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THE FREE VIBRATIONS OF TAPERED RECTANGULAR PLATES USING A NEW SET OF BEAM FUNCTIONS WITH THE RAYLEIGH–RITZ METHOD

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In this paper, the free vibrations of a wide range of non-uniform rectangular plates in one or two directions are considered. The domain of the plate is bounded by $x = \alpha_1 a$, a and $y = \beta_1 b$, b in rectangular co-ordinates. The thickness of the plate is continuously varying and proportional to the power function $x^s y^t$. A variety of tapered rectangular plates can be described by giving the taper factors, s and t, different values. s and t may be given arbitrary real numbers if both $\alpha_1 \neq 0$ and $\beta_1 \neq 0$ or arbitrary non-negative numbers if $\alpha_1 = 0$ or $\beta_1 = 0$. The uniform rectangular plate is a special case by letting both s and t equal to zero. A new set of admissible functions which are the static solutions of the tapered beam (or a strip taken from the tapered rectangular plate), under an arbitrary static load expanded into a Taylor series, is developed. Unlike conventional admissible functions, the set of static beam functions will vary appropriately with the thickness variation of the plate. The eigenfrequency equation is obtained by the use of the Rayleigh–Ritz method. A general computer program has been compiled and some numerical results are tabulated. On the basis of comparison with available results in the literature, it is shown that the first few eigenfrequencies can be obtained with good accuracy by using only a small number of terms of the static beam functions.

1. INTRODUCTION

As one of the widely used structural component elements in aerospace, civil and ocean engineering systems etc., the rectangular plate has received much attention. Its dynamic behavior is the subject of many papers in the literature. Much of the work reported by Leissa [1, 2] has been concerned with plates of uniform thickness. However, the problem of the vibration of plates with varying

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thickness has received rather less attention, especially for rectangular plates with varying thickness in two directions.

It is well known that the analytical solutions of vibration characteristics are available for rectangular plates of uniform thickness with some particular types of boundary conditions, such as two opposite edges simply supported. However, the solutions of tapered rectangular plates, even for the linearly varying thickness in one direction and the simplest boundary conditions such as all edges simply supported, have not been obtained analytically. In such a case, although the governing partial differential equation can be reduced to a fourth order ordinary differential equation with variable coefficients, at the very end numerical solution techniques have to be used. Some methods and results have been reported by Leissa [3–5]. A review of the literature quickly reveals that reported work about the free vibration of non-uniform rectangular plates mainly focused on those having two opposite edges simply supported and with linearly varying thickness in one direction. Appl and Byers [6] used the method of upper and lower bounds to analyse the fundamental frequency of a rectangular plate with all edges simply supported and with linear thickness variation in one direction, and Gumeniuk [7] used the finite difference technique to obtain a formula for the fundamental frequencies of such plates. Chopra and Durvasula [8] studied the free vibration of simply supported skew plates with linear thickness variation in one direction by using double Fourier sine series to represent the plate deflection and Lagrange's equation to obtain the eigenfrequency equation. In a recent paper, Kukreti *et al.* [9] used the differential quadrature method and the Rayleigh–Ritz method to analyse the fundamental frequencies of simply supported rectangular plates with linearly varying thickness in one direction. Kobayashi and Sonoda [10] described an application of power series expansions to the free vibration and buckling of rectangular plates with two opposite edges simply supported and linearly varying thickness in one direction. Soni and Sankara Rao [11] used a quintic spline technique to analyse the free vibration of rectangular plates having two opposite edges of simply supported and exponentially varying thickness in one direction, and subsequently Gupta and Lal [12] extended this method to include the effect of an elastic foundation, and Zhou [13] presented an asymptotic solution for the free vibration of such a plate. Recently, Bert and Malik [14] used the differential quadrature method to analyse the free vibration of two opposite edges of simply supported rectangular plates with linearly varying thickness in one direction, and the results were obtained with good accuracy. The free vibrations of non-uniform rectangular plates with other boundary conditions were also considered by some investigators. Ashton [15] used beam eigenfunctions, Grossi and Bhat [16] used boundary characteristic orthogonal polynomials as the admissible functions to study the free vibration of linearly tapered rectangular plates with elastic constraints at the edges. Mukhopadhyay [17] presented a semi-analytical solution for the free vibration of non-uniform rectangular plates, in which beam eigenfunctions in one direction, and a finite difference scheme in another direction were used. Ng and Araar [18] used six terms of two-dimensional polynomial functions as the trial function to study the fundamental eigenfrequency and critical load for a clamped rectangular plate with linear thickness variation. Pulmano and Gupta [19] analysed the free vibration of linearly tapered rectangular plates by the use of the finite strip method, using a combination of beam eigenfunctions and six terms of polynomials. Malhotra *et al.* [20] used the conventional beam functions to study the eigenfrequencies of an orthotropic square plate with parabolic thickness variation. Dawe [21] used the finite element method to study the free vibration of rectangular plates with variable thickness. It should also be noted that Laura and co-workers [22–31] systematically and widely investigated the vibrational characteristics of tapered rectangular plates with various boundary supports (including edges elastically restrained against rotation) by the use of several approximate methods such as the Galerkin method, the Ritz method, the Rayleigh's optimization method, the extended Kantorovich method, the differential quadrature method and combinations of these methods, in which simple polynomials were frequently used as the displacement functions of the plates in one or two directions and are really only valid for estimating lower order eigenfrequencies, especially for estimating the fundamental eigenfrequency of the non-uniform plates.

Although there have been several approaches to calculating the vibration characteristics of non-uniform rectangular plates, it is still of great significance to develop a simpler, more versatile, more efficient and/or more accurate method for solving such problems. In this paper, rectangular plates with variable thickness in the form of a power function are considered. The eigenfrequencies of the plates are found by the Rayleigh–Ritz method. A new set of admissible functions are developed from the static solutions of a tapered beam under a Taylor series load. This tapered beam may be considered as a unit width strip taken from the tapered rectangular plate in the longitudinal direction or the transverse direction. Only a set of strip functions in one direction needs to be derived because the thickness variations in the two directions are both in the form of power functions. This set of static beam functions is also applicable to tapered rectangular plates with sharp edges and rectangular plates with uniform thickness in one or two directions. Some numerical results are given and compared with known results. Good agreement has been achieved.

