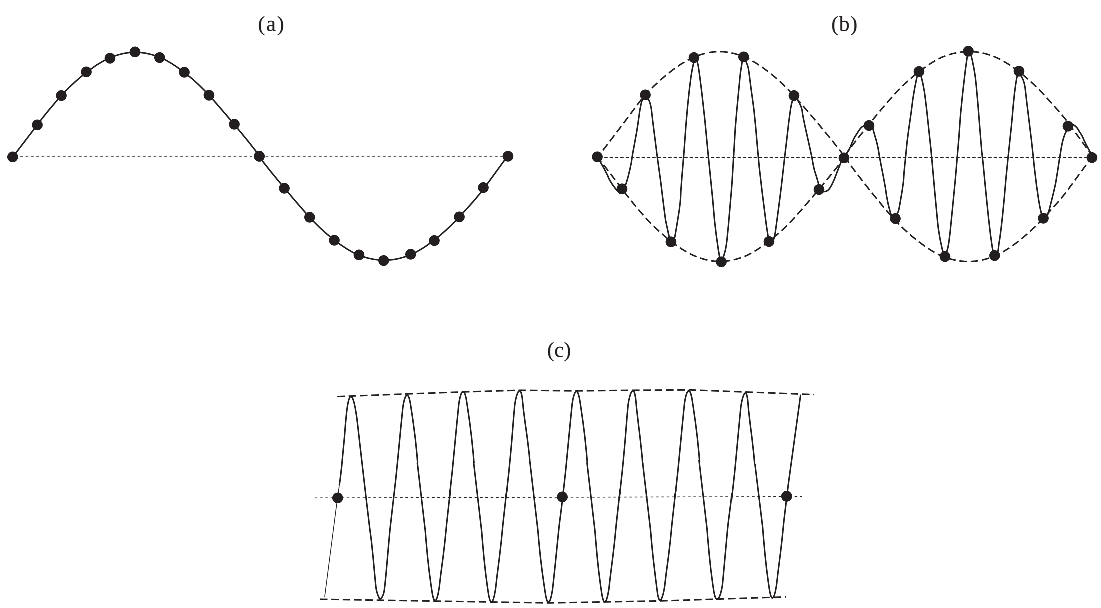

Figure 2. Sketches of the long wave (a), tooth-like (b) and high frequency (c) modes. Displacement of point-like masses and beam are shown by solid circles and solid lines correspondingly. Contour of the long-wave modulation (b, c) is drawn by dotted line.

"tooth" itself, $\psi(y)$ (3.12), and the main frequency, ω_0 (3.13), are determined by the values of a point-like mass, M, and the beam bending rigidity, EI. The rest of the material parameters (beam specific density, ρ, longitudinal load, N, modulus of the springs, c) affect the amplitude, $v_0(x)$, of the antiphase vibrations and the terms of smaller order, ω_1, in particular.

4.3. HIGH FREQUENCY REGION: VIBRATION OF THE BEAM BETWEEN IMMOVABLE SUPPORTS ($\omega \propto \varepsilon^{-2.5}$)

At high frequency, point-like masses are too heavy to move in comparison with the beam. Therefore, the corresponding mode is vibration of the light beam between immovable masses (expressions (3.31)). Because of zero twisting inertia of the point-like masses, the vibrations of the neighbouring segments of the beam are in phase (Figure 2(c)). In the case of large inertia of the masses, vibrations of the beam segments should be independent and one could observe asymptotically localized motion even in the framework of linear analysis.

It should be noted in conclusion that similar analysis is quite possible at arbitrary orders of the physical parameters of the system: another specific density, modulus of the supports, longitudinal load. The choice made is caused by two goals. Clearness of the analysis is the first one. Reach family of the modes is the second. For example, it is clear that in the case when the specific density has the same order of magnitude as the mass, $\rho \propto M$, the inertia of the beam is the same as inertia of the point-like masses and high frequency modes do not exist.

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APPENDIX A: EMPTY FREQUENCY REGIONS

It will be rigorously shown in this Appendix that the rest values of α provide only asymptotically zero solutions of the equation of motion (1).

A.1. NON-SEMIINTEGER α

$$\alpha \neq n/2. \tag{A1}$$

(A1) implies an uncoupled asymptotic series of "elastic" equations

$$\varepsilon^{-4}:\quad A_{-4}u_0 = 0, \qquad \varepsilon^{-3}:\quad A_{-3}u_0 + A_{-4}u_1 = 0.$$

$$\varepsilon^{-2}:\quad A_{-2}u_0 + A_{-3}u_1 + A_{-4}u_2 = 0, \tag{A2.1–3}$$

$$\varepsilon^{-1}:\quad A_{-1}u_0 + A_{-2}u_1 + A_{-3}u_2 + A_{-4}u_3 = 0, \tag{A.2.4}$$

$$e^{0}:\quad A_0u_0 + A_{-1}u_1 + A_{-2}u_2 + A_{-3}u_3 + A_{-4}u_4 = 0, \tag{A2.5}$$

$$e^{1}:\quad A_0u_1 + A_{-1}u_2 + A_{-2}u_3 + A_{-3}u_4 + A_{-4}u_5 = 0, \tag{A.2.6}$$

$$\vdots$$

and so on,

and "inertial" equations

$$\varepsilon^{-2\alpha}:\quad M_0u_0 = 0, \qquad \varepsilon^{-2\alpha+1}:\quad M_0u_1 + M_1u_0 + P_0u_0 = 0, \tag{A.3.1,2}$$

$$\varepsilon^{-2\alpha+2}:\quad M_0u_2 + M_1u_1 + M_2u_0 + P_0u_1 + P_1u_0 = 0, \tag{A.3.3}$$

$$\vdots$$

and so on.