2. THE DEVELOPMENT OF A SET OF TAPERED BEAM FUNCTIONS

Consider a tapered beam with a unit breadth and the continuously variable depth under an arbitrary load $q(x)$, as shown in Figure 1. The length of the sharp ended beam is l. The origin of the co-ordinate system is at the sharp end of the beam and the x-axis is the centre line of the beam. A truncated beam is considered as part of the sharp ended beam and the actual length of the beam is $L = (1-\alpha)l$. α $(0 \leqslant \alpha < 1)$ is referred to as the truncation factor of the beam and taken as 0 for the sharp ended beam. Assume that the depth $h(x)$ of the beam can be described by a power function

$$h(x) = h_0(x/l)^r \qquad (1)$$

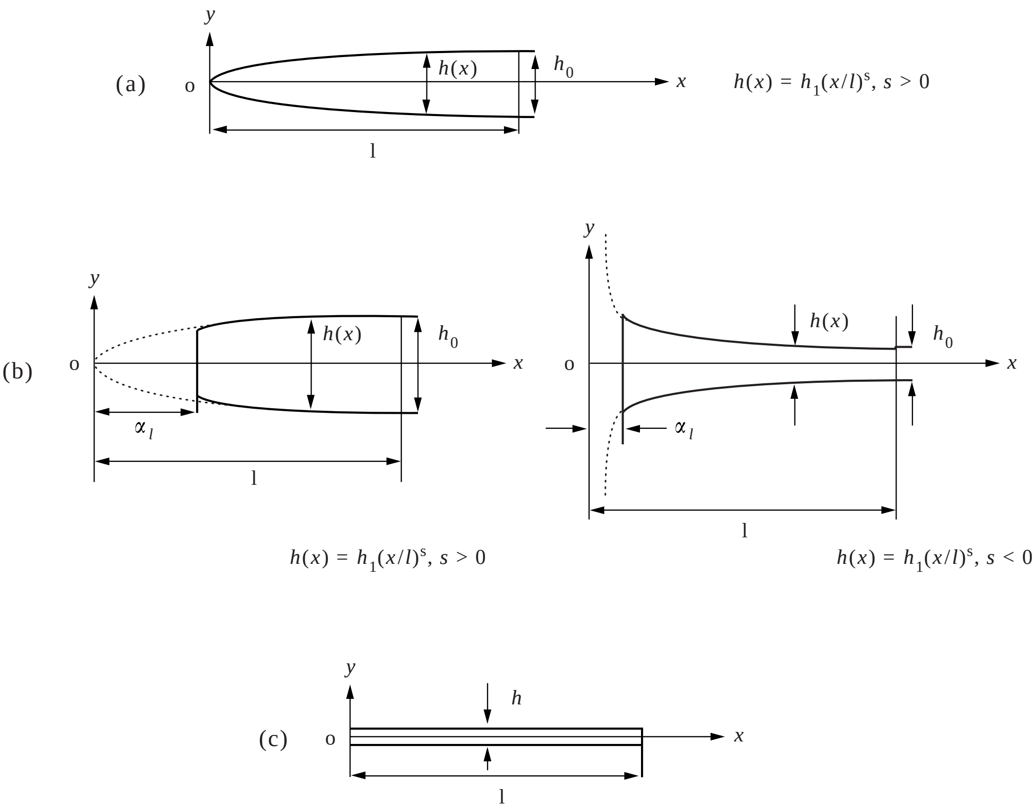

Figure 1. The tapered beams: (a) a sharp ended beam, (b) a truncated beam, (c) a uniform beam.

where h_0 is the depth of the beam at the end $x = l$. The index r, referred to as the taper factor of the beam, is an arbitrary real number for a truncated beam or an arbitrary non-negative number for a sharp ended beam or is equal to zero for a uniform beam.

It is assumed that the breadth and depth dimensions of the beam are small compared with its length and that the Bernoulli–Euler beam theory is valid. It is well-known that the static deflection y of the tapered beam must satisfy the fourth order linear ordinary differential equation

$$\mathrm{d}^2/\mathrm{d}x^2(EI(x)\mathrm{d}^2y/\mathrm{d}x^2) = q(x), \tag{2}$$

where $EI(x)$ is the flexural rigidity of the beam and can be written as

$$EI(x) = Eh^3(x)/12 = (Eh_0^3/12)(x/l)^{3r} = EI_0(x/l)^{3r} \tag{3}$$

in which EI_0 is the flexural rigidity of the beam at the end $x = l$. Introducing a non-dimensional co-ordinate $\xi = x/l$ and designating

$$Q(\xi) = (l^4/EI_0)q(\xi l), \tag{4}$$

equation (2) becomes

$$(\mathrm{d}^2/\mathrm{d}\xi^2)(\xi^{3r}\,\mathrm{d}^2y/\mathrm{d}\xi^2) = Q(\xi). \tag{5}$$

The arbitrary static load $Q(\xi)$ can be expanded, if one wishes, into a Taylor series

$$Q(\xi) = \sum_{i=0}^{\infty} Q_i'\xi^i, \tag{6}$$

where Q_i' are the undetermined constants, which can be determined uniquely if $Q(\xi)$ is a given function.

Substituting equation (6) into equation (5), the solution of the deflection y of the beam may be written in the form of

$$y(\xi) = \sum_{i=0}^{\infty} Q_i y_i(\xi), \tag{7}$$

where Q_i are the constants corresponding to Q_i'. From the theory of linear ordinary differential equation, the solution of equation (5) is made up of two parts: a homogeneous general solution and a non-homogeneous special solution. Substituting equation (7) into equation (5), the solution of the deflection y of the beam may be written as

$$y_i(\xi) = \bar{y}_i(\xi) + \tilde{y}_i(\xi). \tag{8}$$

Substituting equations (6), (7) and (8) into equation (5), the homogeneous general solution $\bar{y}_i(\xi)$ can be obtained analytically as

$$\begin{aligned}
\bar{y}_i(\xi) &= b_0^i + b_1^i\xi + b_2^i\xi^{-3r+2} + b_3^i\xi^{-3(r-1)}, \quad \text{for} \quad r \neq 1/3,\, 2/3,\, 1,\\
\bar{y}_i(\xi) &= b_0^i + b_1^i\xi + b_2^i\xi(\ln\xi - 1) + b_3^i\xi^2, \quad \text{for} \quad r = 1/3,\\
\bar{y}_i(\xi) &= b_0^i + b_1^i\xi + b_2^i\ln\xi + b_3^i\xi(\ln\xi - 1), \quad \text{for} \quad r = 2/3,\\
\bar{y}_i(\xi) &= b_0^i + b_1^i\xi + b_2^i/\xi + b_3^i\ln\xi, \quad \text{for} \quad r = 1,
\end{aligned} \tag{9}$$

where $b_j^i (j = 0,\ 1,\ 2,\ 3)$ are the unknown constants which can be determined uniquely by the boundary conditions of the beam. The non-homogeneous special solution $\tilde{y}_i(\xi)$ may also be obtained analytically as follows:

$$\begin{aligned}
&\tilde{y}_i(\xi) = \xi^{-3r+i+4}, \quad \text{for} \quad i \neq 3(r-1),\, 3r-4,\\
&\tilde{y}_i(\xi) = \xi(\ln\xi - 1), \quad \text{for} \quad i = 3(r-1), \quad \tilde{y}_i(\xi) = \ln\xi, \quad \text{for} \quad i = 3r-4.
\end{aligned} \tag{10}$$

By using $\xi_0 = \alpha$ and $\xi_1 = 1$ to represent the two ends of the beams, the boundary conditions of the tapered beam can be written as

$$y_i(\xi_j) = 0, \quad (\mathrm{d}y_i/\mathrm{d}\xi)|_{\xi=\xi_j} = 0, \quad j = 0, 1 \tag{11a}$$

for the clamped ends and

$$y_i(\xi_j) = 0, \quad (EI(\xi)\ \mathrm{d}^2 y_i/\mathrm{d}\xi^2)|_{\xi=\xi_j} = 0, \quad j = 0, 1 \tag{11b}$$

for the simply supported ends and

$$(EI(\xi)\ \mathrm{d}^2 y_i/\mathrm{d}\xi^2)|_{\xi=\xi_j} = 0, \quad [(\mathrm{d}/\mathrm{d}\xi)(EI(\xi)\ \mathrm{d}^2 y_i/\mathrm{d}\xi^2)]|_{\xi=\xi_j} = 0, \quad j = 0, 1, \tag{11c}$$

for the free ends.

For a truncated beam, by substituting equation (8) into equations (11), the coefficient equations about b^i_j ($j = 0, 1, 2, 3$) may be written in matrix form as

$$\mathbf{F}\mathbf{B}^i = \mathbf{G}^i \tag{12}$$

where

$$\mathbf{B}^i = [b^i_0, b^i_1, b^i_2, b^i_3]^{\mathrm{T}}, \quad \mathbf{G}^i = [g^i_1, g^i_2, g^i_3, g^i_4]^{\mathrm{T}},$$

$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix}. \tag{13}$$

According to the actual boundary conditions of the truncated beam, there are

$$f_{11} = 1, \quad f_{12} = \alpha, \quad f_{13} = \alpha^{-3r+2}, \quad f_{14} = \alpha^{-3(r-1)}, \quad f_{21} = 0, \quad f_{22} = 1,$$

$$f_{23} = (-3r+2)\alpha^{-3r+1}, \quad f_{24} = -3(r-1)\alpha^{-3r+2}, \quad \text{for} \quad r \neq 1/3, 2/3, 1,$$
$$f_{13} = \alpha(\ln\alpha - 1), \quad f_{23} = \ln\alpha, \quad \text{for} \quad r = 1/3,$$
$$f_{13} = \ln\alpha, \quad f_{14} = \alpha(\ln\alpha - 1), \quad f_{23} = 1/\alpha, \quad f_{24} = \ln\alpha, \quad \text{for} \quad r = 2/3,$$
$$f_{14} = \ln\alpha, \quad f_{24} = 1/\alpha, \quad \text{for} \quad r = 1,$$

$$g^i_1 = -\alpha^{-3r+i+4}, \quad g^i_2 = -(-3r+i+4)\alpha^{-3r+i+3}, \quad \text{for} \quad i \neq 3(r-1), 3r-4,$$

$$g^i_1 = -\alpha(\ln\alpha - 1), \quad g^i_2 = -\ln\alpha, \quad \text{for} \quad i = 3(r-1),$$

$$g^i_1 = -\ln\alpha, \quad g^i_2 = -1/\alpha, \quad \text{for} \quad i = 3r-4, \tag{14a}$$

for the clamped left end, and

$$f_{11}=1,\quad f_{12}=\alpha,\quad f_{13}=\alpha^{-3r+2},\quad f_{14}=\alpha^{-3(r-1)},\quad f_{21}=0,\quad f_{22}=0,$$

$$f_{23}=(-3r+2)(-3r+1)\alpha^{-3r},$$

$$f_{24}=-3(r-1)(-3r+2)\alpha^{-3r+2},\quad \text{for}\quad r\neq 1/3,\,2/3,\,1,$$
$$f_{13}=\alpha(\ln\alpha-1),\quad f_{23}=1/\alpha,\quad \text{for}\quad r=1/3,$$

$$f_{13}=\ln\alpha,\quad f_{14}=\alpha(\ln\alpha-1),\quad f_{23}=-1/\alpha^2,\quad f_{24}=1/\alpha,\quad \text{for}\quad r=2/3,$$

$$f_{14}=\ln\alpha,\quad f_{24}=-1/\alpha^2,\quad \text{for}\quad r=1,$$

$$g_1^i=-\alpha^{-3r+i+4},$$

$$g_2^i=-(-3r+i+4)(-3r+i+3)\alpha^{-3r+i+2},\quad \text{for}\quad i\neq 3(r-1),\,3r-4,$$

$$g_1^i=-\alpha(\ln\alpha-1),\quad g_2^i=-1/\alpha,\quad \text{for}\quad i=3(r-1),$$