Equations (A.2.1–4) are identical with (3.1.1–4). Therefore, u_0, u_1, u_2 and u_3 should also be y-independent. Moreover, the next chain of equations (A.2) is of the same structure, which provide u_l y-independence for arbitrary l. Exploring this conclusion in the next chain of asymptotic equations, (A.3), provide zero u_l values.

A.2. NEGATIVE α

The case of non-semiinteger α has been analyzed in the section A.1. That is why α can be assumed to be negative, but semiinteger. For example, at $\alpha = 0{\cdot}5$ an asymptotic series of equations is of the following structure (the general case of semiinteger negative values of α can be treated similarly). The four highest order equations of the asymptotic series are the same as (A.2.1–4). Hence, u_0, u_1, u_2 and u_3 are y-independent. Accounting for this circumstance one can rewrite the fifth one as an equation for u_4:

$$EI\frac{\partial^4 u_0}{\partial x^4} - N\frac{\partial^2 u_0}{\partial x^2} + c\sum_{j=-\infty}^{\infty}\delta(y-j)u_0 + EI\frac{\partial^4 u_4}{\partial y^4} = 0, \tag{A.4}$$

which can be solved if and only if $u_0 = 0$. Then the next equation transverses into equation (A.4), but for u_5 and u_1 instead of u_4 and u_0, which provides $u_1 = 0$, etc.

A.3. $\alpha = 0{\cdot}5,\ 1,\ 1{\cdot}5$

These cases are similar, therefore might be analyzed in one item. If $\alpha = 0{\cdot}5$, equations for the terms of the asymptotic series (2.2) are of the following structure. The three first equations of the asymptotic chain are the same as that of the chain (3.1). Therefore the first three terms, u_0, u_1, u_2, depend on x only.

Requirement of u_3 finite values in conjunction with the fourth equation gives

$$\varepsilon^{-1}: \quad A_{-1}u_0 + A_{-2}u_1 + A_{-3}u_2 + A_{-4}u_3 - \mathrm{M}_0 u_0 = 0 \qquad \text{(A.5.1)}$$

and provides zero u_0 and y-independence of u_3. Substitution of these results into equation (A.5.2) yields

$$\varepsilon^{0}: \quad A_0 u_0 + A_{-1}u_1 + A_{-2}u_2 + A_{-3}u_3 + A_{-4}u_4 - \mathrm{M}_0 u_1 - \mathrm{M}_1 u_0 - \mathrm{P}_0 u_0 = 0, \qquad \text{(A.5.2)}$$

making it identical to equation (A.5.1) and correspondingly proves that $u_1 = 0$ and u_4 is y-independent, etc.

Requirement of u_3 finite values in conjunction with equation (A.5.1) provide zero u_0 and y-independence of u_3. Substitution of these results into equation (A.5.2) makes it identical to equation (A.5.1) and, correspondingly, proves that $u_1 = 0$ and u_4 is y-independent, etc.

Increases in α shifts the terms $M\delta(y)b_0u_0$, $M\delta(y)(b_0u_1 + b_1u_0) + \rho b_0u_0$, $M\delta(y)(b_0u_2 + b_1u_1 + b_1u_1) + \rho(b_0u_1 + b_1u_0)$, to higher order asymptotic equations. In particular these terms should be moved from equations (A.5.1), (A.5.2), . . . at $\alpha = 0{\cdot}5$ to equations (5c.1.3), (5c.1.4), . . . at $\alpha = 1$ and to equations (5c.1.2), (5c.1.3), . . . at $\alpha = 1{\cdot}5$. However, these shifts will not change the manner of analysis as well as the consequence about trivial asymptotic solutions.

A.4. ULTRA HIGH FREQUENCY REGION: $\alpha > 2{\cdot}5$

As it was mentioned in section A.2, α can be considered to be semiinteger: $\alpha = 3, 3{\cdot}5, \ldots$ Under this suggestion the highest order of asymptotic equations is not less than ε^{-6} and at least the first two of them are of the form:

$$Mb_0 \sum_{j=-\infty}^{\infty} \delta(y-j)u_0 = 0,$$

$$Mb_0 \sum_{j=-\infty}^{\infty} \delta(y)u_1 + Mb_1 \sum_{j=-\infty}^{\infty} \delta(y-j)u_0 + \rho b_0 u_0 = 0. \qquad \text{(A.6.1, 2)}$$

It follows from equations (A.6.1.1) u_0 is zero at integer arguments. By accounting for this fact in equation (A.6.1.2) implies zero values of u_1 at integer y and zero u_0 at arbitrary value of the argument. Obviously, substitution of these consequences into the next asymptotic equations transforms them into (A.6.1.1), (A.6.1.2), but for u_1, u_2. Therefore, one obtains: $u_1 = 0$, etc.