$$g_1^i=-\ln\alpha,\quad g_2^i=1/\alpha^2,\quad \text{for}\quad i=3r-4, \tag{14b}$$

for the simply supported left end, and

$$f_{11}=0,\quad f_{12}=0,\quad f_{13}=(-3r+2)(-3r+1)\alpha^{-3r},$$

$$f_{14}=-3(r-1)(-3r+2)^{-3r+1},\quad f_{21}=0,\quad f_{22}=0,$$
$$f_{23}=0,\quad f_{24}=-3(r-1)(-3r+2),\quad \text{for}\quad r\neq 1/3,\,2/3,\,1,$$
$$f_{13}=1/\alpha,\quad f_{23}=0,\quad \text{for}\quad r=1/3,$$

$$f_{13}=-1/\alpha^2,\quad f_{14}=1/\alpha,\quad f_{23}=0,\quad f_{24}=1,\quad \text{for}\quad r=2/3,$$

$$f_{14}=-1/\alpha^2,\quad f_{24}=-1,\quad \text{for}\quad r=1,$$

$$g_1^i=-(-3r+i+4)(-3r+i+3)\alpha^{-3r+i+2},$$

$$g_2^i=-(-3r+i+4)(-3r+i+3)(i+2)\alpha^{i+1},\quad \text{for}\quad i\neq 3(r-1),\,3r-4,$$

$$g_1^i=-1/\alpha,\quad g_2^i=-(3r-1)\alpha^{3r-2},\quad \text{for}\quad i=3(r-1),$$

$$g_1^i=1/\alpha^2,\quad g_2^i=(3r-2)\alpha^{3(r-1)},\quad \text{for}\quad i=3r-4, \tag{14c}$$

for the free left end. Identically, the boundary equations at the right end of the truncated beam may also be obtained by letting $\alpha=1$ in the corresponding boundary equations at the left end of the beam, i.e.,

$$f_{3j}=f_{1j}(\alpha=1),\quad f_{4j}=f_{2j}(\alpha=1),\quad j=1,2,3,4,$$
$$g_3^i=g_1^i(\alpha=1),\quad g_4^i=g_2^i(\alpha=1). \tag{15}$$

The solution of equation (12) may be written as follows

$$\mathbf{B}^i = \mathbf{F}^{-1}\mathbf{G}^i \tag{16}$$

Only one inverse calculation for matrix **F** is needed because the summing variable i is not included in matrix **F**. Computational cost is greatly reduced.

The above analysis is valid for the truncated beams. However, for a sharp ended beam, the sharp end ($\alpha = 0$) cannot sustain a bending moment or a shearing force. Hence one has

$$b_2^i = 0, \quad b_3^i = 0. \tag{17}$$

The deflection and the rotation angle of the beam should be finite at any arbitrary co-ordinate $\xi (0 \leqslant \xi \leqslant 1)$. Hence there is a limit to the beginning order of the Taylor series load as follows:

$$i > 3(r-1). \tag{18}$$

In such a case, the static solution of deflection y for the sharp ended beam may be written as

$$y(\xi) = \sum_{i=I}^{\infty} Q_i' y_i(\xi), \tag{19}$$

where

$$I = \max\{\mathrm{Int}(3r-2), 0\}, \quad y_i(\xi) = b_0^i + b_1^i \xi + \xi^{i+4-3r}, \tag{20}$$

in which Int is the integer function. For the cantilevered beam with a sharp end, the coefficients b_0^i and b_1^i can be directly obtained from the boundary conditions of the beam at the clamped end as

$$b_0^i = i + 3(1-r), \quad b_1^i = -i - 4 + 3r. \tag{21}$$

It should be noted that for the beams with rigid body motion, the coefficients $b_j^i (j = 0, 1, 2, 3)$ cannot be uniquely decided by the approach described above either for truncated beams or for sharp ended beams. However, one may consider the total displacement of the beam as a superposition of the rigid body motion and the deflection of the cantilevered beam. For example, for the free–free truncated beam, the expression (7) of the deflection y may be rewritten as

$$y(\xi) = \sum_{i=-2}^{\infty} Q_i' y_i(\xi), \tag{22}$$

where

$$y_{-2}(\xi) = 1, \quad y_{-1}(\xi) = 1 - \xi \tag{23}$$

and other $y_i(\xi)(i = 0, 1, 2, \ldots)$ are those of the cantilevered truncated beam clamped at the end $\xi = 1$. For the free–simply supported truncated beam, the expression (7) of the deflection y may be rewritten as

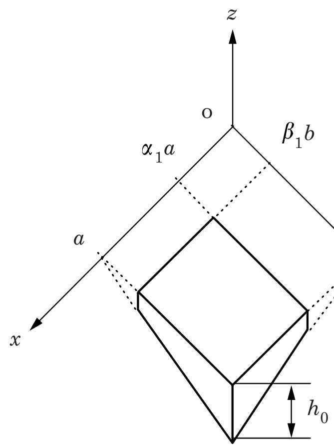

Figure 2. A rectangular plate with variable thickness in two directions.

$$y(\xi) = \sum_{i=-1}^{\infty} Q'_i y_i(\xi), \tag{24}$$

where

$$y_{-1}(\xi) = 1 - \xi \tag{25}$$

and other $y_i(\xi)(i = 0, 1, 2, \ldots)$ are those of the cantilevered truncated beam clamped at the end $\xi = 1$. And for the simply supported–free truncated beam, the expression (24) of the deflection y is still valid but

$$y_{-1}(\xi) = \alpha - \xi \tag{26}$$

and other $y_i(\xi)(i = 0, 1, 2, \ldots)$ are those of the cantilevered truncated beam clamped at the end $\xi = \alpha$.

For the free–free beam with a sharp end, the expression (19) of the deflection y may be rewritten as

$$y(\xi) = \sum_{i=I-2}^{\infty} Q'_i y_i(\xi), \tag{27}$$

where

$$y_{I-2}(\xi) = 1, \quad y_{I-1}(\xi) = 1 - \xi \tag{28}$$

and other $y_i(\xi)(i = I, I + 1, I + 2, \ldots)$ are those of the cantilevered beam with a sharp end. Finally for the free–simply supported beam with a sharp end, the expression (19) of the deflection y may be rewritten as follows

$$y(\xi) = \sum_{i=I-1}^{\infty} Q'_i y_i(\xi), \tag{29}$$

where

$$y_{I-1}(\xi) = 1 - \xi \tag{30}$$

and other $y_i(\xi)(i = I, I + 1, I + 2, \ldots)$ are those of the cantilevered beam with a sharp end.

3. THE RAYLEIGH–RITZ METHOD

A tapered rectangular plate, as shown in Figure 2, lies in the x-y plane and is bounded by edges $x = \alpha_1 a$, a, $y = \beta_1 b$, b where $\alpha_1 (0 \leqslant \alpha_1 < 1)$ and $\beta_1 (0 \leqslant \beta_1 < 1)$ are referred to as the truncation factors of the plate in the x and y directions respectively. The side lengths of the plate are A and B in the x and y directions respectively, in which $A = (1 - \alpha_1)a$, $B = (1 - \beta_1)b$. If the plate has a sharp edge in the x direction then $\alpha_1 = 0$, and if the plate has a sharp edge in the y direction then $\beta_1 = 0$. The truncated plate is part of the sharp ended plate. It is assumed that the thickness $h(x)$ of the plate may be described by a power function as

$$h(x) = h_0 (x/a)^s (y/b)^t, \tag{31}$$

where h_0 is the thickness of the plate at the point $x = a$ and $y = b$, while s and t are referred to as the taper factors of the plate in the x and y directions respectively. s and t may be given arbitrary real numbers if $\alpha_1 \neq 0$ and $\beta_1 \neq 0$, and s or t may be arbitrary non-negative numbers if $\alpha_1 = 0$ or $\beta_1 = 0$. It is clear that equation (31) can describe a variety of non-uniform rectangular plates by giving s and t various values. Some common rectangular plates with variable thickness are listed in Table 1 as examples.

It is assumed that the largest thickness of the plate is small compared to its boundary dimensions and that the classical plate theory is valid, such that the strain energy U and the kinetic energy T of the plate are given by

TABLE 1

Some common rectangular plates with variable thickness

Type of non-uniform rectangular plates	Taper factors s	Taper factors t
Uniform plate	0	0
Linearly tapered plate in the x direction	1	0
Linearly tapered plate in the y direction	0	1
Linearly tapered plate in both directions	1	1
Parabolically tapered plate in the x direction	2	0
Parabolically tapered plate in the y direction	0	2
Parabolically tapered plate in both directions	2	2

$$U = \frac{1}{2}\int_{\alpha_1 a}^{a}\int_{\beta_1 b}^{b} D(x, y)\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} \mathrm{d}y\, \mathrm{d}x,$$

$$T = \frac{1}{2}\int_{\alpha_1 a}^{a}\int_{\beta_1 b}^{b} \rho h(x, y)\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}y\, \mathrm{d}x, \tag{32}$$

where $D(x, y) = Eh^3(x, y)/[12(1-\nu^2)]$ is the flexural rigidity of the plate, E is the Young's modulus, w is the deflection of the plate, ρ is the material density, h is the plate thickness and ν is the Poisson's ratio.

For free vibration, the deflection of the plate may be expressed as

$$w(x, y, t) = W(x, y)\, \mathrm{e}^{\mathrm{i}\omega t}, \tag{33}$$

where $W(x, y)$ is the modal shape function of the plate, ω is the radian eigenfrequency of the plate, t is the time and $\mathrm{i} = \sqrt{-1}$.

Introducing next non-dimensional co-ordinates

$$\xi = x/a, \quad \eta = y/b \tag{34}$$

and substituting equations (31), (33) and (34) into equations (32), the maximum strain energy and the maximum kinetic energy of the plate may be written as

$$U_{\max} = \frac{b}{2a^3} D_0 \int_{\alpha_1}^{1}\int_{\beta_1}^{1} \xi^{3s}\eta^{3t}\left\{\left(\frac{\partial^2 W}{\partial \xi^2}\right)^2 + 2\gamma^2\frac{\partial^2 W}{\partial \xi^2}\frac{\partial^2 W}{\partial \eta^2} + \gamma^4\left(\frac{\partial^2 W}{\partial \eta^2}\right)^2 - 2(1-\nu)\gamma^2\left[\frac{\partial^2 W}{\partial \xi^2}\frac{\partial^2 W}{\partial \eta^2} - \left(\frac{\partial^2 W}{\partial \xi \partial \eta}\right)^2\right]\right\} \mathrm{d}\eta\, \mathrm{d}\xi,$$

$$T_{\max} = \frac{ab}{2}\rho h_0 \omega^2 \int_{\alpha_1}^{1}\int_{\beta_1}^{1} \xi^s \eta^t W^2\, \mathrm{d}\eta\, \mathrm{d}\xi, \tag{35}$$

in which, $D_0 = Eh_0^3/[12(1-\nu^2 0)]$ is the flexural rigidity of the plate at the point $\xi = 1$, $\eta = 1$. $\gamma = a/b = \Gamma(1-\beta_1)/(1-\alpha_1)$ where $\Gamma = A/B$ is the aspect ratio of the plate.

Assuming that the variables in the modal shape function $W(\xi, \eta)$ are separable, the modal function $W(\xi, \eta)$ of the plate may be expressed in terms of a series as

$$W(\xi, \eta) = \sum_{m=M_1}^{\infty}\sum_{n=N_1}^{\infty} A_{mn}\varphi_m(\xi)\psi_n(\eta) \tag{36}$$

where $\phi_m(\xi)$ and $\psi_n(\eta)$ are the appropriate admissible functions which satisfy at least the geometric boundary conditions, and if possible, all the boundary conditions in the Rayleigh–Ritz method. A_{mn} are the unknown coefficients. M_1

and N_1 are the beginning orders of the admissible functions $\phi_m(\xi)$ and $\psi_n(\eta)$ respectively, which are dependent on the practical case to be investigated.

Substituting equation (36) into equation (35) and minimizing the total potential energy of the plate with respect to the coefficients A_{mn}

$$(\partial/\partial A_{mn})(U_{\max} - T_{\max}) = 0 \tag{37}$$

will lead to the next eigenfrequency equation

$$\sum_{m=M_1}^{\infty}\sum_{n=N_1}^{\infty}[C_{mnij} - \lambda^2\bar{E}_{mi}\bar{F}_{nj}]A_{mn} = 0, \quad i = M_1, M_1+1, M_1+2, \ldots, \infty$$

$$j = N_1, N_1+1, N_1+2, \ldots, \infty, \tag{38}$$

where

$$C_{mnij} = E_{mi}^{(2,2)}F_{nj}^{(0,0)} + 2\gamma^2(1-\nu)E_{mi}^{(1,1)}F_{nj}^{(1,1)} + \gamma^4E_{mi}^{(0,0)}F_{nj}^{(2,2)} + \nu\gamma^2(E_{mi}^{(0,2)}F_{nj}^{(2,0)} + E_{mi}^{(2,0)}F_{nj}^{(0,2)}), \tag{39}$$

in which

$$\lambda^2 = \rho h_0\omega^2a^4/D_0 = \rho h_0\omega^2A^4/[D_0(1-\alpha_1)^4] = \Omega^2/(1-\alpha_1)^4,$$

$$E_{mi}^{(p,q)} = \int_{\alpha_1}^{1}\xi^{3s}(\mathrm{d}^p\varphi_m/\mathrm{d}\xi^p)(\mathrm{d}^q\varphi_i/\mathrm{d}\xi^q)\,\mathrm{d}\xi,$$

$$F_{nj}^{(p,q)} = \int_{\beta_1}^{1}\eta^{3t}(\mathrm{d}^p\psi_n/\mathrm{d}\eta^p)(\mathrm{d}^q\psi_j/\mathrm{d}\eta^q)\,\mathrm{d}\eta, \quad p, q = 0, 1, 2,$$

$$\bar{E}_{mi} = \int_{\alpha_1}^{1}\xi^s\varphi_m\varphi_i\,\mathrm{d}\xi, \quad \bar{F}_{nj} = \int_{\beta_1}^{1}\eta^t\psi_n\psi_j\,\mathrm{d}\eta. \tag{40}$$

Truncating m, n, i, j in equation (38), the solution yields the eigenfrequencies of vibration of the plate together with the coefficients for the modal shapes (36). Here the static tapered beam functions presented in the last section are taken as the admissible functions of the tapered rectangular plate, i.e.,

$$\varphi_m(\xi) = y_m(\xi), \quad \psi_n(\eta) = y_n(\eta), \tag{41}$$

where $y_m(\xi)$ are the mth static tapered beam functions which satisfy the thickness variation and the boundary conditions of the plate in the x direction and $y_n(\eta)$ are the nth static tapered beam functions which satisfy those in the y direction. Correspondingly, the taper factor r of the beam should be replaced by the taper factors s and t of the plate in the x and y directions respectively, while the truncation factor α of the beam should be replaced by the truncation factors α_1 and β_1 of the plate in the x and y directions respectively. In addition, the length l of the sharp ended beam should be replaced by the lengths a and b of the sharp ended plate in the x and y directions respectively and the length L of the truncated beam should be replaced by the lengths A and B of the truncated plate in the x and y directions respectively. M_1 and N_1 correspond to the beginning orders of $y_m(\xi)$ and $y_n(\xi)$ respectively.

4. NUMERICAL EXAMPLES

In order to demonstrate the applicability and the accuracy of the proposed approach, equation (38) has been used to generate results for several tapered rectangular plates in one or two directions with a Poisson's ratio $\nu = 0{\cdot}3$. The Chebyshev–Gauss numerical quadrature with 40 points is applied to the integrations in equation (40) for the truncated rectangular plates, although the exact values of these integrations may also be obtained analytically. However analytical values of the integrations in equation (40) are given for the sharp ended rectangular plates. Any desired accuracy may be theoretically achieved simply by increasing the number of terms of admissible functions. However, the convergency study shows that there is a limit to increasing the numbers of terms of the admissible functions, especially for plates with higher truncation factors.

TABLE 2

The fundamental eigenfrequency parameters Ω_1 of a linearly tapered square plate in one direction, the values in ( ) are from reference [14] and the values in [ ] are from reference [10], which have been transformed

Boundary conditions	Terms $m \times n$	Truncation factor α_1				
		1/10	1/5	1/3	1/2	5/7
SSSS	1 × 1	10·329	11·537	12·998	14·736	16·909
	2 × 2	9·7183	11·123	12·750	14·615	16·875
	3 × 3	9·6930	11·108	12·739	14·607	16·867
	4 × 4	9·6921	11·107	12·739	14·607	16·867
	5 × 5	9·6919	11·107	12·739	14·607	16·867
	–	–	–	–	[14·604]	(16·864)
FCFC	4 × 4	5·5227	5·4136	5·5042	5·7781	6·2436
CCCC	4 × 4	16·005	19·117	22·549	26·306	30·668
SSFF	4 × 5	3·8414	4·8469	5·8961	6·9967	8·2294
FCFF	4 × 5	4·5818	4·2466	3·9851	3·7857	3·6239
CCFF	4 × 5	9·8464	11·791	13·924	16·254	18·954
FFSS	5 × 4	5·6030	5·8677	6·4111	7·2072	8·2759
	–	–	–	–	–	(8·2400)
FFCC	5 × 4	9·1616	10·326	12·324	15·035	18·457
FFFC	5 × 4	2·3914	2·4342	2·5363	2·7185	3·0158
CCSS	4 × 4	12·912	15·403	18·153	21·167	24·671
	–	–	–	–	[21·160]	–
SSCC	4 × 4	13·814	16·008	18·512	21·341	24·719
FFFF	6 × 6	8·0031	8·3679	9·1066	10·158	11·573

Furthermore, the actual accuracy is also dependent on the number of significant figures used by the computer. Quadruple precision is used in the numerical computation. For the sake of brevity, four capital letters are used to represent the boundary conditions of the plate. The first two represent the boundary conditions of the plate in the x direction and others represent those in the y direction.

The fundamental eigenfrequency parameters Ω_1 of a linearly tapered square plate in the x direction for different boundary conditions and truncation factor α_1 are given in Table 2. The zero eigenfrequencies for the plate with fully free edges are not listed here. A convergency study has been carried out for a simply supported square plate with linearly variable thickness in the x direction. It is found that for such a plate, good accuracy may be obtained by using only two terms of the admissible functions in each direction, and the higher the truncation factor is, the more rapid is the convergency. From the table, one can see that the eigenfrequency parameters increase with the increase of the truncation factor of the plate except for the FCFF plate. Comparing the results with those obtained from the literature shows that the proposed method has good accuracy and rapid convergency. Furthermore, it should be pointed out that both the accuracy and the convergency of the present method are better than those using conventional admissible functions for constant section beams such as vibrating beam functions [2]. This is especially true for plates with larger taper factors and smaller truncation factors, as seen in Appendix A.

TABLE 3

The first five eigenfrequency parameters Ω_i (i = 1, 2, …, 5) of a uniform square plate, the values in ( ) are from reference [2]

Boundary conditions	Mode number i				
	1	2	3	4	5
SSSS	19·743	49·354	49·354	78·971	98·733
	(19·739)	(49·348)	(49·348)	(78·957)	(98·691)
CCCC	36·004	73·432	73·432	108·27	131·83
	(35·992)	(73·413)	(73·413)	(108·27)	(131·64)
SSFF	9·6605	16·162	36·725	39·052	46·883
	(9·6314)	(16·135)	(36·726)	(38·945)	(46·738)
CCFF	22·227	26·485	43·625	61·337	67·403
	(22·272)	(26·529)	(43·664)	(61·466)	(67·549)
CFCF	6·9465	24·029	26·673	47·757	63·023
	(6·9421)	(24·034)	(26·681)	(47·785)	(63·039)
CFFF	3·4853	8·5143	21·350	27·298	31·032
	(3·4917)	(8·5246)	(21·429)	(27·331)	(31·111)
SFSF	3·4011	17·398	19·369	38·283	51·300
	(3·3687)	(17·407)	(19·367)	(38·291)	(51·324)
FFFF	13·476	19·656	24·318	34·871	34·872
	(13·489)	(19·789)	(24·432)	(35·024)	(35·024)

The proposed approach is also suitable for the free vibration of a uniform square plate which is considered as the special case, by letting both the taper factors s and t of the plate equal to zero. The first five eigenfrequency parameters Ω_i ($i = 1, 2, \ldots, 5$) of a uniform square plate with different boundary conditions are given in Table 3. The three zero eigenfrequencies are not listed here for the plate with fully free edges. Six terms of the static uniform beam functions are used in each direction; however, eight terms are used for the FF boundary conditions and seven terms are used for the SF boundary conditions. The results are compared with those in reference [2]. Good agreement has been observed.

TABLE 4

The first five eigenfrequency parameters Ω_i ($i = 1, 2, \ldots, 5$) *of a simply supported rectangular plate with variable thickness in one direction, the values in ( ) are from reference* [14] *and the values in* [ ] *are from reference* [10], *which have been transformed*

Γ	s	α_1	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
1/2	1/2	1/5	9·1607	14·7317	23·821	31·085	36·591
		1/3	9·8543	15·808	25·623	33·474	39·391
		1/2	10·586	16·957	27·527	35·978	42·333
		5/7	11·397	18·241	29·639	38·745	45·587
	1	1/5	6·6773	11·107	17·566	22·748	26·791
		1/3	7·8023	12·739	20·402	26·551	31·353
		1/2	9·0507	14·607	23·589	30·782	36·271
			[8·9690]	[14·564]	[23·542]	–	–
		5/7	10·518	16·867	27·371	35·766	42·096
			(10·518)	(16·864)	(27·361)	(35·764)	(41·978)
1	1/2	1/5	14·732	36·275	36·591	58·765	70·562
		1/3	15·808	39·207	39·391	64·146	77·286
		1/2	16·956	42·247	42·333	67·788	83·985
		5/7	18·241	45·566	45·588	72·951	91·004
	1	1/5	11·107	25·773	26·960	43·797	46·569
		1/3	12·729	30·588	31·353	50·603	57·739
		1/2	14·607	35·881	36·271	58·265	69·895
			[14·604]	[35·877]	[36·267]	–	–
		5/7	16·863	41·984	42·097	67·422	83·372
			(16·864)	(41·978)	(42·090)	(67·411)	(83·379)
2	1/2	1/5	36·274	58·765	95·021	116·05	148·30
		1/3	39·206	63·146	102·36	128·96	158·94
		1/2	42·246	67·788	110·05	141·56	170·06
		5/7	45·571	72·951	118·54	154·32	182·61
	1	1/5	25·773	42·796	69·975	72·429	106·39
		1/3	30·587	50·603	81·451	92·896	126·10
		1/2	35·881	58·264	94·286	15·49	146·40
			[36·199]	[58·415]	[94·325]	–	–
		5/7	41·997	67·406	109·47	140·48	169·30
			(41·978)	(67·411)	(109·40)	(140·59)	(168·19)

TABLE 5

The fundamental eigenfrequency parameters Ω_1 of a linearly tapered square plate in two directions

Boundary conditions	Truncation factors $\alpha_1 = \beta_1$				
	1/10	1/5	1/3	1/2	5/7
SSSS	4·9012	6·3235	8·2490	10·814	14·410
CCCC	7·1689	10·180	14·137	19·225	26·122
SSFF	2·4695	3·1188	3·9983	5·2384	7·0409
CCFF	4·3535	5·6920	7·8389	11·033	15·731
SSCC	6·2352	8·5537	11·620	15·602	21·056
FCFF	2·7323	2·7848	2·8369	2·9390	3·1379
FCFC	2·9607	3·3238	3·8426	4·5517	5·5558
FSFS	2·5305	2·5282	2·5561	2·6558	2·8948
SCFF	3·9783	4·9642	6·4756	8·5954	11·476
SSFS	3·3917	3·9119	4·7768	6·1450	8·3076
SSFC	3·6918	4·3213	5·2857	6·7621	9·0722
SCSC	5·9928	8·1347	10·972	14·662	19·723
CCFC	4·3891	5·7630	7·9585	11·255	16·257
CSFF	2·9495	3·9026	5·2837	7·3492	10·503
CSCS	5·8290	7·9679	10·819	14·537	19·644
SSCS	5·3463	7·1167	9·5091	12·682	17·112
CCCS	6·7371	9·3804	12·827	17·247	23·237
SSSC	5·6508	7·4916	9·9075	13·045	17·359
CCSC	6·5551	9·1185	12·521	16·951	23·029
SCCS	6·1031	8·2406	11·043	14·688	19·713
SCFS	3·9900	4·9855	6·5306	8·7638	12·020
CSFS	3·6769	4·4524	5·7695	7·8892	11·305

The first five eigenfrequency parameters Ω_i ($i = 1, 2, \ldots, 5$) of a simply supported rectangular plate with variable thickness in the x direction for different aspect ratio Γ, truncation factor α_1 and taper factor s are given in Table 4. Five terms of the static beam functions are used in each direction. It is shown that the eigenfrequency parameters decrease with the increase of the taper factor s but increase with the increase of the truncation factor α_1 and the aspect ratio Γ. The results are compared with those from references [14] and [10]. Good agreement has been observed.

The fundamental eigenfrequency parameters Ω_1 of a linearly tapered square plate with the same taper factors in both directions and with different boundary conditions are given in Table 5. Three terms of the static tapered beam functions are used in each direction, but four terms are used for the FF boundary conditions. It can be seen from the table that the plate with fully clamped edges always gives the highest eigenfrequency parameters when compared with other boundary conditions. This is because the plate with fully clamped edges is the

TABLE 6

The eigenfrequency parameters Ω_i $(i = 1, 2, \ldots, 5)$ *of a cantilevered tapered rectangular plate with a sharp edge*

Γ	s	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
1/2	1/2	4·5753	5·4874	8·1950	12·676	18·450
	3/4	4·8972	5·6584	7·7130	11·017	15·390
	1	5·2879	5·7794	7·3100	9·6956	12·735
	3/2	5·3163	5·6426	6·3526	7·3536	8·4926
1	1/2	4·5443	7·8693	16·893	19·276	22·424
	3/4	4·9530	7·5765	14·421	17·200	19·490
	1	5·2521	7·2608	12·233	15·106	16·719
	3/2	5·3841	6·3471	8·3834	10·372	10·573
3/2	1/2	4·5180	10·798	18·635	26·674	30·359
	3/4	4·9224	10·098	16·878	22·546	23·543
	1	5·2181	9·2905	14·923	18·183	18·991
	3/2	5·3505	7·3181	10·307	10·579	12·561
2	1/2	4·4986	13·891	18·623	31·694	43·245
	3/4	4·8994	12·834	16·862	26·139	33·237
	1	5·1917	11·509	14·906	21·562	24·159
	3/2	5·3217	8·3162	10·495	12·243	13·996

stiffest. It can also be seen from the table that the eigenfrequency parameters generally increase with the increase of the truncation factor of the plate.

The first five eigenfrequency parameters Ω_i $(i = 1, 2, \ldots, 5)$ of a cantilevered tapered rectangular plates with a sharp edge are tabulated in Table 6. The results for different taper factor s and aspect ratio Γ are given. Six terms of the static tapered beam functions in the x direction and eight terms of the static uniform beam functions in the y direction are used.

5. CONCLUDING REMARKS

A new set of admissible functions which are the static solutions of a tapered beam, or a strip with unit breadth taken from the rectangular plate under consideration in the longitudinal or transverse directions, under Taylor series loads, is established for the free vibration analysis of a wide range of non-uniform rectangular plates. Some numerical results are tabulated and compared with those available from the literature, and good agreement has been observed. It can be seen that unlike the conventional admissible functions, this set of static tapered beam functions can appropriately vary with the thickness variation of the plate. The basic concept to form the set of static beam functions is theoretically sound, and very clear and simple, and requires no complicated mathematical knowledge. This approach is suitable for both tapered rectangular plates and uniform rectangular plates with various boundary conditions and may be also directly applied in the case of edges elastically restrained. Although in general it is possible to improve accuracy by increasing the number of terms of the admissible functions, a large number of terms will however lead to the appearance of an ill-conditioned eigenfrequency equation. Fortunately the first few eigenfrequencies can be obtained with good accuracy by using only a small number of terms of the static beam functions.

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APPENDIX A

The comparison of the first two eigenfrequency parameters $\Omega_i = \sqrt{\rho b h_0 / E I_0} L^2 \omega_i$ $(i = 1, 2)$, using the static beam functions (for tapered beams) presented in this paper and the conventional vibrating beam functions

Taper factor r	Truncation factor α	Method	Number of terms	Ω_1	Ω_2
Clamped–clamped beam					
2	0·1	VBF†	4	8·2585	22·2946
			8	6·6971	15·5012
		Present	4	5·7931	12·1383
	0·3	VBF	4	9·5950	26·4739
			8	8·9333	22·9517
		Present	4	8·8191	22·4504
1	0·1	VBF	4	10·9905	31·7723
			8	10·1738	28·1277
		Present	4	9·8847	27·0355
	0·3	VBF	4	13·6359	38·0864
			8	13·4976	37·1310
		Present	4	13·4835	37·0727
Simply–simply supported beam					
2	0·1	VBF	4	1·3793	12·7737
			8	0·92885	9·2279
		Present	4	0·73041	7·7919
	0·3	VBF	4	2·8981	15·9297
			8	2·7617	14·3940
		Present	4	2·7474	14·2523
1	0·1	VBF	4	4·1244	20·0607
			8	3·9317	18·4365
		Present	4	3·8895	18·1251
	0·3	VBF	4	5·7663	24·4316
			8	5·7466	24·1088
		Present	4	5·7454	24·0949

†VBF denotes vibrating beam functions.

(for constant section beams), for the beams with length L, width b and linearly ($r = 1$) or parabolically ($r = 2$) varying height.